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Gap Functions for Vector Equilibrium Problems via Conjugate Duality*

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Abstract: This paper deals with the so-called perturbation approach in the conjugate duality for vector optimization on the basis of weak orderings. As applications, we investigate some new set-valued gap functions for vector equilibrium problems.

Key words: Conjugate duality; Perturbation approach; Vector equilibrium problems; Set-valued gap functions

1 Introduction

Tanino and Sawaragi [12] (see also [9]) developed conjugate duality for vector optimization by introducing new concepts of conjugate maps and set-valued subgradients based on Pareto efficiency. Furthermore, by using the concept of the supremum of a set on the basis of weak orderings, the conjugate duality theory was extended to a partially ordered topological vector space by Tanino [14] and to set-valued vector optimization problems by Song [10], [11], respectively.

Dealing with conjugacy notions in the framework of set-valued optimization, the so-called perturbation approach in the conjugate duality (see [15]) has been extended to the constrained vector optimization problems (cf. [2]). As applications, rewriting the vector variational inequality in the form of a vector optimization problem, new set-valued gap functions for the vector variational inequality have been introduced.

By using a special perturbation function, the Fenchel-type dual problem for vector optimization has been obtained and based on this investigation some set-valued mappings have been introduced in order to apply them to variational principles for vector equilibrium problems (see [3]). Notice that variational principles for vector equilibrium problems have been investigated first in [4] and [5]. Some related results in the scalar case can be found in [1] and [6].

In this paper we consider two additional perturbation functions implying the Lagrange and Fenchel-Lagrange type dual problems, respectively.

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This paper is organized as follows. In Section 2 we give some preliminary results dealing with conjugate duality for vector optimization and stability criteria. On the basis of two special perturbation functions different dual problems are introduced in Section 3. In order to state the strong duality, we use in Section 3 general results due to Song. Finally, as applications some new gap functions for vector equilibrium problems related to conjugate duality are introduced in Section 4.

2 Mathematical preliminaries

Let Y be a real topological vector space partially ordered by a pointed closed convex cone C with a nonempty interior int C in Y. For any $\xi, \mu \in Y$, we use the following ordering relations:

$$\begin{aligned} \xi \leq \mu \quad &\Leftrightarrow \quad \mu - \xi \in C; \\ \xi < \mu \quad &\Leftrightarrow \quad \mu - \xi \in \operatorname{int} C; \\ \xi \not< \mu \quad &\Leftrightarrow \quad \mu - \xi \notin \operatorname{int} C. \end{aligned}$$

The relations $\geq$, $>$ and $\not>$ are defined similarly. Let us now introduce the weak maximum and weak supremum of a set Z in the space $\overline{Y}$ induced by adding to Y two imaginary points $+\infty$ and $-\infty$. We suppose that $-\infty < y < +\infty$ for $y \in Y$. Moreover, we use the following conventions

$$(\pm\infty) + y = y + (\pm\infty) = \pm\infty \text{ for all } y \in Y, \quad (\pm\infty) + (\pm\infty) = \pm\infty,$$

$$\lambda(\pm\infty) = \pm\infty \text{ for } \lambda > 0 \text{ and } \lambda(\pm\infty) = \mp\infty \text{ for } \lambda < 0.$$

The sum $+\infty + (-\infty)$ is not considered, since we can avoid it.

For a given set $Z \subseteq \overline{Y}$, we define *the set* $A(Z)$ *of all points above* Z and *the set* $B(Z)$ *of all points below* Z by

$$A(Z) = \left\{ y \in \overline{Y} \mid y > y' \text{ for some } y' \in Z \right\}$$

and

$$B(Z) = \left\{ y \in \overline{Y} \mid y < y' \text{ for some } y' \in Z \right\},$$

respectively. Clearly $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$.

