Stability results for scattered data interpolation on the rotation group

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Fourier analysis on the rotation group $SO(3)$ expands each function into the orthogonal basis of Wigner-D functions. Recently, fast and reliable algorithms for the evaluation of finite expansion of such type, referred to as nonequispaced FFT on $SO(3)$, have become available. Here, we consider the minimal norm interpolation of given data by Wigner-D functions. We prove bounds on the conditioning of this problem which rely solely on the number of Fourier coefficients and the separation distance of the sampling nodes. The reconstruction of N^3 Fourier coefficients from M well separated samples is shown to take only $\mathcal{O}(N^3 \log^2 N + M)$ floating point operations.

Math Subject Classifications: 65T50, 65F10, 43A75, 41A05, 15A60.
Key words and phrases: scattered data interpolation, iterative methods, FFTs.

1 Introduction

Scattered data interpolation and approximation on various domains is a practical problem with many important applications in science and engineering. In our particular setting, we are interested in functions defined on the rotation group $SO(3)$, cf. [26, 24]. Recent applications include protein-protein docking problems [3] and texture analysis in crystallography [25]. Given a set of measurements $(\boldsymbol{G}_j, y_j) \in SO(3)\times\mathbb{C}$, $j = 0, \ldots, M-1$, discrete least squares approximation by Wigner-D functions (similar to the complex exponentials on the circle) relies on two ingredients: a fast Fourier transform on the rotation group, see [9, 19] and estimates on the involved condition numbers by means of Marcinkiewicz-Zygmund inequalities [13, 6, 8, 22].

On the other hand, interpolation by radial basis functions on $\mathbb{R}^d$ has become a mature tool during the last decade, see e.g. [27] and references therein. Recent generalizations to other domains include manifolds like the Euclidean spheres [17, 12, 10] or compact groups like $SO(3)$ [7, 5, 4]. Central themes in the study of such methods are their convergence rates and the conditioning of proposed solution schemes, see [21, 23] for a trade-off principle.

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We are interested in the condition numbers of interpolation matrices and follow the seminal papers [18, 1] to prove explicit bounds for the extremal eigenvalues of the interpolation problem with respect to the separation distance of the sampling nodes. More specific, the simple constraint that the polynomial degree is bounded from below by a constant multiple of the inverse separation distance turns out to be a sharp condition that allows for polynomial interpolation, cf. Theorem 3.5. Our result implies that N^3 Fourier coefficients can be computed from M well separated samples in $\mathcal{O}(N^3 \log^2 N + M)$ floating point operations, see Corollary 3.6. Moreover, Corollary 3.8 generalizes and improves a recent result [17, Theorems 2.8, 3.6] on the deterioration of the smallest eigenvalue for interpolation with minimal Sobolev norm. The proof of our main result relies on a packing argument [5, Lemma 5.1], the construction of strongly localized polynomials on the rotation group by using a smoothness-decay principle in Fourier analysis [14, 16, 11], and a simple eigenvalue estimate by the Gershgorin circle theorem.

2 Prerequisite

Let $SO(3) := \{\boldsymbol{G} \in \mathbb{R}^3 : \boldsymbol{G}^T\boldsymbol{G} = \boldsymbol{I}, \det \boldsymbol{G} = 1\}$ denote the (compact semisimple Lie) group of rotations in the Euclidean space $\mathbb{R}^3$, cf. [26, 24]. The parameterization of $SO(3)$ in terms of Euler angles $(\phi_1, \theta, \phi_2) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$ allows the following representation of rotations

$$\boldsymbol{G} = \boldsymbol{G}(\phi_1, \theta, \phi_2) = \boldsymbol{R}_z(\phi_1)\boldsymbol{R}_y(\theta)\boldsymbol{R}_z(\phi_2)$$

where

$$\boldsymbol{R}_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{R}_y(t) = \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & 1 & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix}.$$

Moreover, $SO(3)$ can be identified with the three dimensional projective space such that $SO(3) \ni \boldsymbol{G} \mapsto \omega\boldsymbol{x}$ with rotation axis $\boldsymbol{x}$, i.e. $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{x}$, $\|\boldsymbol{x}\| = 1$, and rotation angle $\omega \in [0, \pi]$. In particular, this yields the translation invariant metric

$$\mathrm{d}(\boldsymbol{G}, \boldsymbol{H}) := \omega(\boldsymbol{H}^{-1}\boldsymbol{G}).$$

