Nitsche finite element method for parabolic problems

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Abstract

This paper deals with a method for the numerical solution of parabolic initial-boundary value problems in two-dimensional polygonal domains Ω which are allowed to be non-convex. The Nitsche finite element method (as a mortar method) is applied for the discretization in space, i.e. non-matching meshes are used. For the discretization in time, the backward Euler method is employed. The rate of convergence in some H^1-like norm and in the L_2-norm is proved for the semi-discrete as well as for the fully discrete problem. In order to improve the accuracy of the method in presence of singularities arising in case of non-convex domains, meshes with local grading near the reentrant corner are employed for the Nitsche finite element method. Numerical results illustrate the approach and confirm the theoretically expected convergence rates.

1 Introduction

The mathematical modeling of many problems in science and engineering leads to time-dependent differential equations. Therefore, methods for the approximate solution of initial-boundary value problems for parabolic or hyperbolic equations are of special interest. For solving parabolic problems numerically, the finite difference method (see [28] for an overview) as well as combinations of spatial discretization by finite elements with some finite difference time stepping method (see e.g. [23, 29]) or discontinuous Galerkin method (see e.g. [15, 22, 29]) are applied.
In this paper, a combination of the Nitsche finite element method (as a mortar method) with the backward Euler method for solving initial-boundary value problems for the heat equation in 2D-domains is defined and analyzed. The finite element method with Nitsche mortaring has been investigated for several classes of elliptic problems in 2D, see e.g. [4, 12, 16, 17, 18, 19, 25, 27]. For solving elliptic problems in axisymmetric domains in 3D, a combination of Nitsche mortaring with the approximating Fourier method is presented in [20, 21]. The finite element method with Nitsche mortaring provides several advantages. Since this method is based on a decomposition of the original domain into subdomains with non-matching triangulations, the mesh generation in these subdomains can be carried out independently from each other. Moreover, it allows different discretization techniques in the subdomains. Further, in comparison with the Lagrange multiplier mortar technique (see e.g. [5, 8, 11, 32]), the saddle point problem, the inf–sup condition and the calculation of additional variables (Lagrange multipliers) are circumvented. Concerning the implementation of the Nitsche finite element method, existing software tools for the standard finite element method can be slightly modified since the bilinear forms in the variational equation differ only by interface terms, see Section 2 for more details.

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The aim of this paper is to derive convergence results for the presented approach which is applied to solve initial-boundary value problems for the heat equation in polygonally bounded domains. Thereby, convex domains as well as domains with reentrant corners are taken into account. As it is known from [14, chapter 5], reentrant corners of the domain cause singularities of the solution which can be represented by means of the singularities of the corresponding elliptic problem. The approximation errors of the investigated approach are estimated in the L_2- and $\{1, h\}$-norms. The latter is an H^1-like, mesh-dependent norm which is introduced because of the discontinuity of the approximate solution along the interface of the subdomains provided with non-matching meshes. In order to obtain the error estimates, the Ritz projection (cf. [1, 29, 31]) is now adapted to the bilinear form occurring in the Nitsche finite element discretization. Moreover, the knowledge on singularities of the solutions of elliptic problems in non-convex polygonal domains ([13, 14]) and their approximation by finite elements is used. Some a-priori estimates for the norms of the exact solution of the parabolic problem and its derivatives in time, given in [9, 29], allow to state the error estimates in such way that only norms of the given data are involved. It can be shown that the presented method yields the same convergence order as the combination of the standard finite element method with the backward Euler method, cf. [29, chapter 19]. In case of a solution with singularities, an appropriate grading of the mesh around the reentrant corner leads to the same convergence order of the semi-discretization (in space) and of the fully discrete method as in case of a regular solution. Moreover, the convergence order of discretization in time is not affected by singularities.
In [6], using some results of [1], the Nitsche mortaring for parabolic problems with regular solutions and under more restrictive assumptions than in our paper is considered.
The paper is organized as follows. In Section 2, the model problem is given and its semi-discretization (in space) by finite elements with Nitsche mortaring is described. The next section deals with approximation properties of the Ritz projection and error estimates for the semi-discretization in case of regular solutions (i.e. convex domains). Section 4 contains respective estimates for solutions with singularities arising in case of non-convex domains, where meshes with local grading are employed. In Section 5, the fully discrete method is defined and its convergence is investigated. Finally, in Section 6 two numerical examples illustrating the approach and the convergence rates are presented.

2 The model problem and its semi-discretization

We consider the following initial-boundary value problem for the heat equation,

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega, \text{ for } 0 < t \leq T \\ u &= 0 \quad \text{on } \partial\Omega, \text{ for } 0 < t \leq T \\ u(\cdot, t=0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

with $u = u(x,t)$, as a model problem, where $\Omega \subset \mathbb{R}^2$ is supposed to be a polygonally bounded domain. In the following we assume that the compatibility condition $u_0 = 0$ on $\partial\Omega$ is satisfied.
Subsequently, for functions defined on X, let $H^s(X)$ ($s \geq 0$, s real, $H^0 = L_2$) denote the usual Sobolev-Slobodetskiĭ space. The usual scalar product in $L_2(X)$ will be denoted

by $(\cdot,\cdot)$. Further, let $v \in H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$, with duality pairing $\langle \cdot\,,\cdot\rangle$ over the space $L_2(\Omega)$. Moreover, we shall need the spaces $\dot{H}^s(\Omega)$ (see [9, 29]). For $s \geq -1$, $\dot{H}^s(\Omega)$ denotes the space of functions defined by

$$\|v\|_{\dot{H}^s(\Omega)} = \Big(\sum_{j=1}^{\infty} \lambda_j^s \langle v, \varphi_j\rangle^2\Big)^{1/2} < \infty, \;\; v \in H^{-1}(\Omega), \tag{2}$$

where λ_j $(j = 1, 2, \ldots)$ are the eigenvalues and φ_j $(j = 1, 2, \ldots)$ the corresponding orthonormal eigenfunctions of the operator $-\Delta$.
For an arbitrary Banach space B, let $L_2(0, T; B)$ be the space of functions $u : (0, T) \to B$ satisfying

$$\|u\|_{L_2(0,T;B)} := \Big(\int_0^T \|u(t)\|_B^2 \, dt\Big)^{1/2} < \infty. \tag{3}$$

For some given $f \in L_2(0, T; L_2(\Omega))$, a function $u = u(x, t)$ is called a weak solution of the problem (1) if the relation

$$(u_t, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{4}$$

holds with $u \in L_2(0, T; H_0^1(\Omega))$ and $u_t \in L_2(0, T; H^{-1}(\Omega))$ and if $u(\cdot, t = 0) = u_0 \in L_2(\Omega)$ (see e.g. [15, 23, 29]).
In order to define an approximate solution to the problem (1) (resp. (4)), we first define a semi-discretization in space, i.e., we approximate the solution $u(x, t)$ of (1) by means of a function $u_h(x, t)$ which, for each fixed t, belongs to a finite element space. For this semi-discretization, the Nitsche finite element method will be employed. For the characterization of this method we shall need a subdivision of Ω into subdomains. Throughout this paper we restrict ourselves to the case of two subdomains Ω^1, Ω^2, with some interface Γ,

$$\overline{\Omega} = \overline{\Omega}^1 \cup \overline{\Omega}^2, \quad \Omega^1 \cap \Omega^2 = \emptyset, \quad \Gamma = \overline{\Omega}^1 \cap \overline{\Omega}^2.$$

Moreover, assume that these subdomains are polygonally bounded. There are different cases for the position of the two subdomains: Figure 1(a) shows the case $\partial\Omega^i \cap \Gamma \neq \emptyset$ for $i = 1, 2$, and in Figure 1(b) we have $\partial\Omega \cap \Gamma = \emptyset$, $\Gamma = \partial\Omega^2$. In view of the subdivision of Ω

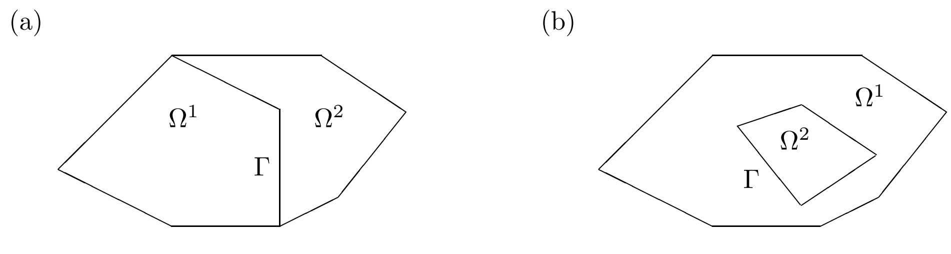

Figure 1: Domain Ω with subdomains Ω^1, Ω^2

we introduce the restrictions $v^i := v|_{\Omega^i}$ of some function v on Ω^i as well as the vectorized form $v = (v^1, v^2)$, i.e. $v^i(x) = v(x)$ holds for $x \in \Omega^i$ $(i = 1, 2)$. It should be noted that for simplicity we use here the same symbol v for denoting the function on Ω as well as the vector (v^1, v^2). Using this notation we obtain that the solution of the BVP (1) is equivalent to the solution of the following problem: Find (u^1, u^2) such that

$$\begin{aligned} u^i_t - \Delta u^i &= f \quad \text{in } \Omega^i \;\; (i=1,2), \;\; \text{for } 0 < t \le T \\ u^i &= 0 \quad \text{on } \partial\Omega^i \cap \partial\Omega \;\; (i=1,2), \;\; \text{for } 0 < t \le T \\ u^i(\cdot, t=0) &= u_0 \quad \text{in } \Omega^i \;\; (i=1,2) \\ u^1 &= u^2 \quad \text{on } \Gamma, \;\; \text{for } 0 < t \le T \\ \frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} &= 0 \quad \text{on } \Gamma, \;\; \text{for } 0 < t \le T, \end{aligned} \tag{5}$$

are satisfied, where n_i $(i = 1, 2)$ denotes the outward normal to $\partial\Omega^i \cap \Gamma$.
Further we introduce the 'broken' space V_0 by

$$V_0 = V_0^1 \times V_0^2, \;\; \text{with} \;\; V_0^i := \{v \in H^1(\Omega^i) : v|_{\partial\Omega^i \cap \partial\Omega} = 0\} \;\; \text{for } i = 1, 2$$

(note that $V_0^i = H^1(\Omega^i)$ if $\partial\Omega^i \cap \partial\Omega = \emptyset$).
Now we describe the finite element discretization of (5) with non-matching meshes. We cover Ω^i $(i = 1, 2)$ by a triangulation $\mathcal{T}_h^i$ $(i = 1, 2)$ consisting of triangles K $(K = \overline{K})$, where $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are independent of each other. Moreover, compatibility of the nodes of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ along the 'mortar interface' $\Gamma = \partial\Omega^1 \cap \partial\Omega^2$ is not required, i.e., non-matching meshes on Γ are admitted. Let h denote the mesh parameter of the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$, with $0 < h \le h_0$ and sufficiently small h_0. Take e.g. $h = \max\{h_K : K \in \mathcal{T}_h\}$, where h_K denotes the diameter of the triangle K. In the sequel, positive constants C occurring in the inequalities are generic constants.
Since in the next section, the case of a regular solution of (5) will be considered, we start with quasi-uniform meshes, i.e., we suppose that the following assumption on the triangulations $\mathcal{T}_h^i$ $(i = 1, 2)$ is fulfilled.