Definition 2.1

(i) A point $\widehat{y} \in \overline{Y}$ *is said to be a weak maximal point of* $Z \subseteq \overline{Y}$ *if* $\widehat{y} \in Z$ *and* $\widehat{y} \notin B(Z)$*, that is, if* $\widehat{y} \in Z$ *and there is no* $y' \in Z$ *such that* $\widehat{y} < y'$*.*

(ii) A point $\widehat{y} \in \overline{Y}$ *is said to be a weak supremal point of* $Z \subseteq \overline{Y}$ *if* $\widehat{y} \notin B(Z)$ *and* $B(\{\widehat{y}\}) \subseteq B(Z)$*, that is, if there is no* $y \in Z$ *such that* $\widehat{y} < y$ *and if the relation* $y' < \widehat{y}$ *implies the existence of some* $y \in Z$ *such that* $y' < y$*.*

Weak minimal and weak infimal points can be defined analogously. The set of all weak maximal (minimal) and weak supremal (infimal) points of Z is denoted by $\mathrm{WMax}\, Z$ ($\mathrm{WMin}\, Z$) and $\mathrm{WSup}\, Z$ ($\mathrm{WInf}\, Z$), respectively. Remark that $\mathrm{WMax}\, Z = Z \cap \mathrm{WSup}\, Z$. Moreover it holds $-\mathrm{WMax}(-Z) = \mathrm{WMin}\, Z$ and $-\mathrm{WSup}(-Z) = \mathrm{WInf}\, Z$. For more properties of these sets we refer to [13] and [14].

Now we give some definitions of the conjugate mapping and the subgradient of a set-valued mapping based on the weak supremum and the weak maximum of a set. Let X be another real topological vector space and let $\mathcal{L}(\mathrm{X}, \mathrm{Y})$ be the space of all linear continuous operators from X to Y. For $x \in X$ and $l \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$, $\langle l, x\rangle$ denotes the value of l at x.

Definition 2.2 *(see [14]). Let $G : X \rightrightarrows \overline{Y}$ be a set-valued mapping.*

(i) A set-valued mapping $G^* : \mathcal{L}(\mathrm{X}, \mathrm{Y}) \rightrightarrows \overline{Y}$ defined by

$$G^*(T) = \mathit{WSup} \bigcup_{x \in X} \Big[\langle T, x\rangle - G(x)\Big], \quad \text{for } T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$$

is called the conjugate mapping of G.

(ii) A set-valued mapping $G^{**} : X \rightrightarrows \overline{Y}$ defined by

$$G^{**}(x) = \mathit{WSup} \bigcup_{T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})} \Big[\langle T, x\rangle - G^*(T)\Big], \quad \text{for } x \in X$$

is called the biconjugate mapping of G.

(iii) $T \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ is said to be a subgradient of the set-valued mapping G at $(x_0; y_0)$ if $y_0 \in G(x_0)$ and

$$\langle T, x_0\rangle - y_0 \in \mathit{WMax} \bigcup_{x \in X} \Big[\langle T, x\rangle - G(x)\Big].$$

The set of all subgradients of G at $(x_0; y_0)$ is called *the subdifferential of G at* $(x_0; y_0)$ and is denoted by $\partial G(x_0; y_0)$. If $\partial G(x_0; y_0) \neq \varnothing$ for every $y_0 \in G(x_0)$, then G is said to be *subdifferentiable at* x_0.

Let X and Y be real topological vector spaces. Assume that $\overline{Y}$ is the extended space of Y and $h : X \to Y \cup \{+\infty\}$ is a given function. We consider the vector optimization problem

$$(P) \qquad \mathrm{WInf}\{h(x) |\, x \in X\}.$$

Based on a perturbation approach (see [14]), a dual problem to (P) can be defined as follows

$$(D) \qquad \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(U, Y)} \Big[-\Phi^*(0, \Lambda)\Big],$$

where $\Phi : X \times U \to Y \cup \{+\infty\}$ is called a perturbation function having the property that

$$\Phi(x,0) = h(x), \ \forall x \in X.$$

Here, U is another real topological vector space. Moreover, the conjugate mapping of Φ is

$$\Phi^*(T,\Lambda) = \mathrm{WSup}\left\{\langle T,x\rangle + \langle \Lambda,u\rangle - \Phi(x,u) |\ x \in X,\ u \in U\right\}$$

for $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ and $\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})$.

Proposition 2.1 *[14].(Weak duality)*
For any $x \in X$ and $\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})$ it holds

$$\Phi(x,0) \notin B\Big(-\Phi^*(0,\Lambda)\Big).$$

Definition 2.3 *[14]. The primal problem (P) is said to be stable with respect to Φ if the value mapping $\Psi : U \rightrightarrows \overline{Y}$ defined by*

$$\Psi(u) = \mathit{WInf}\Big\{\Phi(x,u) |\ x \in X\Big\}$$

is subdifferentiable at 0.

Theorem 2.1 *[14], [10]. If the problem (P) is stable with respect to Φ, then*

$$\mathit{WInf}(P) = \mathit{WSup}(D) = \mathit{WMax}(D).$$

Let us now mention some definitions and assertions related to the stability.