Now, let a sampling set $\mathcal{X} := \{\boldsymbol{G}_j \in SO(3) : j = 0, \ldots, M-1\}$, $M \in \mathbb{N}$, be given and measure its "nonuniformity" by the separation distance

$$q_{\mathcal{X}} := \min_{0 \le j < l < M} \mathrm{d}(\boldsymbol{G}_j, \boldsymbol{G}_l).$$

The sampling set $\mathcal{X}$ is called q-separated for some $0 < q \le \pi$ if $q_{\mathcal{X}} \ge q$. Moreover, we decompose the sampling set $\mathcal{X} \subset SO(3)$ into shells

$$R_{\mathcal{X},q,m} := \{\boldsymbol{G} \in \mathcal{X} : mq \le \mathrm{d}(\boldsymbol{G}, \boldsymbol{I}) < (m+1)q\}, \qquad m \in \mathbb{N}. \tag{2.1}$$

For measurable functions $f : SO(3) \to \mathbb{C}$, the normalized Haar integral is given by

$$\int_{SO(3)} f(\boldsymbol{G}) \mathrm{d}\mu(\boldsymbol{G}) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(\phi_1, \theta, \phi_2) \sin(\theta) \mathrm{d}\phi_2 \mathrm{d}\theta \mathrm{d}\phi_1.$$

A function only depending on the rotation angle $\omega = \omega(\boldsymbol{G})$ is called conjugate invariant (or central) and the above integral simplifies to

$$\int_{SO(3)} f(\boldsymbol{G}) \mathrm{d}\mu(\boldsymbol{G}) = \frac{2}{\pi} \int_0^\pi f(\omega) \sin^2\left(\frac{\omega}{2}\right) \mathrm{d}\omega.$$

In analogy to the complex exponentials $\mathrm{e}^{\mathrm{i}kx}$ on the circle, the Wigner-D functions $D_l^{k,k'}$ of degree $l \in \mathbb{N}_0$ and orders $k, k' = -l, \dots, l$ are the key to Fourier analysis on the rotation group. First, let the space of square integrable functions on the unit sphere $f : \mathbb{S}^2 \to \mathbb{C}$ be decomposed into the mutual orthogonal spaces of spherical harmonics of degree $l \in \mathbb{N}_0$ and let $\{Y_l^k : \mathbb{S}^2 \to \mathbb{C} : k = -l, \dots, l\}$ denote an orthonormal basis for each of them, see [15] for details. Then, the Wigner-D functions are defined pointwise by

$$D_l^{k,k'}(\boldsymbol{G}) := \int_{\mathbb{S}^2} Y_l^{k'}\left(\boldsymbol{G}^{-1}\boldsymbol{\xi}\right) \overline{Y_l^k(\boldsymbol{\xi})} \mathrm{d}\mu_{\mathbb{S}^2}(\boldsymbol{\xi}) \quad \text{and} \quad \int_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}(\boldsymbol{\xi}) = 4\pi.$$

They form an orthogonal basis of $L^2(SO(3))$, are normalized by $\|D_l^{k,k'}\|_{L^2}^2 = 1/(2l+1)$, and every $f \in L^2(SO(3))$ obeys the series expansion $f = \sum_{l\in\mathbb{N}_0} \sum_{k,k'=-l}^{l} \hat{f}_l^{k,k'} D_l^{k,k'}$ with Fourier-Wigner coefficients

$$\hat{f}_l^{k,k'} = (2l+1) \int_{SO(3)} f(\boldsymbol{G}) \overline{D_l^{k,k'}(\boldsymbol{G})} \mathrm{d}\mu(\boldsymbol{G}).$$

For later reference we define a family of Sobolev spaces $H_s^2 \subset L^2(SO(3))$, $s > 2$, with inner product and norm

$$\langle f, g \rangle_{H_s^2} := \sum_{l\in\mathbb{N}_0} \sum_{k,k'=-l}^{l} (1+l)^s \hat{f}_l^{k,k'} \overline{\hat{g}_l^{k,k'}}, \qquad \|f\|_{H_s^2} = \sqrt{\langle f, f \rangle_{H_s^2}}.$$