Assumption 1a

(i) For $i = 1, 2$, it holds $\overline{\Omega}^i = \cup_{K \in \mathcal{T}_h^i} K$, and two arbitrary triangles $K, K' \in \mathcal{T}_h^i$ $(K \ne K')$ are either disjoint or have a common vertex, or a common edge.

(ii) The mesh in $\overline{\Omega}^i$ $(i = 1, 2)$ is quasi-uniform, i.e. the relation

$$\max_{K \in \mathcal{T}_h^i} h_K \, (\min_{K \in \mathcal{T}_h^i} \rho_K)^{-1} \le C \quad (i = 1, 2) \tag{6}$$

holds for $h \in (0, h_0]$, where ρ_K denotes the diameter of inscribed circle of K.

For $i = 1, 2$ and according to V_0^i we introduce finite element spaces V_{0h}^i of functions v_h^i on $\overline{\Omega}^i$ by

$$V_{0h}^i \; := \; \{ \, v_h^i \in C(\overline{\Omega}^i) : \; v_h^i \in \mathbb{P}_1(T) \; \forall \, K \in \mathcal{T}_h^i, \; v_h^i|_{\partial\Omega^i \cap \partial\Omega} = 0 \, \},$$

i.e., employ linear finite elements. The finite element space V_{0h} of vectorized functions v_h with components v_h^i on Ω^i is given by

$$V_{0h} \; := \; V_{0h}^1 \times V_{0h}^2 = \{ v_h = (v_h^1, v_h^2) : \; v_h^1 \in V_{0h}^1, v_h^2 \in V_{0h}^2 \}.$$

It should be pointed out that the functions v_h in V_{0h} are in general not continuous across Γ. Further we introduce a triangulation $\mathcal{E}_h$ of the mortar interface Γ by intervals E ($E = \overline{E}$), i.e., $\Gamma = \cup_{E\in\mathcal{E}_h} E$, where h_E denotes the diameter of E. We suppose that two segments E, E' are either disjoint or have a common endpoint. A natural choice for the triangulation $\mathcal{E}_h$ is $\mathcal{E}_h := \mathcal{E}_h^1$ or $\mathcal{E}_h := \mathcal{E}_h^2$, where $\mathcal{E}_h^1$ and $\mathcal{E}_h^2$ denote the triangulations of Γ defined by the traces of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on Γ, resp.:

$$\mathcal{E}_h^i := \{E : E = \partial K \cap \Gamma,\ \text{if } E \text{ is a segment},\ K \in \mathcal{T}_h^i\} \quad \text{for } i = 1, 2. \tag{7}$$

Subsequently we use real parameters α_1 and α_2 with

$$0 \le \alpha_i \le 1 \quad (i = 1, 2), \quad \alpha_1 + \alpha_2 = 1. \tag{8}$$

The asymptotic behaviour of the triangulations $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and of $\mathcal{E}_h$ should be consistent on Γ in the following sense.

Assumption 2

(i) For $E \in \mathcal{E}_h$ and $K \in \mathcal{T}_h^i$ with $\partial K \cap E \neq \emptyset$, $i = 1$ and $i = 2$, there are positive constants C_1 and C_2 independent of h_K, h_E and h ($0 < h \le h_0$) such that the following condition is satisfied

$$C_1 h_K \le h_E \le C_2 h_K. \tag{9}$$

(ii) In the special case $\mathcal{E}_h := \mathcal{E}_h^i$ and $\alpha_i := 1$ (cf. (7), (8)), where $i = 1$ or $i = 2$, for $E \in \mathcal{E}_h$ and $K \in \mathcal{T}_h^{3-i}$ with $\partial K \cap E \neq \emptyset$, instead of relation (9) the following condition is required

$$C_1 h_K \le h_E. \tag{10}$$

Relation (9) means that the diameter h_K of the triangle K touching the interface Γ at E is asymptotically equivalent to the diameter of the segment E, i.e. the equivalence of h_K, h_E is required only locally. In contrast, condition (10) is weaker and admits even locally at Γ a different asymptotics of the triangles $T_1 \in \mathcal{T}_h^1$, $T_2 \in \mathcal{T}_h^2$: $T_1 \cap T_2 \neq \emptyset$.

For the Nitsche finite element approximation of the function $u(t) = u(\cdot, t)$ we shall need bilinear forms $\mathcal{B}_h(\cdot,\cdot)$ and functionals $\mathcal{F}(t)$. The definitions of $\mathcal{B}_h(\cdot,\cdot)$ and $\mathcal{F}(t)$ are motivated by the related definitions in case of elliptic problems (cf. [4, 16, 17, 18, 27]). Thus we introduce

$$\begin{aligned} \mathcal{B}_h(u(t), v) &:= \sum_{i=1}^{2} (\nabla u^i(t), \nabla v^i) - \Big\langle \alpha_1 \frac{\partial u^1(t)}{\partial n_1} - \alpha_2 \frac{\partial u^2(t)}{\partial n_2}, v^1 - v^2 \Big\rangle_\Gamma \\ &\quad - \Big\langle \alpha_1 \frac{\partial v^1}{\partial n_1} - \alpha_2 \frac{\partial v^2}{\partial n_2}, u^1(t) - u^2(t) \Big\rangle_\Gamma + \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} (u^1(t) - u^2(t), v^1 - v^2)_{L_2(E)} \\ \langle \mathcal{F}(t), v \rangle &:= (f(t), v), \ \text{with } u(t), v \in V_0,\ t \in (0, T]. \end{aligned} \tag{11}$$

Here, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the $H_*^{-\frac{1}{2}}(\Gamma) \times H_*^{\frac{1}{2}}(\Gamma)$-duality pairing, where $H_*^{\frac{1}{2}}(\Gamma)$ (also written $H_{00}^{\frac{1}{2}}(\Gamma)$) is defined as the trace space of $H_0^1(\Omega)$ provided with the quotient norm (see e.g. [13]), and $H_*^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H_*^{\frac{1}{2}}(\Gamma)$. If $\partial\Omega^i \cap \partial\Omega = \emptyset$ holds for $i = 1$ or $i = 2$, we have $H_*^{\frac{1}{2}}(\Gamma) = H^{\frac{1}{2}}(\Gamma)$. Moreover, γ is a sufficiently large positive constant (the restriction of γ will be given subsequently) and α_1 as well as α_2 are taken from (8). The Nitsche finite element approximation $u_h : [0,T] \to V_{0h}$ of $u(t)$ is defined to be the solution of the equation

$$(u_{h,t}(t), v_h) + \mathcal{B}_h(u_h(t), v_h) = \langle \mathcal{F}(t), v_h\rangle \quad \forall\, v_h \in V_{0h},\ \forall t \in (0,T] \tag{12}$$

satisfying the initial condition

$$u_h(0) = u_{0h}, \tag{13}$$

where u_{0h} is some approximation of u_0 in V_{0h}.
In order to state the boundedness and ellipticity of the forms $\mathcal{B}_h(\cdot,\cdot)$ we impose the restriction $\gamma > C_I$ on the parameter γ from (11), where the constant C_I is taken from the estimate (see [17, ineq. (17)])

$$\sum_{E\in\mathcal{E}_h} h_E \left\|\alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2}\right\|^2_{L_2(E)} \le C_I \sum_{i=1}^{2} \alpha_i^2 \|\nabla v_h^i\|^2_{L_2(\Omega^i)} \quad \text{for } v_h \in V_{0h}, \tag{14}$$

with α_1, α_2 from (8). Moreover, we shall need the h-dependent norm $\|\cdot\|_{1,h}$ defined by

$$\|v_h\|^2_{1,h} := \sum_{i=1}^{2} \|\nabla v_h^i\|^2_{L_2(\Omega^i)} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|v_h^1 - v_h^2\|^2_{L_2(E)}. \tag{15}$$

Lemma 1 *Let Assumptions 1a and 2 be satisfied for $\mathcal{T}_h^i$ $(i = 1,2)$ and for $\mathcal{E}_h$. Then there exists a constant $\mu_1 > 0$ such that the following estimate holds,*

$$|\mathcal{B}_h(w_h, v_h)| \le \mu_1 \|w_h\|_{1,h} \|v_h\|_{1,h} \quad \forall w_h, v_h \in V_{0h}. \tag{16}$$

If the constant γ in (11) is independent of h and fulfills $\gamma > C_I$ (C_I from (14)), then the inequality

$$\mathcal{B}_h(v_h, v_h) \ge \mu_2 \|v_h\|^2_{1,h} \quad \forall v_h \in V_{0h} \tag{17}$$

holds with a positive constant μ_2. The constants μ_1, μ_2 are independent of h.