For a given set-valued mapping $G : X \rightrightarrows Y \cup \{+\infty\}$, we have

- *effective domain of G:* $\operatorname{dom} G = \{x \in X |\ G(x) \neq \varnothing,\ G(x) \neq \{+\infty\}\}$,
- *epigraph of G:* $\operatorname{epi} G = \{(x,y) \in X \times Y |\ y \in G(x) + C\}$.

In particular, if $g : X \to Y \cup \{+\infty\}$ is a vector-valued function, then its effective domain and epigraph are defined as

$$\begin{aligned} \operatorname{epi}\ g &= \{(x,y) \in X \times Y |\ g(x) \leq y\}, \\ \operatorname{dom}\ g &= \{x \in X |\ g(x) \neq +\infty\}, \end{aligned}$$

respectively. The function g is said to be *proper* if $g(x) \in X \cup \{+\infty\}$ and $g \not\equiv +\infty$.

A set-valued mapping $G : X \rightrightarrows Y \cup \{+\infty\}$ is said to be C-convex if its epigraph is convex. A given set-valued mapping $G : X \rightrightarrows Y \cup \{+\infty\}$ is C-convex if and only if for all $\lambda \in [0,1]$ and $x_1, x_2 \in X$

$$\lambda G(x_1) \cap Y + (1-\lambda) G(x_2) \cap Y \subseteq G(\lambda x_1 + (1-\lambda)x_2) \cap Y + C.$$

In particular, if $g : X \to Y \cup \{+\infty\}$ is a proper vector-valued function, then g is C-convex if and only if for all $\lambda \in (0,1)$ and $x_1, x_2 \in X,\ x_1 \neq x_2$

$$\lambda g(x_1) + (1-\lambda) g(x_2) \in g(\lambda x_1 + (1-\lambda)x_2) + C.$$

Proposition 2.2 *[10]. Let $G : X \rightrightarrows Y \cup \{+\infty\}$ be a C-convex set-valued mapping with $int(epi\ G) \neq \varnothing$. If $x_0 \in int(dom\ G)$ and $G(x_0) \subseteq WInf G(x_0)$, then G is subdifferentiable at x_0.*

Definition 2.4

(i) A set-valued mapping $G : X \rightrightarrows Y \cup \{+\infty\}$ is said to be C-Hausdorff lower continuous at $x_0 \in X$ if for every neighborhood V of zero in Y there exists a neighborhood U of zero in X such that

$$G(x_0) \subseteq G(x) + V + C,\ \forall x \in (x_0 + U) \cap dom\ G.$$

(ii) A set-valued mapping $G : X \rightrightarrows Y \cup \{+\infty\}$ is said to be weakly C-upper bounded on a set $A \subseteq X$ if there exists a point $b \in Y$ such that $(x, b) \in epi\ G,\ \forall x \in A$.

Let us remark that G is weakly C-upper bounded on a set $A \subseteq X$ if and only if there exists a point $b \in Y$ such that $G(x) \cap (b - C) \neq \varnothing,\ \forall x \in A$.

Proposition 2.3 *[10]. Let $G : X \rightrightarrows Y \cup \{+\infty\}$ be a set-valued mapping.*
1. Then the following assertions are equivalent.

(i) $int(epi\ G) \neq \varnothing$.

(ii) $\exists x_0 \in int(dom\ G)$ such that G is weakly C-upper bounded on some neighborhood of x_0.

2. If G is C-Hausdorff lower continuous on $int(dom\ G)$, then (i) and (ii) hold.

Proposition 2.4 *[14]. If the perturbation function $\Phi : X \times U \to Y \cup \{+\infty\}$ is C-convex, then the value mapping Ψ is a C-convex set-valued mapping.*

Proposition 2.5 *(cf. [11]). Let $\Phi : X \times U \to Y \cup \{+\infty\}$ be a C-convex vector-valued function and the value mapping Ψ be weakly C-upper bounded on a neighborhood of zero in U. Then the problem (P) is stable with respect to Φ.*

Remark 2.1 Proposition 2.5 was proved in [11] in the more general case when $\Phi : X \times U \to Y \cup \{+\infty\}$ is a set-valued mapping.