One of the more remarkable properties of the Wigner-D functions is the addition theorem

$$\sum_{k,k'=-l}^{l} D_l^{k,k'}(\boldsymbol{G}) \overline{D_l^{k,k'}(\boldsymbol{H})} = U_{2l}\left(\cos \frac{\mathrm{d}(\boldsymbol{G}, \boldsymbol{H})}{2}\right), \tag{2.2}$$

where $U_l(\cos\omega) = \sin((l+1)\omega)/\sin(\omega)$ denotes the l-th Chebyshev polynomial of second kind. For a sampling set $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \dots, M-1\}$, we call

$$\boldsymbol{D} = \left(D_l^{k,k'}(\boldsymbol{G}_j)\right)_{j,(l,k,k')} \in \mathbb{C}^{M \times d_N}$$

the nonequispaced Fourier matrix on the rotation group. Given a vector of Fourier coefficients $\hat{\boldsymbol{f}} \in \mathbb{C}^{d_N}$, $d_N = \frac{1}{6}(2N+1)(2N+2)(2N+3)$, we call

$$f(\boldsymbol{G}) = \sum_{l=0}^{N} \sum_{k,k'=-l}^{l} \hat{f}_l^{k,k'} D_l^{k,k'}(\boldsymbol{G})$$

the corresponding polynomial on the rotation group. Its evaluation at the sampling nodes $\mathcal{X} \subset SO(3)$ can be written in matrix vector from by $\boldsymbol{f} = (f(\boldsymbol{G}_j))_{j=0,\dots,M-1} = \boldsymbol{D}\hat{\boldsymbol{f}}$.

In what follows, we study the underdetermined interpolation of scattered data on $SO(3)$ by polynomials. Let $M < d_N$, a sampling set $\mathcal{X} = \{\boldsymbol{G}_j \in SO(3) : j = 0, \dots, M-1\}$, values $y_j \in \mathbb{C}$ for $j = 0, \dots, M-1$, and weights $\hat{w}_l > 0$ for $l = 0, \dots N$, be given. Then, the minimal norm interpolation problem is given by

$$\min_{\hat{\boldsymbol{f}}} \sum_{l=0}^{N} \sum_{k,k'=-l}^{l} \frac{\left|\hat{f}_l^{k,k'}\right|^2}{\hat{w}_l} \quad \text{s.t.} \quad f\left(\boldsymbol{G}_j\right) = y_j \text{ for } j = 0 \dots M-1. \tag{2.3}$$

3 Results

Lemma 3.1. *Every q-separated sampling set $\mathcal{X} \subset SO(3)$ has cardinality*

$$M \leq \frac{109\pi}{2q^3}.$$

Moreover, there exists a q-separated sampling set of cardinality

$$M \geq \frac{6\pi}{q^3}.$$

Given a q-separated sampling set, its decomposition into shells $R_{\mathcal{X},q,m}$, cf. (2.1)*, allows for the cardinality estimate*

$$|R_{\mathcal{X},q,m}| \leq 141 m^2.$$

Proof. Let $B_r = \{\boldsymbol{G} \in SO(3) : \mathrm{d}(\boldsymbol{G}, \boldsymbol{I}) \leq r\}$ denote the cap of rotation angle $r \in (0, \pi]$ around the identity with measure

$$\mu\left(B_r\right) = \int_{B_r} \mathrm{d}\mu\left(\boldsymbol{G}\right) = \frac{2}{\pi} \int_0^r \sin^2\left(\frac{t}{2}\right) \mathrm{d}t \tag{3.1}$$

be given. From [5, Lemma 5.1], we know that $\mu\left(B_{q/2}\right) \geq \frac{2}{\pi}\frac{q^3}{109}$ for $0 < q \leq \pi$. Now lets assume, we decompose the whole $SO(3)$ into such caps, where each cap has a sampling node $\boldsymbol{G}_j$ as center. Since the sampling set is q separated, the number of nodes is bounded by

$$M \leq \frac{\mu\left(SO(3)\right)}{\mu\left(B_{q/2}\right)} \leq \frac{109\pi}{2q^3}.$$

Regarding the second claim, we presume the contrary, i.e., let a q-separated sampling set $\mathcal{X}$ with $M < \frac{6\pi}{q^3}$ nodes be given. Around each node, we place a cap of latitude q and obtain

$$\mu\left(SO(3) \setminus \bigcup_{j=0}^{M-1} B_q(\boldsymbol{G}_j)\right) \geq \mu\left(SO(3)\right) - \sum_{j=0}^{M-1} \mu\left(B_q(\boldsymbol{G}_j)\right) \geq 1 - M\frac{q^3}{6\pi} > 0,$$

where we estimated $\mu(B_q(\boldsymbol{G}_j)) \leq \frac{q^3}{6\pi}$ which is due to $\sin(t/2) \leq t/2$ in (3.1). Hence, there exists a point $G \in SO(3)$ such that $\mathcal{X} \cup \{G\}$ remains q-separated.