For the proof we refer to [17].
Furthermore, for functions $w \in V_0$ satisfying $\frac{\partial w^i}{\partial n_i} \in L_2(\Gamma)$ $(i = 1,2)$, the estimate

$$|\mathcal{B}_h(w, v_h)| \le \mu_3 \|w\|_{h,\Omega} \|v_h\|_{1,h} \tag{18}$$

(see also [17]) can be stated for all functions $v_h \in V_{0h}$. Here, the norm $\|\cdot\|_{h,\Omega}$ is defined by

$$\|w\|^2_{h,\Omega} := \sum_{i=1}^{2} \left\{ \|\nabla w^i\|^2_{L_2(\Omega^i)} + \sum_{E\in\mathcal{E}_h} h_E \left\|\alpha_i \frac{\partial w^i}{\partial n_i}\right\|^2_{L_2(E)} \right\} + \sum_{E\in\mathcal{E}_h} h_E^{-1} \|w^1 - w^2\|^2_{L_2(E)}.$$

The norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{h,\Omega}$ are equivalent on the space V_{0h}:

$$\|v_h\|_{1,h} \le \|v_h\|_{h,\Omega} \le C \|v_h\|_{1,h} \quad \forall v_h \in V_{0h}. \tag{19}$$

3 Convergence of the semi-discretization: case of a regular solution

In the following we also consider the elliptic problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{20}$$

which is the corresponding stationary problem assigned to (1). The variational formulation of (20) reads:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \quad a(u,v) := (\nabla u, \nabla v) = (f,v) \quad \forall v \in H_0^1(\Omega). \tag{21}$$

Throughout this section we assume that the domain Ω is convex. Then, it is well-known that for $f \in L_2(\Omega)$, the solution u of (21) belongs to the space $H^2(\Omega)$.
In order to derive convergence estimates for the semi-discretization, we introduce for $v \in V_0$ the Ritz projection $R_h v \in V_{0h}$. Usually the Ritz projection is defined by means of the bilinear form $a(\cdot,\cdot)$ from (21), see e.g. [23, 29]. But we introduce it by means of the h-dependent bilinear form $\mathcal{B}_h(\cdot,\cdot)$ (see (11)) of the Nitsche finite element approach,

$$\mathcal{B}_h(R_h v, v_h) = \mathcal{B}_h(v, v_h) \quad \forall v_h \in V_{0h}. \tag{22}$$

Moreover, let $I_h v := (I_h v^1, I_h v^2)$, where $I_h v^i$ $(i = 1,2)$ denotes the usual Lagrange interpolant of v^i in the space V_{0h}^i. In the next two lemmas, estimates of the error $R_h v - v$ are given. They generalize the results well-known for the bilinear form $a(\cdot,\cdot)$ like in [23, 29].

Lemma 2 *Let Assumptions 1a and 2 be fulfilled for $\mathcal{T}_h^i$ $(i = 1,2)$ and for $\mathcal{E}_h$. Moreover, assume that $\gamma > C_I$ holds (see Lemma 1). Then, for $v \in H^2(\Omega) \cap H_0^1(\Omega)$, the function $R_h v$ from (22) satisfies the estimate*

$$\|R_h v - v\|_{1,h} \le Ch\|v\|_{H^2(\Omega)}. \tag{23}$$

Proof: First we obtain the inequality

$$\|R_h v - v\|_{1,h} \le \|R_h v - I_h v\|_{1,h} + \|v - I_h v\|_{1,h} \le \|R_h v - I_h v\|_{1,h} + \|v - I_h v\|_{h,\Omega}. \tag{24}$$

Using the abbreviation $\chi := R_h v - I_h v$, the first term on the right-hand side of (24) can be estimated by means of relations (17), (22), and (18):

$$\begin{aligned} \|\chi\|_{1,h}^2 &\le C\mathcal{B}_h(\chi,\chi) = C(\mathcal{B}_h(R_h v,\chi) - \mathcal{B}_h(I_h v,\chi)) = C(\mathcal{B}_h(v,\chi) - \mathcal{B}_h(I_h v,\chi)) \\ &= C(\mathcal{B}_h(v - I_h v,\chi) \le C\|v - I_h v\|_{h,\Omega}\|\chi\|_{1,h}. \end{aligned} \tag{25}$$

The interpolation error can be bounded by $\|v - I_h v\|_{h,\Omega} \le Ch\|v\|_{H^2(\Omega)}$, which follows from [17, proof of Theorem 2]). This, together with the estimates (24) and (25) leads to (23). ■

Lemma 3 *Under the assumptions of Lemma 2, for $v \in H^2(\Omega) \cap H_0^1(\Omega)$ the following estimate holds,*

$$\|R_h v - v\|_{L_2(\Omega)} \le Ch^2\|v\|_{H^2(\Omega)}. \tag{26}$$

Proof: We introduce the auxiliary elliptic problem: find $\tilde{v} \in H_0^1(\Omega)$ such that

$$a(\tilde{v}, w) = (v - R_h v, w) \quad \forall\, w \in H_0^1(\Omega), \tag{27}$$

with $a(\cdot,\cdot)$ from (21). Owing to the assumptions on the domain Ω, the solution $\tilde{v}$ of this problem belongs to $H^2(\Omega)$. The Nitsche finite element approximation $\tilde{v}_h \in V_{0h}$ of $\tilde{v}$ is then given by (cf. [17]): $\mathcal{B}_h(\tilde{v}_h, v_h) = (v - R_h v, v_h)\ \forall v_h \in V_{0h}$. As a result of [17, Lemma 1], the solution $\tilde{v}$ is consistent with this variational equation, i.e.

$$\mathcal{B}_h(\tilde{v}, v_h) = (v - R_h v, v_h) \quad \forall v_h \in V_{0h}. \tag{28}$$

Taking into account the definitions of $\mathcal{B}_h(\cdot,\cdot)$ and $a(\cdot,\cdot)$ and using v as a test function in (27), we obtain

$$\mathcal{B}_h(\tilde{v}, v) = a(\tilde{v}, v) = (v - R_h v, v). \tag{29}$$

Choosing $v_h := R_h v$ in (28) and using (29) we are led to

$$\begin{aligned} \|v - R_h v\|^2_{L_2(\Omega)} &= (v - R_h v, v) - (v - R_h v, R_h v) = \mathcal{B}_h(\tilde{v}, v) - \mathcal{B}_h(\tilde{v}, R_h v) \\ &= \mathcal{B}_h(v - R_h v, \tilde{v} - I_h \tilde{v}), \end{aligned} \tag{30}$$

where the last equality follows from symmetry of $\mathcal{B}_h(\cdot,\cdot)$ and relation (22) with $v_h := I_h\tilde{v}$. Employing the Hölder and Cauchy-Schwarz inequalities, the interpolation error estimate $\|I_h\tilde{v} - \tilde{v}\|_{h,\Omega} \le Ch\|\tilde{v}\|_{H^2(\Omega)}$ as well as the a priori estimate $\|\tilde{v}\|_{H^2(\Omega)} \le C\|v - R_h v\|_{L_2(\Omega)}$ of the solution $\tilde{v}$, the term on the right-hand side of (30) can be bounded as follows,

$$\mathcal{B}_h(v - R_h v, \tilde{v} - I_h\tilde{v}) \le \|v - R_h v\|_{h,\Omega}\|\tilde{v} - I_h\tilde{v}\|_{h,\Omega} \le Ch\|v - R_h v\|_{h,\Omega}\|v - R_h v\|_{L_2(\Omega)}. \tag{31}$$

Therefore it remains to find an estimate for $\|v - R_h v\|_{h,\Omega}$. Inserting $I_h v$ and using (19) we obtain

$$\|v - R_h v\|_{h,\Omega} \le \|v - I_h v\|_{h,\Omega} + C\|I_h v - R_h v\|_{1,h}. \tag{32}$$

Thanks to (17), (22), and (18) we get $\|I_h v - R_h v\|^2_{1,h} \le C\mathcal{B}_h(I_h v - R_h v, I_h v - R_h v) = C\mathcal{B}_h(I_h v - v, I_h v - R_h v) \le C\|I_h v - v\|_{h,\Omega}\|I_h v - R_h v\|_{1,h}$. This implies $\|I_h v - R_h v\|_{1,h} \le C\|I_h v - v\|_{h,\Omega}$, and together with the interpolation error estimate $\|I_h v - v\|_{h,\Omega} \le Ch\|v\|_{H^2(\Omega)}$ as well as relations (30)–(32) we obtain (26). ■

In the following, the error between the solutions of the semi-discrete and continuous problems is estimated in the L_2-norm and the norm $\|\cdot\|_{1,h}$. These error estimates are based on the splitting of the error (see e.g. [23, 29]):

$$u_h(t) - u(t) = \theta(t) + \rho(t), \ \text{ with } \ \theta = u_h - R_h u, \ \rho = R_h u - u, \tag{33}$$

and R_h defined by relation (22).
We require in the following that the solution (resp. the given data) of the parabolic problem has such a regularity that all norms arising on the right-hand sides of the estimates are finite.

Lemma 4 *Let the assumptions of Lemma 2 be satisfied. Then, for the solutions u and u_h from (1) and (12), with $u_{0h} = R_h u_0$, the following error estimate holds,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \le Ch^2\Big\{\|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)} d\tau\Big\}, \quad \text{for } t \le T. \tag{34}$$

Proof: In view of Lemma 3 and the fact that $u \in H^2(\Omega)$, the summand $\rho(t)$ occurring in the splitting (33) can be bounded by

$$\begin{aligned} \|\rho(t)\|_{L_2(\Omega)} &= \|R_h u(t) - u(t)\|_{L_2(\Omega)} \le Ch^2 \|u(t)\|_{H^2(\Omega)} \\ &\le Ch^2 \Big(\|u(0)\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)} d\tau \Big). \end{aligned} \tag{35}$$

In order to find an estimate for the remaining summand $\theta(t)$ we use (12) and (22) leading to

$$\begin{aligned} &(\theta_t, v_h) + \mathcal{B}_h(\theta, v_h) = (u_{h,t}, v_h) + \mathcal{B}_h(u_h, v_h) - ((R_h u)_t, v_h) - \mathcal{B}_h(R_h u, v_h) \\ &\quad = (f, v_h) - (R_h u_t, v_h) - \mathcal{B}_h(u, v_h) = (u_t, v_h) - (R_h u_t, v_h) = -(\rho_t, v_h) \end{aligned} \tag{36}$$

for $v_h \in V_{0h}$. With the special choice $v_h := \theta$ this yields

$$(\theta_t, \theta) + \mathcal{B}_h(\theta, \theta) = -(\rho_t, \theta),$$

and by means of (17) and the Cauchy-Schwarz inequality we get

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L_2(\Omega)} = \|\theta\|_{L_2(\Omega)} \frac{d}{dt} \|\theta\|_{L_2(\Omega)} \le \|\rho_t\|_{L_2(\Omega)} \|\theta\|_{L_2(\Omega)}.$$

After dividing by $\|\theta\|_{L_2(\Omega)}$ and integrating this implies in consideration of the assumption $u_{0h} = R_h u_0$ (i.e. $\theta(0) = 0$),

$$\|\theta(t)\|_{L_2(\Omega)} \le \|\theta(0)\|_{L_2(\Omega)} + \int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau = \int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau, \tag{37}$$

and thanks to (26), the norm of ρ_t can be bounded as follows,

$$\|\rho_t\|_{L_2(\Omega)} = \|R_h u_t - u_t\|_{L_2(\Omega)} \le Ch^2 \|u_t\|_{H^2(\Omega)}. \tag{38}$$

Finally, the assertion is a result of relations (33) and (35)–(38). ■

Lemma 5 *Under the assumptions of Lemma 2, the solutions u and u_h from (1) and (12), with $u_{0h} = R_h u_0$, satisfy the following error estimate,*

$$\|u_h(t) - u(t)\|_{1,h} \le Ch \Big\{ \|u_0\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)} d\tau \Big\}, \quad \textit{for } t \le T. \tag{39}$$