3 The constrained vector optimization problem

3.1 Different dual problems

Assume that $h : X \to Y \cup \{+\infty\}$ is a given function and $G \subseteq X$. We consider the constrained vector optimization problem

$$(P_c) \qquad \text{WInf}\{h(x) |\, x \in G\}.$$

By using the perturbation function $\Phi_F : X \times X \to Y \cup \{+\infty\}$ defined by

$$\Phi_F(x,u) = \begin{cases} h(x+u), & \text{if } x \in G, \\ +\infty, & \text{otherwise,} \end{cases}$$

the Fenchel dual problem to (P_c) has been stated as follows (cf. [3])

$$(D_F) \qquad \mathrm{WSup} \bigcup_{T \in \mathcal{L}(\mathrm{X},\mathrm{Y})} \mathrm{WInf}\Big\{ -h^*(T) + \{\langle T, x\rangle \mid x \in G\}\Big\}.$$

Proposition 3.1 *(Weak duality)*
For any $x \in G$ and $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$ it holds

$$h(x) \notin B\Big(-\Phi_F^*(0,T)\Big).$$

Let U be a real topological vector space, $D \subseteq U$ be a pointed closed convex cone, $M \subseteq X$ and $g : X \to U \cup \{+\infty\}$. If the feasible set G is given by

$$G = \{x \in M \mid g(x) \in -D\},$$

then one can consider the following two perturbation functions (cf. [2] and [15])

$$\Phi_L : X \times U \to Y \cup \{+\infty\}, \quad \Phi_L(x,u) = \begin{cases} h(x), & x \in M,\ g(x) \in -D + u, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\Phi_{FL} : X \times X \times U \to Y \cup \{+\infty\},$$
$$\Phi_{FL}(x,v,u) = \begin{cases} h(x+v), & x \in M,\ g(x) \in -D + u, \\ +\infty, & \text{otherwise.} \end{cases}$$

In analogy to Proposition 3.3 and Proposition 3.11 in [2], the following assertion can be shown easily.

Proposition 3.2 *Let $\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})$ and $T \in \mathcal{L}(\mathrm{X},\mathrm{Y})$. Then*

(i) $\Phi_L^*(0,\Lambda) = \mathit{WSup}\Big\{\{\langle \Lambda, u\rangle \mid u \in D\} + \{\langle \Lambda, g(x)\rangle - h(x) \mid x \in M\}\Big\}.$

(ii)

$$\begin{aligned} \Phi_{FL}^*(0,T,\Lambda) &= \mathit{WSup}\Big\{\{\langle \Lambda, u\rangle \mid u \in D\} \\ &+ \{\langle T, v\rangle - h(v) \mid v \in X\} + \{\langle \Lambda, g(x)\rangle - \langle T, x\rangle \mid x \in M\}\Big\}. \end{aligned}$$

Remark 3.1 According to Proposition 2.6 in [14], we can use for $\Phi_L^*(0,\Lambda)$ and $\Phi_{FL}^*(0,T,\Lambda)$ some equivalent formulations. For instance, for $\Phi_{FL}^*(0,T,\Lambda)$ we have

$$\begin{aligned}\Phi_{FL}^*(0,T,\Lambda) &= \mathrm{WSup}\Big\{\{\langle\Lambda,u\rangle|\ u\in D\} \\ &+ \{\langle T,v\rangle - h(v)|\ v\in X\} + \{\langle\Lambda,g(x)\rangle - \langle T,x\rangle|\ x\in M\}\Big\} \\ &= \mathrm{WSup}\Big\{\mathrm{WSup}\{\langle\Lambda,u\rangle|\ u\in D\} \\ &\dot{+}\ h^*(T) + \{\langle\Lambda,g(x)\rangle - \langle T,x\rangle|\ x\in M\}\Big\}.\end{aligned}$$

As a consequence of Proposition 3.2 can be stated the Lagrange dual problem to (P_c)

$$\begin{aligned}(D_L)\qquad &\mathrm{WSup}\bigcup_{\Lambda\in\mathcal{L}(\mathrm{U},\mathrm{Y})}\Big[-\Phi_L^*(0,\Lambda)\Big] \\ = &\mathrm{WSup}\bigcup_{\Lambda\in\mathcal{L}(\mathrm{U},\mathrm{Y})}\mathrm{WInf}\Big\{\{-\langle\Lambda,u\rangle|\ u\in D\} + \{h(x) - \langle\Lambda,g(x)\rangle|\ x\in M\}\Big\}\end{aligned}$$