We refer to [5, Lemma 5.1] for the last assertion. $\square$

Lemma 3.2. *The optimal interpolation problem* (2.3) *is equivalent to the normal equations of second kind*

$$\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^{\mathsf{H}}\tilde{\boldsymbol{f}} = \boldsymbol{y}, \qquad \hat{\boldsymbol{f}} = \hat{\boldsymbol{W}}\boldsymbol{D}^{\mathsf{H}}\tilde{\boldsymbol{f}}, \tag{3.2}$$

where the weighting matrix is given by $\hat{\boldsymbol{W}} := \operatorname{diag}(\tilde{\boldsymbol{w}}) \in \mathbb{R}^{d_N \times d_N}$ *for the vector* $\tilde{\boldsymbol{w}} = (\tilde{w}_l^{k,k'})_{l=0,\ldots,N,|k|,|k'|\leq l}$ *with* $\tilde{w}_l^{k,k'} = \hat{w}_l$, $l = 0,\ldots,N, |k|,|k'| \leq l$.

Moreover, let the trigonometric polynomial $K_N : [-\pi,\pi] \to \mathbb{R}$ *and its corresponding interpolation matrix* $\boldsymbol{K} = (k_{i,j})_{i,j=0,\ldots,M-1}$ *be given by*

$$K_N(t) := \sum_{l=0}^{N} \hat{w}_l U_{2l}(\cos(t/2)), \qquad k_{i,j} := K_N(\mathrm{d}(\boldsymbol{G}_i,\boldsymbol{G}_j)). \tag{3.3}$$

Then, we have the identity

$$\boldsymbol{K} = \boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^{\mathsf{H}} \tag{3.4}$$

Proof. The first assertion is due to [2, Thm. 1.1.2] for the matrix $\boldsymbol{D}\hat{\boldsymbol{W}}^{1/2}$. The second assertion follows from the addition theorem (2.2), i.e.,

$$(\boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^{\mathsf{H}})_{i,j} = \sum_{l=0}^{N} \hat{w}_l \sum_{k,k'=-l}^{l} D_l^{k,k'}(\boldsymbol{G}_i)\, \overline{D_l^{k,k'}(\boldsymbol{G}_j)} = \sum_{l=0}^{N} \hat{w}_l U_{2l}\left(\cos \frac{\mathrm{d}(\boldsymbol{G}_i,\boldsymbol{G}_j)}{2}\right). \quad \square$$

Definition 3.3. *Let the normalized B-spline of order* $\beta \in \mathbb{N}$ *be defined by* $g_\beta : [-\frac{1}{2},\frac{1}{2}] \to \mathbb{R}$, $g_\beta(z) := \beta N_\beta(\beta z + \frac{\beta}{2})$, *with the cardinal B-spline given by*

$$N_{\beta+1}(z) = \int_{z-1}^{z} N_\beta(\tau)\,\mathrm{d}\tau,\ \beta \in \mathbb{N}, \qquad N_1(z) = \begin{cases} 1 & 0 < z < 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, for $N \in \mathbb{N}$ *let* $\|g_\beta\|_{1,N} := \sum_{l=-N}^{N} g_\beta(\frac{l}{2(N+1)})$ *denote a discrete norm of* g.