Proof: Owing to (15) we have

$$\begin{aligned} &\|u_h(t) - u(t)\|^2_{1,h} = \\ &\sum_{i=1}^{2} \|\nabla(u_h^i(t) - u^i(t))\|^2_{L_2(\Omega^i)} + \sum_{E \in \mathcal{E}_h} h_E^{-1} \|u_h^1(t) - u^1(t) - (u_h^2(t) - u^2(t))\|^2_{L_2(E)}. \end{aligned} \tag{40}$$

For any $v_h \in V_{0h}$ the first term on the right-hand side of (40) can be estimated as follows,

$$\begin{aligned}\sum_{i=1}^{2}\|\nabla(u_h^i(t)-u^i(t))\|^2_{L_2(\Omega^i)} &\leq 2\sum_{i=1}^{2}(\|\nabla(u_h^i(t)-v_h^i)\|^2_{L_2(\Omega^i)}+\|\nabla(v_h^i-u^i(t))\|^2_{L_2(\Omega^i)})\\ &=: 2\sum_{i=1}^{2}(S_1^i+S_2^i), \end{aligned} \tag{41}$$

where S_1^i and S_2^i are abbreviations for the corresponding norm terms. Since the mesh is supposed to be quasi-uniform in $\overline{\Omega}^i$, the term S_1^i from (41) may be bounded by means of an inverse inequality (see e.g. [10]), i.e. we obtain for $i=1,2$:

$$S_1^i \leq Ch^{-2}\|u_h^i(t)-v_h^i\|^2_{L_2(\Omega^i)} \leq Ch^{-2}(\|u_h^i(t)-u^i(t)\|^2_{L_2(\Omega^i)}+\|u^i(t)-v_h^i\|^2_{L_2(\Omega^i)}). \tag{42}$$

Setting $v_h = I_h u(t)$, i.e. $v_h^i = I_h u^i(t)$ $(i=1,2)$, using Lemma 4 for estimating the first summand on the right-hand side of (42) and employing some interpolation error estimate for the second summand we arrive at

$$S_1^i \leq Ch^2\Big\{\|u_0^i\|^2_{H^2(\Omega^i)}+\Big[\int_0^t\|u_t^i\|_{H^2(\Omega^i)}d\tau\Big]^2+\|u^i(t)\|^2_{H^2(\Omega^i)}\Big\},\ \ i=1,2. \tag{43}$$

For estimating S_2^i $(i=1,2)$ from (41), these terms will be considered in combination with some terms arising from the second summand in (40). With the notation

$$S_3^i=\sum_{E\in\mathcal{E}_h}h_E^{-1}\|u_h^i(t)-v_h^i\|^2_{L_2(E)},\ \ S_4^i=\sum_{E\in\mathcal{E}_h}h_E^{-1}\|v_h^i-u^i(t)\|^2_{L_2(E)},\ \ i=1,2,$$

we obtain for the second summand in (40)

$$\sum_{E\in\mathcal{E}_h}h_E^{-1}\|u_h^1(t)-u^1(t)-(u_h^2(t)-u^2(t))\|^2_{L_2(E)} \leq 2\sum_{i=1}^{2}(S_3^i+S_4^i). \tag{44}$$

Taking into account that $v_h = I_h u(t)$ and using the estimate for $\|u-I_h u\|_{h,\Omega}$ from [17, Proof of Theorem 2] leads to

$$\sum_{i=1}^{2}(S_2^i+S_4^i) \leq Ch^2\sum_{i=1}^{2}\|u^i(t)\|^2_{H^2(\Omega^i)} \leq Ch^2\|u(t)\|^2_{H^2(\Omega)}. \tag{45}$$

Hence it remains to find an estimate for S_3^i from (44). The summation over $E\in\mathcal{E}_h$ can be rewritten such that the estimates of $u_h^i(t)-I_h u^i(t)$, $i=1$ or $i=2$, involve the side F of the triangle $T\subset\overline{\Omega}^i$ $(T=T_F)$ with $T_F\cap\Gamma=F\in\mathcal{E}_h^i$ ($\mathcal{E}_h^i$ from (7)):

$$S_3^i=\sum_{E\in\mathcal{E}_h}h_E^{-1}\|u_h^i(t)-I_h u^i(t)\|^2_{L_2(E)} \leq C\sum_{F\in\mathcal{E}_h^i}h_F^{-1}\|u_h^i(t)-I_h u^i(t)\|^2_{L_2(F)}. \tag{46}$$

Then we get by means of [30, Theorem 3] for $i=1,2$:

$$\|u_h^i(t)-v_h^i\|^2_{L_2(F)} \leq C(h_F^{\perp})^{-1}\|u_h^i(t)-v_h^i\|^2_{L_2(T_F)}, \tag{47}$$

where $h_F^\perp$ is the height of the triangle T_F over the side F. Using $h_F^{-1} \leq Ch^{-1}$ and $(h_F^\perp)^{-1} \leq Ch^{-1}$, relations (46) and (47) imply

$$\begin{aligned} S_3^i \;&\leq\; C \sum_{\substack{K\in\mathcal{T}_h^i:\\ T\cap\Gamma\neq\emptyset}} h^{-2}\|u_h^i(t) - I_h u^i(t)\|^2_{L_2(T_F)} \leq Ch^{-2}\|u_h^i(t) - I_h u^i(t)\|^2_{L_2(\Omega_i)} \\ &\leq\; Ch^{-2}\{\|u_h^i(t) - u^i(t)\|^2_{L_2(\Omega_i)} + \|u^i(t) - I_h u^i(t)\|^2_{L_2(\Omega_i)}\}. \end{aligned}$$

Then, employing once more Lemma 4 and some interpolation error estimate we obtain

$$S_3^i \leq Ch^2\Big\{\|u_0^i\|^2_{H^2(\Omega^i)} + \Big[\int_0^t \|u_t^i\|_{H^2(\Omega^i)} d\tau\Big]^2 + \|u^i(t)\|^2_{H^2(\Omega^i)}\Big\},\;\; i = 1, 2. \tag{48}$$

Finally, relations (40), (41), (43)–(45), and (48), together with

$$\|u^i(t)\|_{H^2(\Omega^i)} \leq \|u_0^i\|_{H^2(\Omega^i)} + \int_0^t \|u_t^i\|_{H^2(\Omega^i)} d\tau,$$

lead to the desired estimate. ■

The terms on the right-hand sides of (34) and (39) still comprise norms of the derivative of the solution u. The next aim is to establish an estimate in terms of data of the problem. For this purposes we apply some results from [29].

Theorem 1 *Let Assumptions 1a and 2 be fulfilled for $\mathcal{T}_h^i$ ($i = 1, 2$) and for $\mathcal{E}_h$. Moreover, assume that $\gamma > C_I$ holds (see Lemma 1), and let the function g be defined by*

$$g := u_t(0) = f(0) + \Delta u_0. \tag{49}$$

Then the solutions u and u_h from (1) and (12), with $u_{0h} = R_h u_0$, satisfy the following error estimates,

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \leq Ch^2\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\} \quad (t \leq T), \tag{50}$$

$$\|u_h(t) - u(t)\|_{1,h} \leq Ch\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\} \quad (t \leq T), \tag{51}$$

for any $\varepsilon \in (0, \frac{1}{2})$ and with $C = C(\varepsilon, T)$.

Proof: According to [29, Lemma 19.1], the estimate

$$\int_0^t (\|u_t\|_{H^2(\Omega)} + \|u_{tt}\|_{L_2(\Omega)})\, d\tau \leq C\,\Big\{\|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}, \quad t \leq T, \tag{52}$$

with $C = C(\varepsilon, T)$ holds for $\varepsilon \in (0, \frac{1}{2})$ and convex domains Ω. This, together with Lemmas 4 and 5, yields the assertion. ■

Consequently, if for the semi-discretization of the initial-boundary value problem (1) the Nitsche finite element method is applied, then the same convergence rate as in case of a semi-discretization with the standard finite element method is achieved, see [29, chapter 19].

4 Convergence of the semi-discretization: solution with singularities

Throughout this section we consider non-convex domains Ω. For simplicity we assume that there is only one reentrant corner P, with angle β, $\pi < \beta < 2\pi$. Then, according to [13, 14], the solution u of the elliptic problem (21) in general does not belong to $H^2(\Omega)$, but admits a splitting into a regular and a singular part:

$$u = w + s, \text{ with } w \in H^2(\Omega), \; s = s(r,\theta) = \eta(r) r^\lambda \sin(\lambda\theta), \; \lambda := \frac{\pi}{\beta}, \; \frac{1}{2} < \lambda < 1. \quad (53)$$

In (53), (r,θ) denote polar coordinates with respect to the reentrant corner, and $\eta(r)$ is a smooth cut-off function with

$$0 \le \eta \le 1, \;\; \eta = 1 \text{ for } 0 \le r \le \frac{r_0}{3}, \;\; \eta = 0 \text{ for } r \ge \frac{2r_0}{3}.$$

For the singular part s of the solution holds $s \in H^{1+\lambda-\varepsilon}(\Omega)$ with any $\varepsilon > 0$. Furthermore, according to [9, 29], the singular part s belongs to the space $\dot{H}^2(\Omega)$ defined by (2). This regularity statement will be essentially needed for subsequent estimates.
As it is shown e.g. in [2, 3, 7, 24, 26], the convergence rate of the standard finite element method on quasi-uniform meshes is reduced when this method is applied for the solution of boundary value problems with singularities of the type (53). This gives reason to modify the assumptions on the meshes given in Section 2 such that meshes with some local grading are admitted. Instead of Assumption 1a we suppose from now on that the following assumption is fulfilled.

Assumption 1b

(i) For $i = 1, 2$, it holds $\overline{\Omega}^i = \cup_{K \in \mathcal{T}_h^i} K$, and two arbitrary triangles $K, K' \in \mathcal{T}_h^i$ $(K \ne K')$ are either disjoint or have a common vertex, or a common edge.