and the Fenchel-Lagrange dual problem

$$\begin{aligned}(D_{FL})\qquad &\mathrm{WSup}\bigcup_{(T,\Lambda)\in\mathcal{L}(\mathrm{X},\mathrm{Y})\times\mathcal{L}(\mathrm{U},\mathrm{Y})}\Big[-\Phi_{FL}^*(0,T,\Lambda)\Big] \\ = &\mathrm{WSup}\bigcup_{(T,\Lambda)\in\mathcal{L}(\mathrm{X},\mathrm{Y})\times\mathcal{L}(\mathrm{U},\mathrm{Y})}\mathrm{WInf}\Big\{\{h(v) - \langle T,v\rangle|\ v\in X\} \\ + &\{-\langle\Lambda,u\rangle|\ u\in D\} + \{\langle T,x\rangle - \langle\Lambda,g(x)\rangle|\ x\in M\}\Big\},\end{aligned}$$

respectively.

Proposition 3.3 *(Weak duality)*

(i) For any $x\in G$ and $T\in\mathcal{L}(\mathrm{X},\mathrm{Y})$ it holds

$$h(x)\notin B\Big(-\Phi_L^*(0,\Lambda)\Big).$$

(ii) For any $x\in G$ and $(T,\Lambda)\in\mathcal{L}(\mathrm{X},\mathrm{Y})\times\mathcal{L}(\mathrm{U},\mathrm{Y})$ it holds

$$h(x)\notin B\Big(-\Phi_{FL}^*(0,T,\Lambda)\Big).$$

3.2 Stability and strong duality

This subsection deals with some stability assertions associated with the presented perturbation functions as special cases of general results due to Song [10] and [11]. In order to investigate stability criteria, let us notice that the value mappings with respect to Φ_F, Φ_L and Φ_{FL} turn out to be

$$
\begin{aligned}
\Psi_L : U \rightrightarrows \overline{Y},\ \Psi_L(u) &= \mathrm{WInf}\Big\{\Phi_L(x,u)|\ x \in X\Big\} \\
&= \mathrm{WInf}\Big\{h(x)|\ x \in M,\ g(x) \in -D + u\Big\}; \\
\Psi_F : X \rightrightarrows \overline{Y},\ \Psi_F(v) &= \mathrm{WInf}\Big\{\Phi_F(x,v)|\ x \in X\Big\} \\
&= \mathrm{WInf}\Big\{h(x+v)|\ x \in G\Big\}; \\
\Psi_{FL} : X \times U \rightrightarrows \overline{Y},\ \Psi_{FL}(v,u) &= \mathrm{WInf}\Big\{\Phi_{FL}(x,v,u)|\ x \in X\Big\} \\
&= \mathrm{WInf}\Big\{h(x+v)|\ x \in M,\ g(x) \in -D + u\Big\},
\end{aligned}
$$

respectively.

Proposition 3.4 *Let $M \subseteq X$ be a convex set and $h : X \to Y \cup \{+\infty\}$, $g : X \to U$ be C- and D-convex functions, respectively. Then the value mappings Ψ_L, Ψ_F and Ψ_{FL} are convex.*

Proof: Under the stated assumptions of convexity one can easy verify that the perturbation functions Φ_L, Φ_F and Φ_{FL} are convex. Then the desired assertions follow from Proposition 2.4. □

Theorem 3.1 *Let $M \subseteq X$ be a convex set and $h : X \to Y \cup \{+\infty\}$, $g : X \to U$ be C- and D-convex functions, respectively. Suppose that the value mapping Ψ_F (resp. Ψ_L and Ψ_{FL}) is weakly C-upper bounded on a neighborhood of zero in X. Then the problem (P_c) is stable with respect to Φ_F (resp. Φ_L and Φ_{FL}).*

Proof: By Proposition 3.4 the value mapping Ψ_F (resp. Ψ_L and Ψ_{FL}) is convex. Then the stability of the problem (P_c) follows from Proposition 2.5. □

Proposition 3.5 *If there exists some $x_0 \in$ dom $h \cap G$ such that the function h is weakly C-upper bounded on some neighborhood of x_0, then the value mapping Ψ_F is weakly C-upper bounded on some neighborhood of zero in X.*