Lemma 3.4. *Let* $N,\beta \in \mathbb{N}$, $N \geq \beta - 1 \geq 1$, *and* $B_{\beta,N}(t) = \sum_{l=0}^{N} \hat{w}_l U_{2l}(\cos(t/2))$ *with*

$$0 < \hat{w}_l := \frac{1}{\|g_\beta\|_{1,N}} \begin{cases} g_\beta\left(\frac{N}{2(N+1)}\right) & l = N, \\ g_\beta\left(\frac{l}{2(N+1)}\right) - g_\beta\left(\frac{l+1}{2(N+1)}\right) & 0 \leq l < N \end{cases} \tag{3.5}$$

be given. The following localization property holds true for $t \in (0,\pi]$:

$$|B_{\beta,N}(t)| \leq c_\beta\, |(N+1)t|^{-\beta}, \qquad c_\beta := \frac{(2^\beta - 1)\,\zeta(\beta)\,\beta^\beta}{2^{\beta-1} - \zeta(\beta)\,\pi^{-\beta}}, \tag{3.6}$$

with the normalization $B_{\beta,N}(0) = 1$.

Proof. We first note that the Chebyshev polynomials of first kind $T_l(\cos(x)) = \cos(lx)$, $l \in \mathbb{N}_0$, are related to the Chebyshev polynomials of second kind by

$$T_l = \frac{1}{2}(U_l - U_{l-2}), \qquad U_{2l} = \sum_{k=0}^{l}(2-\delta_{0,k})T_{2k}.$$

In conjunction with (3.5), we obtain

$$\begin{aligned}\sum_{l=0}^{N}\hat{w}_l U_{2l} &= \sum_{l=0}^{N}\sum_{k=0}^{l}\hat{w}_l(2-\delta_{0,k})T_{2k}\\ &= \sum_{k=0}^{N}\sum_{l=k}^{N}\hat{w}_l(2-\delta_{0,k})T_{2k}\\ &= \sum_{k=0}^{N}(2-\delta_{0,k})T_{2k}\sum_{l=k}^{N}\hat{w}_l\\ &= \sum_{k=0}^{N}(2-\delta_{0,k})\frac{1}{\|g_\beta\|_{1,N}}g_\beta\left(\frac{k}{2(N+1)}\right)T_{2k}.\end{aligned}$$

Applying the simple equality $T_{2l}(\cos(t/2)) = \cos(2lt/2) = T_l(\cos(t))$, we arrive at

$$B_{\beta,N}(t) = \sum_{l=0}^{N}(2-\delta_{0,l})\frac{1}{\|g_\beta\|_{1,N}}g_\beta\left(\frac{l}{2(N+1)}\right)T_l(\cos(t))$$

from which the assertion follows by [8, Lemma 4.6]. □

We finally obtain the following result on the condition number of the interpolation problem (2.3).

Theorem 3.5. *Let $q > 0$ and $\mathcal{X} \subset SO(3)$ be a q-separated sampling set of cardinality $M \in \mathbb{N}$. Moreover, let $N, \beta \in \mathbb{N}$, $N \geq \beta - 1 \geq 3$ be given and define the weights $\hat{w}_k$ by (3.5) and the kernel matrix $\boldsymbol{K} \in \mathbb{R}^{M\times M}$ by (3.3). Then, the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_M$ of $\boldsymbol{K}$ satisfy $\lambda_1 \leq 1 \leq \lambda_M$ and*

$$|\lambda_j - 1| \leq 141\zeta(\beta-2)c_\beta\left((N+1)q\right)^{-\beta}, \qquad j = 1,\ldots,M, \tag{3.7}$$

where c_β is given in (3.6). In particular, the matrix $\boldsymbol{D} \in \mathbb{C}^{M\times d_N}$ has full rank M whenever

$$N + 1 > 19/q$$

and this condition is optimal in the sense that there is another q-separated sampling set $\mathcal{X}' \subset SO(3)$ of cardinality M and a constant $C_1 > 0$ such that for $N + 1 \leq C_1/q$ the matrix $\boldsymbol{D}'$ has rank less than M.

Proof. The first assertion follows from $\sum_{j=1}^{M} \lambda_j = M B_{\beta,N}(0) = M$. Moreover, the Gershgorin circle theorem yields for every $1 \le r \le M$ and some $1 \le l \le M$ that

$$|\lambda_r - k_{l,l}| \le \sum_{j=1, j \neq l}^{M} |B_{\beta,N}(\mathrm{d}(\boldsymbol{G_j}, \boldsymbol{G_l}))| .$$