(ii) The mesh in $\overline{\Omega}^i$ $(i = 1, 2)$ is shape regular, i.e., the following relation holds (ρ_K : radius of inscribed circle of K),

$$h_K {\rho_K}^{-1} \le C \text{ for } K \in \mathcal{T}_h^i, \; h \in (0, h_0]. \quad (54)$$

Relation (54) means that the triangulations $\mathcal{T}_h^i$ $(i = 1, 2)$ do not have to be quasi-uniform in general. Moreover, for providing a framework for graded meshes, we introduce the real grading parameter μ, $0 < \mu \le 1$, the grading function R_i $(i = 0, 1, \ldots, n)$ with some real constant $d > 0$, and the step size h_i for the mesh associated with layers $[R_{i-1}, R_i] \times [0, \theta_0]$ around the reentrant corner P:

$$R_i := d(ih)^{\frac{1}{\mu}} \;\; (i = 0, 1, \ldots, n), \quad h_i := R_i - R_{i-1} \;\; (i = 1, 2, \ldots, n). \quad (55)$$

Here $n := n(h)$ denotes an integer of the order h^{-1}, $n := [\delta h^{-1}]$ for some real $\delta > 0$ ($[\cdot]$ means the integer part). We shall choose d and δ such that $\frac{2}{3} r_0 < R_n < r_0$ holds.
Using the step size h_i $(i = 1, 2, \ldots, n)$ from (55) we define a mesh which is graded locally in the neighbourhood of the vertex P of the reentrant corner and quasi-uniform in the

remaining part of the domain Ω. The diameter h_K of a triangle $K \in \mathcal{T}_h$ is now characterized by the mesh size h $(0 < h \le h_0)$, by the distance R_K of K from P, and by the grading parameter μ, with fixed μ: $0 < \mu \le 1$. The properties of $\mathcal{T}_h$ are summarized in the following assumption.

Assumption 3

Let the triangulation $\mathcal{T}_h$ satisfy Assumption 1b. Furthermore, $\mathcal{T}_h$ is provided with a local grading around the vertex P of the reentrant corner such that $h_K := \operatorname{diam} K$ depends on the distance R_K of K from P, $R_K := \operatorname{dist}(K, P)$ in the following way:

$$\begin{aligned} c_1 h^{\frac{1}{\mu}} \ &\le\ h_K \le c_1^{-1} h^{\frac{1}{\mu}} && \text{for } K \in \mathcal{T}_h : R_K = 0, \\ c_2 h R_K^{1-\mu} \ &\le\ h_K \le c_2^{-1} h R_K^{1-\mu} && \text{for } K \in \mathcal{T}_h : 0 < R_K < R_g, \\ c_3 h \ &\le\ h_K \le c_3^{-1} h && \text{for } K \in \mathcal{T}_h : R_g \le R_K, \end{aligned} \tag{56}$$

with some constants c_i, $0 < c_i \le 1$ $(i = 1, 2, 3)$ and some real R_g, $0 < \underline{R}_g < R_g < \overline{R}_g$, where $\underline{R}_g, \overline{R}_g$ are fixed and independent of h.

In (56), R_g is the radius of the sector with mesh grading, and w.l.o.g. we may assume $R_g = R_n$. The value $\mu = 1$ yields a quasi-uniform mesh in the whole domain Ω, i.e. the relation $\max_{K \in \mathcal{T}_h^i} h_K (\min_{K \in \mathcal{T}_h^i} \rho_K)^{-1} \le C$ $(i = 1, 2)$ holds instead of (54). Owing to Assumption 3, the asymptotic behaviour of h_K is determined by the relations

$$\begin{aligned} &\varepsilon_1 h_j \le h_K \le \varepsilon_1^{-1} h_j \quad \text{for } K \in \mathcal{T}_h : R_{j-1} \le R_K \le R_j \ \ (j = 1, 2, \ldots, n), \\ &\varepsilon_2 h \le h_K \le \varepsilon_2^{-1} h \quad \text{for } K \in \mathcal{T}_h : R_n \le R_K, \end{aligned} \tag{57}$$

where $0 < \varepsilon_l \le 1$ $(l = 1, 2)$ holds, and h_j, R_j as well as n are taken from (55). Examples of meshes with local grading as described in Assumption 3 will be given in Section 6.

It should be noted that the total number of nodes of $\mathcal{T}_h$ satisfying Assumption 3 is always of the order $\mathcal{O}(h^{-2})$. In [7, 17, 24, 26], related methods of mesh grading are given. In [9], a mesh grading is described which guarantees an optimal convergence rate even in the $C(\overline{\Omega})$-norm.

Under the Assumptions 1b, 2, and 3, the definitions of the spaces V_{0h}^i and V_{0h} remain the same as in Section 2. Moreover, the Nitsche finite element approximation $u_h : [0, T] \to V_{0h}$ of the solution u is defined by (12), (13) as before. The statement concerning boundedness and ellipticity of $\mathcal{B}_h(\cdot, \cdot)$ (see Lemma 1) is also valid in case of graded meshes, cf. [17].

Now we turn to error estimates of the semi-discretization. In view of the splitting (33) of the error $u_h - u$, we need estimates for $u - R_h u$ in the case that the solution u of (21) has singularities.

Lemma 6 *Let u be the solution of (21), where the representation (53) holds. Further, let Assumptions 1b, 2, and 3 be satisfied for $\mathcal{T}_h^i$ $(i = 1, 2)$ and for $\mathcal{E}_h$. If $\gamma > C_I$ (see Lemma 1), then the inequalities*

$$\|R_h u - u\|_{1,h} \le C \kappa(h, \mu) \|\Delta u\|_{L_2(\Omega)} \tag{58}$$

$$\|R_h u - u\|_{L_2(\Omega)} \le C \kappa^2(h, \mu) \|\Delta u\|_{L_2(\Omega)} \tag{59}$$

hold, with

$$\kappa(h,\mu) = \begin{cases} h^{\frac{\lambda}{\mu}} & \textit{for } \lambda < \mu \le 1 \\ h|\ln h|^{1/2} & \textit{for } \mu = \lambda \\ h & \textit{for } 0 < \mu < \lambda. \end{cases} \tag{60}$$

Proof: We find by analogy to the proof of Lemma 2 that

$$\|R_h u - u\|_{1,h} \le C\|u - I_h u\|_{h,\Omega}. \tag{61}$$

Further, due to [17, Lemma 7] and since $-\Delta u = f \in L_2(\Omega)$ we obtain

$$\|u - I_h u\|_{h,\Omega} \le C\kappa(h,\mu)\|f\|_{L_2(\Omega)} = C\kappa(h,\mu)\|\Delta u\|_{L_2(\Omega)}, \tag{62}$$

with $\kappa(h,\mu)$ from (60). Then, inequalities (61) and (62) lead to (58).
In order to prove (59), we observe that the estimate

$$\mathcal{B}_h(u - R_h u, \tilde{u} - I_h \tilde{u}) \le C\|u - R_h u\|_{h,\Omega}\|\tilde{u} - I_h\tilde{u}\|_{h,\Omega} \le C\|u - I_h u\|_{h,\Omega}\|\tilde{u} - I_h\tilde{u}\|_{h,\Omega} \tag{63}$$

holds. This inequality is obtained by analogy to the proof of Lemma 3, now with $\tilde{u}$ as the solution of the auxiliary elliptic problem with the right-hand side $u - R_h u$. Using again [17, Lemma 7] we are led to (59). ■

Further, according to [29, Lemma 19.3], the estimate $\|\Delta u\|_{L_2(\Omega)} \le C\|u\|_{\dot{H}^2(\Omega)}$ holds with $\|\cdot\|_{\dot{H}^2(\Omega)}$ defined by (2). This, together with Lemma 6 yields the inequalities

$$\|R_h u - u\|_{1,h} \le C\kappa(h,\mu)\|u\|_{\dot{H}^2(\Omega)}, \quad \|R_h u - u\|_{L_2(\Omega)} \le C\kappa^2(h,\mu)\|u\|_{\dot{H}^2(\Omega)}. \tag{64}$$

Now we are ready to give estimates of the error between the solutions of the semi-discrete and continuous problems.

Theorem 2 *Let Assumptions 1b, 2, and 3 be fulfilled for* $\mathcal{T}_h^i$ *(*$i = 1,2$*) and for* $\mathcal{E}_h$*, and let* $\gamma > C_I$ *(see Lemma 1). Then, for the solutions* u *and* u_h *from (1) and (12), with* $u_{0h} = R_h u_0$*, the error estimate*

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \le C\kappa^2(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\,d\tau\Big\} \quad (t \le T) \tag{65}$$

holds, with $\kappa(h,\mu)$ *from (60),* g *from (49) and any* $\varepsilon \in (0,\frac{1}{2})$*. The constant* C *in (65) depends on* ε *and* T*.*

Proof: Taking into account relations (33) and (37), the error can be estimated by

$$\|u_h(t) - u(t)\|_{L_2(\Omega)} \le \|\rho(t)\|_{L_2(\Omega)} + \|\theta(t)\|_{L_2(\Omega)} \le \|\rho(0)\|_{L_2(\Omega)} + 2\int_0^t \|\rho_t\|_{L_2(\Omega)}d\tau. \tag{66}$$

In view of Lemma 6 and the assumption $u_{0h} = R_h u_0$ we obtain

$$\|\rho(0)\|_{L_2(\Omega)} = \|u(0) - R_h u(0)\|_{L_2(\Omega)} \le C\kappa^2(h,\mu)\|\Delta u_0\|_{L_2(\Omega)}, \tag{67}$$

such that it remains to estimate the integral on the right-hand side of (66). Using the second relation from (64) we get

$$\int_0^t \|\rho_t\|_{L_2(\Omega)} d\tau = \int_0^t \|u_t - R_h u_t\|_{L_2(\Omega)} d\tau \le C\kappa^2(h,\mu) \int_0^t \|u_t\|_{\dot{H}^2(\Omega)} d\tau. \tag{68}$$

The right-hand side of this inequality can be bounded in terms of data of the problem when the estimate (see [29, Lemma 19.5] or [9, Lemma 3.1])

$$\int_0^t (\|u_t\|_{\dot{H}^2(\Omega)} + \|u_{tt}\|_{L_2(\Omega)})\, d\tau \le C\left\{\|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\right\}, \quad t \le T, \tag{69}$$

with $C = C(\varepsilon, T)$ is applied. Hence, the assertion of the theorem can be deduced from (66)–(69). ■

Theorem 3 *Under the assumptions of Theorem 2 we have for* $t \le T$ *the error estimate*

$$\begin{aligned} &\|u_h(t) - u(t)\|_{1,h} \qquad (70)\\ &\le C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \Big(\int_0^t \|f_t\|^2_{L_2(\Omega)}\, d\tau\Big)^{1/2} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}, \end{aligned}$$

with $\varepsilon \in (0, \frac{1}{2})$ *and* $C = C(\varepsilon, T)$.