Proof: Since h is weakly C-upper bounded on some neighborhood of $x_0 \in$ dom $h \cap G$, there exists a neighborhood $V_0 \subseteq X$ of zero and $\exists b \in Y$ such that

$$(x_0 + v, b) \in \text{epi } h,\ \forall v \in V_0,$$

or, equivalently,

$$h(x_0 + v) \leq b,\ \forall v \in V_0.$$

Hence $h(x_0 + v) \in b - C,\ \forall v \in V_0$.
On the other hand, by Corollary 2.1 in [14], we obtain that for any $v \in V_0$

$$\{h(x+v)|\ x \in G\} \subseteq \Psi_F(v) \cup A(\Psi_F(v)).$$

In particular, it holds $h(x_0 + v) \in \Psi_F(v) \cup A(\Psi_F(v)),\ \forall v \in V_0$.

a. If $h(x_0+v) \in \Psi_F(v)$, then $(b-C) \cap \Psi_F(v) \neq \varnothing$, $\forall v \in V_0$.

b. If $h(x_0+v) \in A(\Psi_F(v))$, then $\exists \bar{y} \in \Psi_F(v)$ such that $h(x_0+v) > \bar{y}$. Therefore

$$\bar{y} \in h(x_0+v) - \text{int } C \subseteq h(x_0+v) - C \subseteq b - C - C \subseteq b - C,$$

which means that also $(b-C) \cap \Psi_F(v) \neq \varnothing$, $\forall v \in V_0$. The proof is completed.

□

Proposition 3.6 *If there exists some $x_0 \in dom\ h \cap M$ such that $0 \in int(g(x_0)+D)$, then the value mapping Ψ_L is weakly C-upper bounded on some neighborhood of zero in X.*

Proof: As $0 \in \text{int}(g(x_0)+D)$, there exists a neighborhood U_0 of zero such that $u \in g(x_0)+D$, $\forall u \in U_0 \subseteq U$. This means that $g(x_0) \in -D+u$, $\forall u \in U_0$. Let us notice that because $h(x_0) \neq +\infty$, $\exists b \in Y$ such that $h(x_0) \leq b$.
By Corollary 2.1 in [14], for any $u \in U_0$ one has

$$\{h(x) |\ x \in M,\ g(x) \in -D+u\} \subseteq \Psi_L(u) \cup A(\Psi_L(u)).$$

In particular, it holds $h(x_0) \in \Psi_L(u) \cup A(\Psi_L(u))$, $\forall u \in U_0$.

a. If $h(x_0) \in \Psi_L(u)$, then $(b-C) \cap \Psi_L(u) \neq \varnothing$, $\forall u \in U_0$.

b. If $h(x_0) \in A(\Psi_L(u))$, then $\exists \bar{y} \in \Psi_L(u)$ such that $h(x_0) > \bar{y}$. Therefore

$$\bar{y} \in h(x_0) - \text{int } C \subseteq b - C - \text{int } C \subseteq b - C,$$

which means that also $(b-C) \cap \Psi_L(u) \neq \varnothing$, $\forall u \in U_0$.

□

Combining the assumptions of Propositions 3.5 and 3.6, we easy show the following assertion.

Proposition 3.7 *If there exists some $x_0 \in dom\ h \cap M$ such that $0 \in int(g(x_0)+D)$ and the function h is weakly C-upper bounded on some neighborhood of x_0, then the value mapping Ψ_{FL} is weakly C-upper bounded on some neighborhood of zero in X.*

Theorem 3.2 *Let $M \subseteq X$ be a convex set and $h : X \to Y \cup \{+\infty\}$, $g : X \to U$ be C- and D-convex functions, respectively.*

(i) If there exists some $x_0 \in dom\ h \cap G$ such that the function h is weakly C-upper bounded on some neighborhood of x_0, then

$$WInf(P_c) = WSup(D_F) = WMax(D_F).$$

(ii) If there exists some $x_0 \in dom\ h \cap M$ such that $0 \in int(g(x_0)+D)$, then

$$WInf(P_c) = WSup(D_L) = WMax(D_L).$$

(iii) If there exists some $x_0 \in dom\ h \cap M$ such that $0 \in int(g(x_0) + D)$ and the function h is weakly C-upper bounded on some neighborhood of x_0, then

$$\begin{aligned} WInf(P_c) &= WSup(D_F) = WSup(D_L) = WSup(D_{FL}) \\ &= WMax(D_F) = WMax(D_L) = WMax(D_{FL}). \end{aligned}$$

Proof: Under the assumptions and by Theorem 3.1 the problem (P_c) is stable with respect to Φ_F (resp. Φ_L and Φ_{FL}). Therefore according to Theorem 2.1 one obtains the desired assertions. □

4 Gap functions for vector equilibrium problems

Let X and Y be real topological vector spaces. Assume that K is a nonempty convex set in X and $f : K \times K \to Y$ is a bifunction such that $f(x, x) = 0,\ \forall x \in K$. We consider the vector equilibrium problem which consists in finding $x \in K$ such that

$$(VEP) \qquad f(x, y) \not< 0,\ \forall y \in K.$$

By K^p we denote the solution set of (VEP). In analogy to the vector variational inequality, we can give the definition of a gap function for (VEP).