Using the last assertion of Lemma 3.1 and the localization property as shown in Lemma 3.4, we further estimate

$$\begin{aligned} |\lambda_r - 1| &\le \sum_{m=1}^{\lfloor \pi q^{-1} \rfloor} |R_{\mathcal{X},q,m}| \max_{\boldsymbol{G} \in R_{\mathcal{X},q,m}} |B_{\beta,N}(\mathrm{d}(\boldsymbol{I}, \boldsymbol{G}))| \\ &\le \sum_{m=1}^{\infty} 141 m^2 c_\beta \left((N+1)mq\right)^{-\beta} \\ &\le 141 \zeta(\beta - 2) c_\beta \left((N+1)q\right)^{-\beta} . \end{aligned}$$

The last claim is due to $c_4 = 3840\pi^4/719$, i.e., we set $\beta = 4$. The optimality of this condition can be seen as follows: We apply Lemma 3.1 and use the fact that the number of Fourier coefficients is bounded by $d_N \le C_2 N^3$. Hence, there is a constant $C_1 > 0$ such that $N + 1 \le C_1/q$ implies also $d_N < M$ and thus $\operatorname{rank}(\boldsymbol{D}') < M$. ☐

Corollary 3.6. Under the conditions of Theorem 3.5 with $\beta = 4$, the conjugate gradient method applied to (3.2) converges linearly, i.e.,

$$\|\hat{\boldsymbol{e}}_l\|_{\hat{\boldsymbol{W}}^{-1}} \le 2 \left(\frac{19}{(N+1)q} \right)^{4l} \|\hat{\boldsymbol{e}}_0\|_{\hat{\boldsymbol{W}}^{-1}}$$

with the initial error $\hat{\boldsymbol{e}}_0 := \hat{\boldsymbol{W}} \boldsymbol{D}^{\mathsf{H}} \boldsymbol{K}^{-1} \boldsymbol{y}$ and the error $\hat{\boldsymbol{e}}_l := \hat{\boldsymbol{f}}_l - \hat{\boldsymbol{W}} \boldsymbol{D}^{\mathsf{H}} \boldsymbol{K}^{-1} \boldsymbol{y}$ of the l-th iterate $\hat{\boldsymbol{f}}_l$.

Proof. Applying the standard estimate for the convergence of the conjugate gradient method, see e.g. [2, p. 289], yields the assertion. ☐

Remark 3.7. *We solve problem* (3.2) *by a factorized variant of conjugated gradients (CGNE, N for "Normal equation" and E for "Error minimization") [2, p. 269], where we use the nonequispaced fast Fourier transform on the rotation group [19] for fast matrix vector multiplications with* $\boldsymbol{D}$ *and its adjoint* $\boldsymbol{D}^{\mathsf{H}}$. *Note that for* $(N+1)q > 19$ *a constant number of iterations suffices to decrease the error to a certain fraction, i.e., the total number of floating point operations is bounded by the complexity* $\mathcal{O}(N^3 \log^2 N + M)$ *of the fast Fourier transform on* $SO(3)$.

We finally give an estimate on the smallest eigenvalue for an interpolation problem with minimal Sobolev norm. Our result generalizes [17, Theorems 2.8, 3.6] to the rotation group and improves the involved constant which artificially depended on the Sobolev order. Similar techniques have been used in [4, Theorem 5.1].

Corollary 3.8. Let $q > 0$ and $\mathcal{X} \subset SO(3)$ be a q-separated sampling set of cardinality $M \in \mathbb{N}$. Moreover, let $s > 2$ be given and consider the interpolation problem

$$\min_{f \in H_s^2} \|f\|_{H_s^2} \quad \text{s.t.} \quad f(\boldsymbol{G}_j) = y_j \text{ for } j = 0 \ldots M-1. \tag{3.8}$$

This problem is (only) mildly ill-posed in the sense that the smallest eigenvalue $\lambda_1(\boldsymbol{M})$ of the corresponding interpolation matrix

$$\boldsymbol{M} = (m_{i,j})_{i,j=0,\ldots,M-1}, \qquad m_{i,j} = \sum_{l=0}^{\infty} (1+l)^{-s} U_{2l}\left(\cos \frac{\mathrm{d}(\boldsymbol{G}_i, \boldsymbol{G}_j)}{2}\right) \tag{3.9}$$

satisfies

$$\lambda_1(\boldsymbol{M}) \geq (C_3 q)^s \tag{3.10}$$

with some fixed constant $C_3 \geq 1/35$.