Proof: Since we consider meshes which are not quasi-uniform, the technique of the proof of Lemma 5 based on an inverse inequality cannot be applied. Therefore we use the splitting (33) and derive estimates for the norms of $\rho(t)$ and $\theta(t)$.
First we obtain by Lemma 6, relation (58),

$$\|\rho(t)\|_{1,h} \le C\kappa(h,\mu)\|\Delta u(t)\|_{L_2(\Omega)} \le C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \int_0^t \|\Delta u_t\|_{L_2(\Omega)}\, d\tau\Big\}. \tag{71}$$

Using $\|\Delta u_t\|_{L_2(\Omega)} \le C\|u_t\|_{\dot{H}^2(\Omega)}$ and inequality (69), we are led to

$$\begin{aligned} \|\rho(t)\|_{1,h} \;&\le\; C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \int_0^t \|u_t\|_{\dot{H}^2(\Omega)}\, d\tau\Big\} \\ &\le\; C\kappa(h,\mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^t \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}. \end{aligned} \tag{72}$$

Further, relation (36) with $v_h := \theta_t$ yields $(\theta_t, \theta_t) + \mathcal{B}_h(\theta, \theta_t) = -(\rho_t, \theta_t)$, and by means of the Cauchy-Schwarz inequality we obtain

$$\|\theta_t\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}\mathcal{B}_h(\theta,\theta) \le \frac{1}{2}\|\rho_t\|^2_{L_2(\Omega)} + \frac{1}{2}\|\theta_t\|^2_{L_2(\Omega)},$$

which implies

$$\frac{1}{2}\|\theta_t\|^2_{L_2(\Omega)} + \frac{1}{2}\frac{d}{dt}\mathcal{B}_h(\theta,\theta) \le \frac{1}{2}\|\rho_t\|^2_{L_2(\Omega)}.$$

After omitting the first term on the left-hand side, integrating and using the ellipticity of $\mathcal{B}_h(\cdot,\cdot)$ (see Lemma 1) as well as the assumption $u_{0h} = R_h u_0$ (i.e. $\theta(0) = 0$) we arrive at

$$\|\theta(t)\|^2_{1,h} \le C\mathcal{B}_h(\theta(t),\theta(t)) \le C\int_0^t \|\rho_t\|^2_{L_2(\Omega)} d\tau. \tag{73}$$

Lemma 6, relation (59), yields an estimate for the norm of ρ_t leading to

$$\|\theta(t)\|^2_{1,h} \le C\kappa^4(h,\mu)\int_0^t \|\Delta u_t\|^2_{L_2(\Omega)} d\tau,$$

and by the use of [29, Lemma 19.6] for bounding the integral on the right-hand side of this inequality we obtain

$$\|\theta(t)\|^2_{1,h} \le C\kappa^4(h,\mu)\Big\{\|g\|^2_{H_0^1(\Omega)} + \int_0^t \|f_t\|^2_{L_2(\Omega)} d\tau\Big\}. \tag{74}$$

Finally, we deduce from (33), (72), and (74) that the estimate (70) holds. ■

5 Estimates for the fully discrete method

For the discretization in time of the spatially semi-discrete problem (12), the backward Euler method is applied. The constant time step is denoted by k. Further we use the notation $U^n = U_h^n$, where U_h^n means the approximation in V_{0h} of the exact solution $u(t) = u(\cdot,t)$ from (1) at time $t = t_n = nk$, $n = 0, 1, \ldots, N_T$. The fully discrete problem then reads

$$\begin{aligned}(\overline{\partial}U^n, v_h) + \mathcal{B}_h(U^n, v_h) &= \langle \mathcal{F}(t_n), v_h\rangle \quad \forall\, v_h \in V_{0h},\ 1 \le n \le N_T,\\ U^0 &= u_{0h} = R_h u_0,\end{aligned} \tag{75}$$

with $\overline{\partial}U^n = (U^n - U^{n-1})/k$.

First we give error estimates in case of a convex domain, i.e. the results from Section 3 for the semi-discretization will be used. The L_2-norm of the error for the fully discrete method can be bounded as follows.

Theorem 4 *Let Ω be a convex domain and let the assumptions of Lemma 2 be satisfied. Then, for the solution U^n of (75) and the solution $u(t_n)$ of (1), with $0 \le n \le N_T$, we obtain the error estimate*

$$\|U^n - u(t_n)\|_{L_2(\Omega)} \le C(h^2 + k)\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\}, \tag{76}$$

with g from (49), $\varepsilon \in (0, \frac{1}{2})$ and $C = C(\varepsilon, T)$.

Proof: We follow the techniques of [29, Proof of Theorem 1.5], with the necessary modifications due to $\mathcal{B}_h(\cdot,\cdot)$. By analogy to (33), we employ the following splitting of the error

$$U^n - u(t_n) = (U^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) =: \theta^n + \rho^n. \tag{77}$$

The estimate of the term $\rho^n = \rho(t_n)$ can be deduced from relations (35) and (52). Further, for bounding the term θ^n we introduce ω^n as follows,

$$\omega^n = \omega_1^n + \omega_2^n = (R_h - I)\overline{\partial} u(t_n) + (\overline{\partial} u(t_n) - u_t(t_n)). \tag{78}$$

Some calculations analogously to (36) show that ω^n satisfies the relation

$$(\overline{\partial}\theta^n, v_h) + \mathcal{B}_h(\theta^n, v_h) = -(\omega^n, v_h) \quad \forall\, v_h \in V_{0h},\ 1 \le n \le N_T, \tag{79}$$

where $\overline{\partial}\theta^n$ is defined by $\overline{\partial}\theta^n = (\theta^n - \theta^{n-1})/k$. Taking θ^n for v_h in (79) and using the Cauchy-Schwarz inequality we are led to $(\overline{\partial}\theta^n, \theta^n) \le \|\omega^n\|_{L_2(\Omega)} \|\theta^n\|_{L_2(\Omega)}$. Then, by means of the definition of $\overline{\partial}\theta^n$ we get the inequalities

$$\|\theta^n\|^2_{L_2(\Omega)} - (\theta^{n-1}, \theta^n) \le k\|\omega^n\|_{L_2(\Omega)} \|\theta^n\|_{L_2(\Omega)}, \quad \|\theta^n\|_{L_2(\Omega)} \le \|\theta^{n-1}\|_{L_2(\Omega)} + k\|\omega^n\|_{L_2(\Omega)}. \tag{80}$$

Applying the second relation in (80) repeatedly and using $\theta(0) = 0$ (since $u_{0h} = R_h u_0$) as well as (78) we arrive at

$$\|\theta^n\|_{L_2(\Omega)} \le \|\theta^0\|_{L_2(\Omega)} + k\sum_{j=1}^{n} \|\omega^j\|_{L_2(\Omega)} \le k\sum_{j=1}^{n} \|\omega_1^j\|_{L_2(\Omega)} + k\sum_{j=1}^{n} \|\omega_2^j\|_{L_2(\Omega)} =: S_1 + S_2. \tag{81}$$

In order to bound S_1, we use the definition of ω_1^j (cf. (78)) leading to

$$\omega_1^j = (R_h - I)k^{-1} \int_{t_{j-1}}^{t_j} u_t \, d\tau = k^{-1} \int_{t_{j-1}}^{t_j} (R_h - I) u_t \, d\tau, \tag{82}$$

and by means of Lemma 3 (applied to u_t) we obtain

$$k\sum_{j=1}^{n} \|\omega_1^j\|_{L_2(\Omega)} \le \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} Ch^2 \|u_t\|_{H^2(\Omega)} \, d\tau = Ch^2 \int_0^{t_n} \|u_t\|_{H^2(\Omega)} \, d\tau. \tag{83}$$

To find an estimate of the term S_2 from (81) we use

$$k\omega_2^j = u(t_j) - u(t_{j-1}) - ku_t(t_j) = -\int_{t_{j-1}}^{t_j} (\tau - t_{j-1}) u_{tt}(\tau) \, d\tau,$$

which implies

$$k\sum_{j=1}^{n} \|\omega_2^j\|_{L_2(\Omega)} \le \sum_{j=1}^{n} \left\| \int_{t_{j-1}}^{t_j} (\tau - t_{j-1}) u_{tt}(\tau) d\tau \right\|_{L_2(\Omega)} \le k \int_0^{t_n} \|u_{tt}\|_{L_2(\Omega)} d\tau. \tag{84}$$

Relations (81)–(84), together with inequality (52), yield the estimate for θ^n. ■

The next theorem contains the error estimate in the norm $\|\cdot\|_{1,h}$.

Theorem 5 *Under the assumptions of Theorem 4, the following error estimate holds for the solution U^n of (75) and the solution $u(t_n)$ of (1), $0 \le n \le N_T$,*

$$\|U^n - u(t_n)\|_{1,h} \le C(h+k)\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \Big(\int_0^{t_n} \|f_t\|^2_{L_2(\Omega)} d\tau\Big)^{1/2} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\}, \quad (85)$$

with $\varepsilon \in (0, \frac{1}{2})$ and $C = C(\varepsilon, T)$.

Proof: According to (77), we have to find estimates for θ^n and ρ^n. By means of (23) and (52) we obtain

$$\|\rho^n\|_{1,h} \le Ch\Big\{\|u_0\|_{H^2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}. \quad (86)$$

Further, relation (79), now with the special choice $v_h := \overline{\partial}\theta^n$, together with the linearity of $\mathcal{B}_h(\cdot,\cdot)$ and the Cauchy-Schwarz inequality, leads to $\overline{\partial}\mathcal{B}_h(\theta^n, \theta^n) \le \|\omega^n\|^2_{L_2(\Omega)}$. A repeated application of this estimate, taking into account $\theta(0) = 0$ and the ellipticity of $\mathcal{B}_h(\cdot,\cdot)$ (see Lemma 1), yields

$$\|\theta^n\|^2_{1,h} \le C\Big\{k\sum_{j=1}^{n} \|\omega_1^j\|^2_{L_2(\Omega)} + k\sum_{j=1}^{n} \|\omega_2^j\|^2_{L_2(\Omega)}\Big\}. \quad (87)$$

Then, using relations (83) and (84) we are led to

$$k\sum_{j=1}^{n} \|\omega_1^j\|^2_{L_2(\Omega)} + k\sum_{j=1}^{n} \|\omega_2^j\|^2_{L_2(\Omega)} \le Ch^4 \int_0^{t_n} \|u_t\|^2_{H^2(\Omega)}\, d\tau + Ck^2 \int_0^{t_n} \|u_{tt}\|^2_{L_2(\Omega)} d\tau. \quad (88)$$

For the integrand in the first term on the right-hand side of (88) holds (cf. [29, Proof of Lemma 19.1]): $\|u_t\|^2_{H^2(\Omega)} \le C(\|u_{tt}\|^2_{L_2(\Omega)} + \|f_t\|^2_{L_2(\Omega)})$. Therefore it remains to find a bound for the second integral on the right-hand side of (88). Here we make use of [29, Lemma 19.6] which yields, together with inequality (87),

$$\|\theta^n\|_{1,h} \le C(h^2 + k)\Big\{\|g\|_{H_0^1(\Omega)} + \Big(\int_0^{t_n} \|f_t\|^2_{L_2(\Omega)} d\tau\Big)^{1/2}\Big\}. \quad (89)$$

The assertion of Theorem 5 is a consequence of (77), (86), and (89). ■

Now we turn to the case of a non-convex domain, i.e. consider solutions with singularities. For this case, the convergence of the semi-discretization has been analyzed in Section 4. We start with the error estimate for the fully discrete method in the L_2-norm.