Definition 4.1 *(cf. [7] and [8]). A set-valued mapping $\gamma : K \rightrightarrows Y \cup \{+\infty\}$ is said to be a gap function for (VEP) if it satisfies the following conditions:*

(i) $0 \in \gamma(x)$ if and only if $x \in K$ solves the problem (VEP);

(ii) $0 \not> \gamma(y),\ \forall y \in K$.

According to [3], let us remark that $\bar{x} \in K$ is a solution to (VEP) if and only if 0 is a weak minimal point of the set $\{f(\bar{x}, y)|\ y \in K\}$. Rewriting the problem (VEP) into the vector optimization problem

$$(P^{VEP}; x) \qquad \mathrm{WInf}\Big\{f(x, y)|\ y \in K\Big\},$$

where $x \in X$ is fixed, and using the Fenchel dual problem to $(P^{VEP}; x)$, let us introduce the following mapping

$$\gamma_F^{VEP}(x) := \bigcup_{T \in \mathcal{L}(\mathrm{X,Y})} \widetilde{\Phi}_F^*(0, T; x),$$

where $\widetilde{\Phi}_F^*(0, T; x) = \mathrm{WSup}\Big\{\{\langle T, y\rangle - f(x, y)|\ y \in K\} + \{-\langle T, y\rangle|\ y \in K\}\Big\}$, i.e.

$$\gamma_F^{VEP}(x) = \bigcup_{T \in \mathcal{L}(\mathrm{X,Y})} \mathrm{WSup}\Big\{\{\langle T, y\rangle - f(x, y)|\ y \in K\} + \{-\langle T, y\rangle|\ y \in K\}\Big\}.$$

Theorem 4.1 *Let $f(x,\cdot) : K \to Y$ be a convex function for all $x \in K$. Assume that for all $x \in K^p$ there exists some $y_0 \in K$ such that the function $f(x,\cdot)$ is weakly C-upper bounded on some neighborhood of y_0. Then γ_F^{VEP} is a gap function for (VEP).*

Proof: Under the assumptions it is clear that the problem $(P^{VEP};x)$ is stable. Consequently, the desired assertion follows from Lemma 4.1 and Theorem 4.1(i) in [3]. $\square$

Let the ground set K be nonempty and given by

$$K = \{x \in X|\ g(x) \in -D\}, \tag{4.1}$$

where $D \subseteq U$ is a pointed closed convex cone, U is a real topological vector space and $g : X \to U \cup \{+\infty\}$. Let $x \in X$ be fixed. Taking $f(x,\cdot)$ instead of h in (D_L) and (D_{FL}), respectively, the Lagrange and the Fenchel-Lagrange dual problems can be written as follows

$$(D_L^{VEP};x) \qquad \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(U,Y)} \Big[-\widetilde{\Phi}_L^*(0,\Lambda;x)\Big]$$

$$(D_{FL}^{VEP};x) \qquad \mathrm{WSup} \bigcup_{(T,\Lambda) \in \mathcal{L}(X,Y)\times\mathcal{L}(U,Y)} \Big[-\widetilde{\Phi}_{FL}^*(0,T,\Lambda;x)\Big],$$

where

$$\widetilde{\Phi}_L^*(0,\Lambda;x) := \mathrm{WSup}\Big\{\{\langle\Lambda,u\rangle|\ u \in D\} + \{\langle\Lambda,g(y)\rangle - f(x,y)|\ y \in X\}\Big\}, \tag{4.2}$$

and

$$\begin{aligned}\widetilde{\Phi}_{FL}^*(0,T,\Lambda;x) : &= \mathrm{WSup}\Big\{\{\langle T,y\rangle - f(x,y)|\ y \in X\} \\ &+ \{\langle\Lambda,u\rangle|\ u \in D\} + \{\langle\Lambda,g(y)\rangle - \langle T,y\rangle|\ y \in X\}\Big\}. \end{aligned} \tag{4.3}$$