Proof. We start with the polynomial interpolation matrix $\boldsymbol{K}$ *from Theorem 3.5 with* $N = \lfloor 30/q \rfloor - 1$ *and* $\beta = 4$. *Due to* $q \leq \pi$, *the estimate* (3.7) *yields a smallest eigenvalue* $\lambda_1(\boldsymbol{K}) \geq 3/4 > (3/4)^{s/2}$. *Moreover note that the corresponding weights* $\hat{w}_l$, $l = 0, \ldots, N$, *cf.* (3.5), *satisfy* $\hat{w}_l \leq 1$.

Now let $\boldsymbol{c} \in \mathbb{C}^M$ *be given, the assertion follows from the addition theorem* (2.2) *by*

$$\begin{aligned} \sum_{i,j=0}^{M-1} c_i \overline{c_j} m_{i,j} &= \sum_{i,j=0}^{M-1} c_i \overline{c_j} \sum_{l=0}^{\infty} (1+l)^{-s} U_{2l}\left(\cos \frac{\mathrm{d}(\boldsymbol{G}_i, \boldsymbol{G}_j)}{2}\right) \\ &= \sum_{l=0}^{\infty} (1+l)^{-s} \sum_{k,k'=-l}^{l} \left| \sum_{i=0}^{M-1} c_i D_l^{k,k'}(\boldsymbol{G}_i) \right|^2 . \end{aligned} \tag{3.11}$$

We decrease the right hand side by truncating to only N *terms and insert our "nice" weights* $\hat{w}_l$:

$$\begin{aligned} \ldots &\geq \sum_{l=0}^{N} (1+l)^{-s} \hat{w}_l^{-1} \hat{w}_l \sum_{k,k'=-l}^{l} \left| \sum_{i=0}^{M-1} c_i D_l^{k,k'}(\boldsymbol{G}_i) \right|^2 \\ &\geq \min_{r=0,\ldots,N} (1+r)^{-s} \hat{w}_r^{-1} \sum_{l=0}^{N} \hat{w}_l \sum_{k,k'=-l}^{l} \left| \sum_{i=0}^{M-1} c_i D_l^{k,k'}(\boldsymbol{G}_i) \right|^2 \\ &\geq (1+N)^{-s} \sum_{i,j=0}^{M-1} c_i \overline{c_j} k_{i,j}. \end{aligned} \tag{3.12}$$

Finally, we use that the minimal value in (3.11) *is* $\lambda_1(\boldsymbol{M}) \|\boldsymbol{c}\|_2^2$ *and the last expression* (3.12) *can be bounded from below by* $(q/30)^s \cdot \lambda_1(\boldsymbol{K}) \|\boldsymbol{c}\|_2^2$. ☐

Remark 3.9. *For the actual solution of the interpolation problem* (3.8)*, we use a so-called fast summation scheme [20] for multiplication with* $\boldsymbol{M} \in \mathbb{C}^{M \times M}$*. Given* $\varepsilon > 0$*, we approximate* $\boldsymbol{M} \approx \tilde{\boldsymbol{M}}$ *by truncating the series in* (3.9) *to a polynomial degree* $N \in \mathbb{N}$ *such that the remainder fulfills*

$$|m_{i,j} - \tilde{m}_{i,j}| = \sum_{l=N+1}^{\infty} (1+l)^{-s} U_{2l} \left(\cos \frac{\mathrm{d}\left(\boldsymbol{G}_i, \boldsymbol{G}_j\right)}{2} \right) \leq 2 \int_N^{\infty} (1+l)^{1-s} \mathrm{d}l \quad \leq \quad \varepsilon.$$

Due to the addition theorem (2.2) *and the nonequispaced FFT on the rotation group [19], this yields the factorization* $\tilde{\boldsymbol{M}} = \boldsymbol{D}\hat{\boldsymbol{W}}\boldsymbol{D}^{\mathsf{H}}$ *which can be applied to a vector in* $\mathcal{O}(N^3 \log^2 N + M)$ *floating point operations. Since we ask at least for mildly ill-posed matrices* $\boldsymbol{M}, \tilde{\boldsymbol{M}}$*, we moreover force* $N \geq \lfloor 30/q \rfloor - 1$*. Thus, for* q*-separated sampling sets with* $M \geq C_4/q^3$ *nodes, the multiplication with* $\tilde{\boldsymbol{M}}$ *takes* $\mathcal{O}(M \log^2 M)$ *flops.*

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