Theorem 6 *Let Ω be a non-convex domain, with the interior angle β at the reentrant corner, and let the assumptions of Theorem 2 be satisfied. Then, for the solutions U^n and $u(t_n)$ from (75) and (1), the error estimate*

$$\|U^n - u(t_n)\|_{L_2(\Omega)} \le C(\kappa^2(h,\mu) + k)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\} \quad (90)$$

holds with $\varepsilon \in (0, \frac{1}{2})$, $C = C(\varepsilon, T)$, and $\kappa(h, \mu)$ from (60) (with $\lambda := \frac{\pi}{\beta}$).

Proof: In view of (66)-(69), we can state that the term $\rho^n = \rho(t_n)$ from (77) satisfies the following inequality,

$$\|\rho^n\|_{L_2(\Omega)} \leq C\kappa^2(h, \mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\}. \tag{91}$$

The norm of θ^n can be estimated by means of (81), with ω^j defined by analogy to (78). For the term S_1 on the right-hand side of (81) we obtain by using (82) and (68),

$$S_1 = k\sum_{j=1}^{n} \|\omega_1^j\|_{L_2(\Omega)} \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} C\kappa^2(h, \mu)\|u_t\|_{\dot{H}^2(\Omega)} d\tau = C\kappa^2(h, \mu) \int_0^{t_n} \|u_t\|_{\dot{H}^2(\Omega)} d\tau, \tag{92}$$

whereas inequality (84) remains valid for the term $S_2 = k\sum_{j=1}^{n} \|\omega_2^j\|_{L_2(\Omega)}$. Summing up these estimates for S_1 and S_2 and applying (69) yields the desired bound for θ^n. ■

Finally, the error of the fully discrete method is to be estimated in the norm $\|\cdot\|_{1,h}$.

Theorem 7 *Under the assumptions of Theorem 6, the solutions U^n and $u(t_n)$ from (75) and (1) satisfy the estimate*

$$\begin{aligned} &\|U^n - u(t_n)\|_{1,h} \\ &\quad \leq C(\kappa(h, \mu) + k)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H_0^1(\Omega)} + \Big(\int_0^{t_n} \|f_t\|^2_{L_2(\Omega)} d\tau\Big)^{1/2} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)} d\tau\Big\}, \end{aligned} \tag{93}$$

with $\varepsilon \in (0, \frac{1}{2})$, $C = C(\varepsilon, T)$, and $\kappa(h, \mu)$ from (60) (with $\lambda := \frac{\pi}{\beta}$).

Proof: The term ρ^n from (77) can be bounded by means of relations (58), (64), and (69) as follows,

$$\|\rho^n\|_{1,h} \leq C\kappa(h, \mu)\Big\{\|\Delta u_0\|_{L_2(\Omega)} + \|g\|_{H^\varepsilon(\Omega)} + \int_0^{t_n} \|f_t\|_{H^\varepsilon(\Omega)}\, d\tau\Big\}. \tag{94}$$

In order to estimate θ^n, we start from inequality (87). Then, by means of (82), (58), (84), and [29, Lemma 19.6] we obtain

$$\begin{aligned} k\sum_{j=1}^{n} \|\omega_1^j\|^2_{L_2(\Omega)} + k\sum_{j=1}^{n} \|\omega_2^j\|^2_{L_2(\Omega)} &\leq C\kappa^4(h, \mu) \int_0^{t_n} \|\Delta u_t\|^2_{L_2(\Omega)}\, d\tau + Ck^2 \int_0^{t_n} \|u_{tt}\|^2_{L_2(\Omega)} d\tau \\ &\leq C(\kappa^4(h, \mu) + k^2)\Big\{\|g\|^2_{H_0^1(\Omega)} + \int_0^{t_n} \|f_t\|^2_{L_2(\Omega)} d\tau\Big\}. \end{aligned}$$

This, together with (94), yields the assertion of Theorem 7. ■

Theorems 6 and 7 imply that for solutions with singularities, the presented method has the same convergence order as in case of regular solutions if an appropriate mesh grading parameter is chosen.

6 Numerical results

For verifying the convergence rate of the investigated approach, some numerical experiments were carried out. Some hints concerning the implementation can be found in [6]. First we observe the convergence in case of a convex domain, i.e. for regular solutions. We consider some initial-boundary value problem of type (1) in the domain $\Omega = (-1, 1) \times (0, 1)$. Let the two subdomains Ω_i, $i = 1, 2$, be given by $\Omega_1 = (-1, 0) \times (0, 1)$ and $\Omega_2 = (0, 1) \times (0, 1)$, cf. Figure 2. We take $t \in [0, 1]$ for the time interval. The right-hand side f and the initial function u_0 are chosen such that the solution of (1) is

$$u(x, t) = (1 - x_1^2)(1 - x_2)x_2(1 + t)^{-0.8}.$$

The initial mesh shown in Figure 2 is used for the semidiscretization in space. This mesh is refined globally by dividing each triangle into four equal triangles such that the mesh parameters form a sequence $\{h_1, h_2, h_3, h_4\}$ given by $h_{i+1} = 0.5\, h_i$. The mortar parameters (cf. Section 2) are chosen as follows: $\alpha_1 = \alpha_2 = 0.5$ and $\gamma = 6$. The triangulation $\mathcal{E}_h$ of the mortar interface Γ is defined as

$$\mathcal{E}_h := \{E : E = \partial T_1 \cap \partial T_2, \text{if } E \text{ is a segment; } T_i \in \mathcal{T}_h^i \text{ with } T_i \cap \Gamma \neq \emptyset \text{ for } i = 1, 2\}, \quad (95)$$

i.e. the nodes of both triangulations $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ lying on Γ establish the endpoints of the intervals $E \in \mathcal{E}_h$. For the discretization in time we employ three levels k_i, $i = 1, 2, 3$, where $k_1 = \frac{1}{10}$ and $k_{i+1} = 0.5k_i$.

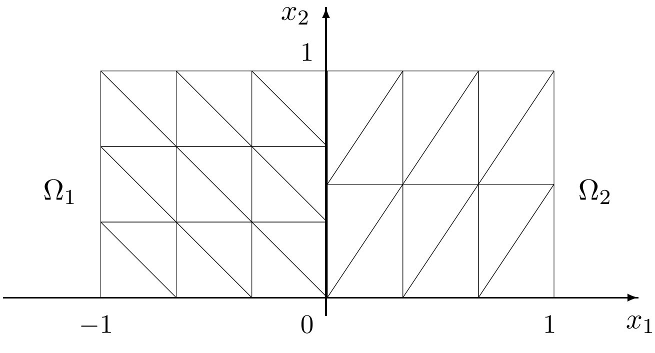

Figure 2: Initial triangulation (first example)

For the approximate measuring of the convergence rates stated in Theorems 4 and 5, the hypothesis for the tests is that

$$\|U^n - u(t_n)\|_{L_2(\Omega)} \approx C_1^{(0)} h^{\sigma_0} + C_2^{(0)} k^{\tau_0}, \quad \|U^n - u(t_n)\|_{1,h} \approx C_1^{(1)} h^{\sigma_1} + C_2^{(1)} k^{\tau_1}, \qquad (96)$$

with U^n defined in Section 5 ($n = 0, 1, \ldots, N_T$). The parameters $C_1^{(i)}$ and $C_2^{(i)}$ ($i = 0, 1$) are assumed to be approximately constant for three consecutive levels of h and k. Then, the exponent σ_0 in (96) can be approximately computed by means of relation

$$\sigma_0 = \log_2 \frac{e_0(h_i, k_j) - e_0(h_{i+1}, k_j)}{e_0(h_{i+1}, k_j) - e_0(h_{i+2}, k_j)}, \tag{97}$$

where $e_0(h_i, k_j)$ denotes the L_2-norm of the approximation error for discretization parameters h_i for the Nitsche finite element method and k_j for the backward Euler method, i.e. three consecutive levels of the mesh parameter h and a fixed time step k are used. An analogous relation holds for the convergence rate σ_1 if in (97) the errors $e_0(\cdot, \cdot)$ are replaced by $e_{1,h}(\cdot, \cdot)$ denoting the approximation errors in the $\{1, h\}$-norm. Further, the exponent τ_0 in (96) is approximately calculated by using the relation

$$\tau_0 = \log_2 \frac{e_0(h_i, k_j) - e_0(h_i, k_{j+1})}{e_0(h_i, k_{j+1}) - e_0(h_i, k_{j+2})}, \tag{98}$$

with $e_0(\cdot, \cdot)$ as in (97), i.e. three consecutive levels of the time step k and a fixed mesh parameter h are included. An analogous relation holds with τ_1 instead of τ_0 and $e_{1,h}(\cdot, \cdot)$ instead of $e_0(\cdot, \cdot)$.

Table 1 shows the approximation errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 1, \ldots, 4$ and $j = 3$ at the level $t_n = T$ as well as the observed convergence rates σ_0^{obs} and σ_1^{obs} which are obtained by using formula (97) and its analogue for σ_1, with $i = 2$ and $j = 3$. According to Theorems 4 and 5, the theoretically expected convergence rates are $\sigma_0 = 2$ and $\sigma_1 = 1$, and the observed rates σ_i^{obs} from Table 1 are approximately equal to σ_i, $i = 1, 2$. In Table 2 we represent the approximation errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 4$ and $j = 1, 2, 3$ at the level $t_n = T$ as well as the observed convergence rates τ_0^{obs} and τ_1^{obs} obtained by formula (98) and its analogue for τ_1, with $i = 4$ and $j = 1$. The observed values confirm approximately the expected convergence rates $\tau_0 = \tau_1 = 1$, cf. Theorems 4 and 5.

level	$e_0(h_i, k_3)$	$e_{1,h}(h_i, k_3)$
h_1	3.13072e-2	1.88140e-1
h_2	1.01964e-2	9.59374e-2
h_3	5.07436e-3	4.92381e-2
h_4	3.88784e-3	2.68348e-2
	$\sigma_0^{\text{obs}} = 2.11$	$\sigma_1^{\text{obs}} = 1.06$

Table 1: Error norms and convergence rates for $h = h_1, \ldots, h_4$ and $k = k_3$

In order to investigate the convergence rate in presence of singularities arising in case of non-convex domains, we consider the initial-boundary value problem (1) in the L-shaped domain $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1) \times (-1, 0]$. We choose $\Omega_1 = (-1, 0) \times (-1, 1)$ and $\Omega_2 = (0, 1) \times (0, 1)$ as subdomains. The time interval is again $[0, 1]$. Let the right-hand side f and the initial function u_0 be given such that the solution of problem (1) is

$$u(x, t) = (1 - x_1^2)(1 - x_2^2)\, r^\lambda \sin(\lambda\theta)(1 + \frac{1}{2}\,\mathrm{e}^{-t}), \ \ \lambda = \frac{2}{3},$$

level	$e_0(h_4, k_j)$	$e_{1,h}(h_4, k_j)$
k_1	3.92313e-3	2.68924e-2
k_2	3.89884e-3	2.68527e-2
k_3	3.88783e-3	2.68348e-2
	$\tau_0^{\text{obs}} = 1.14$	$\tau_1^{\text{obs}} = 1.15$

Table 2: Error norms and convergence rates for $h = h_4$ and $k = k_1, k_2, k_3$

where (r, θ) are polar coordinates centered at $(0, 0)$, cf. Section 4.