Consequently, we can introduce two set-valued mappings

$$\gamma_L^{VEP}(x) := \bigcup_{\Lambda \in \mathcal{L}(U,Y)} \widetilde{\Phi}_L^*(0,\Lambda;x)$$

and

$$\gamma_{FL}^{VEP}(x) := \bigcup_{(T,\Lambda) \in \mathcal{L}(X,Y)\times\mathcal{L}(U,Y)} \widetilde{\Phi}_{FL}^*(0,T,\Lambda;x).$$

Theorem 4.2 *Let the functions $f(x,\cdot) : K \to Y$, $x \in K$ and $g : X \to Y$ be convex. Assume that there exists $y_0 \in K$ such that $0 \in int(g(y_0) + D)$. Then γ_L^{VEP} is a gap function for (VEP).*

Proof:

(i) Let $\bar{x} \in K$ be a solution to (VEP), then by Theorem 3.2(ii), one has

$$0 \in \mathrm{WInf}(P^{VEP}; \bar{x}) = \mathrm{WMax}(D_L^{VEP}; \bar{x}).$$

Consequently,

$$0 \in \mathrm{WMax}[-\gamma_L^{VEP}(\bar{x})].$$

Whence $0 \in \gamma_L^{VEP}(\bar{x})$. Conversely, let

$$0 \in \gamma_L^{VEP}(\bar{x}) = \bigcup_{\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})} \mathrm{WSup}\Big\{\{\langle \Lambda, u\rangle |\ u \in D\} + \{\langle \Lambda, g(y)\rangle - f(\bar{x}, y) |\ y \in X\}\Big\}.$$

Then $\exists \bar{\Lambda} \in \mathcal{L}(\mathrm{U},\mathrm{Y})$ such that

$$0 \in \mathrm{WSup}\Big\{\{\langle \bar{\Lambda}, u\rangle |\ u \in D\} + \{\langle \bar{\Lambda}, g(y)\rangle - f(\bar{x}, y) |\ y \in X\}\Big\},$$

or, equivalently,

$$0 \in \mathrm{WInf}\Big\{\{-\langle \bar{\Lambda}, u\rangle |\ u \in D\} + \{f(\bar{x}, y) - \langle \bar{\Lambda}, g(y)\rangle |\ y \in X\}\Big\}. \tag{4.4}$$

Assume that $0 \notin \mathrm{WMin}\{f(\bar{x}, y) |\ y \in K\}$. This means that $\exists \bar{y} \in K$ such that $f(\bar{x}, \bar{y}) < 0$. In other words, we have

$$f(\bar{x}, \bar{y}) - \langle \bar{\Lambda}, g(\bar{y})\rangle + \langle \bar{\Lambda}, g(\bar{y})\rangle < 0,$$

which contradicts (4.4) since $g(\bar{y}) \in -D$.

(ii) Let $x \in K$ be fixed and $z \in \gamma_L^{VEP}(x)$. Then $\exists \bar{\Lambda} \in \mathcal{L}(\mathrm{U},\mathrm{Y})$ such that

$$z \in \mathrm{WSup}\Big\{\{\langle \bar{\Lambda}, u\rangle |\ u \in D\} + \{\langle \bar{\Lambda}, g(y)\rangle - f(x, y) |\ y \in X\}\Big\}.$$

Choosing $y := x$ and $u := -g(x) \in D$, we obtain that

$$\langle \bar{\Lambda}, -g(x)\rangle + \langle \bar{\Lambda}, g(x)\rangle - f(x, x) = 0$$

is an element of the set defined within the outer braces. Therefore z as an element of the set of the weak supremal points of this set can not be less than zero with respect to the partial ordering given by the cone C, i.e. $z \nless 0$. Consequently, one has $\gamma_L^{VEP}(x) \nless 0,\ \forall x \in K$. □

Analogously, we can verify the following assertion concerning γ_{FL}^{VEP}.

Theorem 4.3 *Let the functions $f(x, \cdot) : K \to Y,\ x \in K$ and $g : X \to Y$ be convex. Assume that there exists some $y_0 \in K$ such that $0 \in int(g(y_0) + D)$ and the function $f(x, y)$ is weakly C-upper bounded with respect to y on some neighborhood of y_0. Then γ_{FL}^{VEP} is a gap function for (VEP).*

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