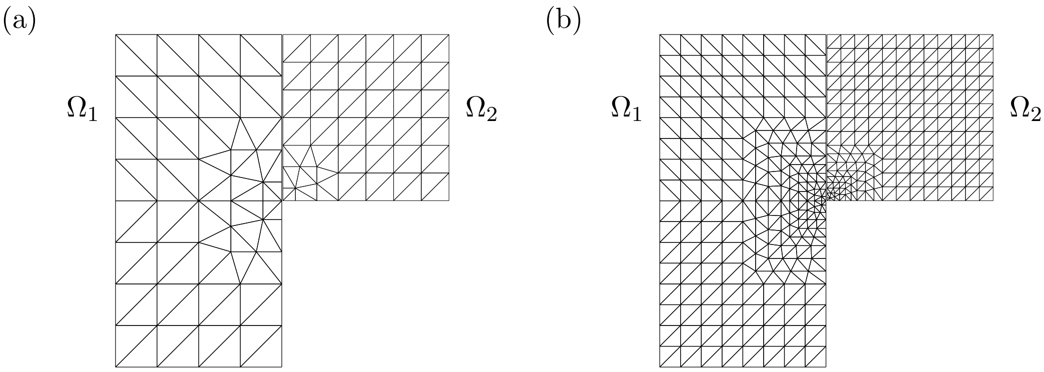

Figure 3: Triangulations on the levels $h = h_1$ and $h = h_2$ (second example)

For the Nitsche finite element discretization, the initial mesh (with mesh parameter h_1) depicted in Figure 3(a) is employed. Near the reentrant corner, this mesh is provided with local grading as defined in Section 4, the grading parameter is $\mu = 0.7\lambda \approx 0.47$. The subsequent meshes (with mesh parameters h_i, $i = 2, 3, 4$, $h_{i+1} = 0.5h_i$) arise by dividing each triangle into four equal triangles in the quasi-uniform part of the mesh and by local grading with $\mu = 0.7\lambda \approx 0.47$ near the reentrant corner, see Figure 3(b) for the mesh with $h = h_2$. The mortar parameters are the same as in the first example, the triangulation $\mathcal{E}_h$ is defined by (95). For the discretization in time we take the three levels k_1, k_2, and k_3 with $k_1 = \frac{1}{20}$, $k_{i+1} = 0.5k_i$. For the computation of approximate convergence rates we use again (97), (98), and analogous relations for σ_1, τ_1.
Table 3 contains the approximation errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 1, \ldots, 4$ and $j = 3$ at the level $t_n = T$ as well as the observed convergence rates σ_0^{obs} and σ_1^{obs}.
The observed values σ_0^{obs}, σ_1^{obs} are approximately equal to the expected convergence rates $\sigma_0 = 2$ and $\sigma_1 = 1$ stated in Theorems 6 and 7. The errors $e_0(h_i, k_j)$ and $e_{1,h}(h_i, k_j)$ for $i = 4$ and $j = 1, 2, 3$ at the level $t_n = T$ are reported in Table 4. The convergence rates resulting from these errors are very close to the expected values $\tau_0 = \tau_1 = 1$.
The numerical examples presented in this paper illustrate that Nitsche mortaring combined with the backward Euler method is a suitable approach for the numerical treatment of initial-boundary value problems for the heat equation in polygonally bounded domains. In particular, for solutions with singularities, the use of meshes with a grading parameter $\mu < \lambda$ leads to the same convergence rates as for regular solutions.

level	$e_0(h_i, k_3)$	$e_{1,h}(h_i, k_3)$
h_1	3.83180e-2	3.82840e-1
h_2	1.46901e-2	1.90226e-1
h_3	9.48033e-3	9.84876e-2
h_4	8.23570e-3	5.32338e-2
	$\sigma_0^{\text{obs}} = 2.07$	$\sigma_1^{\text{obs}} = 1.02$

Table 3: Error norms and convergence rates for $h = h_1, \ldots, h_4$ and $k = k_3$

level	$e_0(h_4, k_j)$	$e_{1,h}(h_4, k_j)$
k_1	8.26364e-3	5.32740e-2
k_2	8.24468e-3	5.32467e-2
k_3	8.23570e-3	5.32338e-2
	$\tau_0^{\text{obs}} = 1.08$	$\tau_1^{\text{obs}} = 1.08$

Table 4: Error norms and convergence rates for $h = h_4$ and $k = k_1, k_2, k_3$

References

[1] D.N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[2] I. Babuška, R.B. Kellogg, J. Pitkäranta. Direct and Inverse Error Estimates for Finite Elements with Mesh Refinements, *Numer. Math.*, 33:447–471, 1979.

[3] I. Babuška, M.B. Rosenzweig. A Finite Element Scheme for Domains with Corners. *Numer. Math.*, 20:1–21, 1972.

[4] R. Becker, P. Hansbo, R. Stenberg. A finite element method for domain decomposition with non-matching grids. *M2AN Math. Model. Numer. Anal.*, 37(2):209–225, 2003.

[5] F. Ben Belgacem. The Mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84:173–197, 1999.

[6] O. Benedix. *Finite-elemente-mortaring nach Nitsche für parabolische Anfangsrandwert-aufgaben.* Diplomarbeit, Technische Universität Chemnitz, 2006.

[7] H. Blum. Numerical treatment of corner and crack singularities. *Finite Element and Boundary Element Techniques from Mathematical and Engineering Point of View* (E. Stein & W. Wendland, eds.), pp. 172–212, Wien-New York: Springer Verlag, 1988.

[8] D. Braess, W. Dahmen, C. Wieners. A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37:48–69, 1999.

[9] P. Chatzipantelidis, R.D. Lazarov, V. Thomée, L.B. Wahlbin. Parabolic finite element equations in nonconvex polygonal domains. *BIT*, 46, Suppl., S113–S143, 2006.

[10] P. G. Ciarlet. *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[11] S. Falletta, B.P. Lamichhane. Mortar elements for a heat transfer problem on sliding meshes. *Calcolo*, submitted.

[12] A. Fritz, S. Hüeber, B. Wohlmuth. A comparison of Mortar and Nitsche Techniques for linear elasticity. *Calcolo*, 41:115–137, 2004.

[13] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston-London-Melbourne, 1985.

[14] P. Grisvard. *Singularities in boundary value problems.* Research Notes in Applied Mathematics 22, Masson, Paris, 1992.

[15] C. Großmann, H.-G. Roos. *Numerische Behandlung partieller Differentialgleichungen.* B. G. Teubner Verlag/GWV Fachverlage GmbH: Wiesbaden, 2005.

[16] P. Hansbo. Nitsche's method for interface problems in computational mechanics. *GAMM-Mitt.*, 28(2):183–206, 2005.

[17] B. Heinrich, K. Pietsch. Nitsche Type Mortaring for some Elliptic Problem with Corner Singularities. *Computing*, 68:217–238, 2002.

[18] B. Heinrich, S. Nicaise. The Nitsche mortar finite-element method for transmission problems with Singularities. *IMA J. Numer. Anal.*, 23:331–358, 2003.

[19] B. Heinrich, K. Pönitz. Nitsche type mortaring for singularly perturbed reaction-diffusion problems. *Computing*, 75(4):257–279, 2005.

[20] B. Heinrich, B. Jung. The Fourier finite-element method with Nitsche mortaring. *IMA J. Numer. Anal.*, 26(3):541–562, 2006.

[21] B. Heinrich, B. Jung. The Fourier-Nitsche-mortaring for elliptic problems with reentrant edges. *Computing*, 80(3):221–246, 2007.

[22] C. Johnson. *Numerical solution of partial differential equations by the finite element method.* Cambridge Univ. Press: Cambridge, 1990.

[23] S. Larsson, V. Thomée. *Partielle Differentialgleichungen und numerische Methoden.* Springer: Berlin Heidelberg, 2005.

[24] L. A. Oganesjan und L. A. Ruchovec. *Variacionno–raznostnye metody rešenija elliptičeskich uravnenij.* Izd. Akad. Nauk Armjanskoj SSR: Erevan, 1979.

[25] K. Pönitz. Finite-Elemente-Mortaring nach einer Methode von J. A. Nitsche für elliptische Randwertaufgaben. PhD thesis, Technische Universität Chemnitz, 2006.

[26] G. Raugel. Résolution numérique par une méthode d'éléments finis du probléme Dirichlet pour le Laplacien dans un polygone. *C. R. Acad. Sci. Paris, Sér. A*, 286(18):A791–A794, 1978.

[27] R. Stenberg. Mortaring by a method of J. A. Nitsche. In: *Computational mechanics. New trends and applications.*, Centro Internac. Métodos Numér. Ing., Barcelona, 1998.

[28] V. Thomée. Finite difference methods for linear parabolic equations. *Handbook of numerical analysis. Vol. I* (P. G. Ciarlet & J. L. Lions, eds.), pp. 5–196, North-Holland, Amsterdam, 2003.

[29] V. Thomée. *Galerkin Finite Element Methods for Parabolic Problems.* Springer Series in Computational Mathematics. Springer: Berlin Heidelberg, 2006 (Second Edition).

[30] T. Warburton, J.S. Hesthaven. On the constants in *hp*-finite element trace inverse inequalities. *Comput. Methods Appl. Mech. Engrg.*, 192:2765–2773, 2003.

[31] M.F. Wheeler. A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal.*, 10:723–759, 1973.

[32] B. I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition.* Lecture Notes in Computational Science and Engineering 17, Springer-Verlag, Berlin Heidelberg, 2001.