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ε - optimality conditions for composed convex optimization problems

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Abstract. The aim of the present paper is to provide a formula for the ε - subdifferential of $f + g \circ h$ different from the ones which can be found in the existent literature. Further we equivalently characterize this formula by using of a so-called closedness type regularity condition expressed by means of the epigraphs of the conjugates of the functions involved. Even more, using the ε - subdifferential formula we are able to derive necessary and sufficient conditions for the ε - optimal solutions of composed convex optimization problems.

Key Words. conjugate functions, composed convex functions, ε - subdifferential, ε - optimality conditions

1 Introduction

In many practical applications it is necessary to solve an optimization problem, i.e. to find a point where the minimal or the maximal value a function can take is attained. Unfortunately this is not always possible, because an optimization problem does not necessarily have an optimal solution (such a situation can occur even if its optimal objective value can be determined). Thus we are forced sometimes to deal not with optimal solutions, but with

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approximate ones. Nevertheless, this is not a major drawback if the approximate solutions we can provide act well to our purposes. Even more, from a computational point of view it is much more advantageous to find approximate solutions as the goal of an algorithm is to deliver an approximate solution and not an optimal one (supposing that there exist an optimal solution, it is seldom possible to find it, but even in such situations this usually means a waste of time and resources). Therefore the study of the approximate solutions of an optimization problem is of great interest from many points of view and many authors have turned their attention to this topic.

It is well-known that a function reaches its minimal value at $\overline{x}$ if and only if $0 \in \partial f(\overline{x})$. Using this property one can easily characterize the optimal solutions of an optimization problem by means of the subdifferential. A similar property holds also for the approximate solutions of an optimization problem, which can be characterized by means of the ε - subdifferentials. From the large amount of works which deal with such a topic we mention here only some of them, namely [9, 11, 12, 15].

Many optimization problems generated by practical fields like location and transports or economics and finance involve composed convex functions. Therefore, in order to be able to characterize the approximate solutions of an optimization problem involving composed convex functions, it is important to provide a formula for the ε - subdifferential of a composed convex function (the interested reader can consult the papers [1, 3, 4, 6–8, 13] for more information regarding the optimization problems involving composed convex functions).

The paper is organized as follows. In the second section we present some notions and results used later. The third section contains the main results of the paper. We provide a formula for the ε - subdifferential of a composed convex function of type $f + g \circ h$. The formula we give is a refinement of the one provided in [16] and, moreover, for $\varepsilon = 0$ we rediscover the subdifferential formula given in [1] and [6]. We prove that the formula we give holds if and only if a closedness type regularity condition expressed using only epigraphs of the conjugates of the functions involved is fulfilled. Further we consider an optimization problem the objective function of which is of type $f + g \circ h$. Using the connection between the ε - subdifferential of a convex function and its conjugate function we are able to point out necessary and sufficient optimality conditions for the ε - optimal solutions of the problem. In the fourth section of the paper special instances of the functions f, g and h are considered and some special cases of our general results are provided.

2 Preliminary notions and results

Let X and Y be two separated locally convex spaces and let X^* and Y^* be their topological dual spaces endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Throughout the entire paper we denote by $\langle x^*, x\rangle = x^*(x)$ the value of the continuous linear functional $x^* \in X^*$ at $x \in X$. For any $K \subseteq Y$ nonempty and closed convex cone we define its *dual cone* as $K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in K\}$. The cone K induces on Y a partial order "$\leq_K$" defined for $x, y \in Y$ by

$$x \leq_K y \Leftrightarrow y - x \in K.$$

To Y we attach an element $\infty_Y \notin Y$ which is the greatest element with respect to $\leq_K$ and we denote $Y^\bullet = Y \cup \{\infty_Y\}$. Then for all $y \in Y^\bullet$ it holds $y \leq_K \infty_Y$. Moreover, the addition and the multiplication with a positive real number can be naturally extended to $Y^\bullet$ by taking

$$y + \infty_Y = \infty_Y + y = \infty_Y \quad \text{and} \quad t\infty_Y = \infty_Y$$

for all $y \in Y$ and $t \geq 0$.

For a given function $f : X \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *effective domain* and by $\mathrm{epi}(f) = \{(x, r) : x \in X, r \in \mathbb{R}, f(x) \leq r\}$ its *epigraph*, respectively. The function f is called *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in X$. The next notion we introduce is that of a conjugate function.

Definition 1 *For $f : X \to \overline{\mathbb{R}}$ an arbitrary function and $C \subseteq X$ a non - empty set, by the conjugate function of f regarding the set C we understand the function*

$$f_C^* : X^* \to \overline{\mathbb{R}}, \quad f_C^*(x^*) = \sup_{x \in C} \Big\{ \langle x^*, x\rangle - f(x) \Big\}.$$

The previous definition obviously implies

$$f_C^*(x^*) + f(x) \geq \langle x^*, x\rangle, \ \forall x \in C, \ \forall x^* \in X^*. \tag{1}$$

When $C = X$ the conjugate regarding the set X turns out to be the classical (Legendre - Fenchel) *conjugate function* of f, denoted by f^*, and the inequality (1) becomes the well-known *Fenchel - Young inequality*

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle, \forall x \in X, \forall x^* \in X^*. \tag{2}$$

For a set $C \subseteq X$ we also consider its *indicator* function $C \subseteq X$

$$\delta_C : X \to \overline{\mathbb{R}}, \quad \delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

and its *support* function

$$\sigma_C : X^* \to \overline{\mathbb{R}}, \quad \sigma_C(x^*) = \sup_{x \in C} \langle x^*, x\rangle.$$

It is not hard to see that

$$f_C^* = (f + \delta_C)^* \quad \text{and} \quad \sigma_C = \delta_C^*. \tag{3}$$

If f is a proper function the ε - *subdifferential* of f at $\overline{x} \in \text{dom}(f)$ is the set

$$\partial_\varepsilon f(\overline{x}) = \left\{ x^* \in X^* : f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \varepsilon, \forall x \in X \right\},$$

where $\varepsilon \geq 0$ is a non - negative real number. For $\varepsilon = 0$ we denote $\partial f(\overline{x}) = \partial_0 f(\overline{x})$ and we say that the function f is *subdifferentiable* at $\overline{x} \in \text{dom}(f)$ if $\partial f(\overline{x}) \neq \emptyset$. It can be easily proved (see, for instance, [16]) that for all $\overline{x} \in \text{dom}(f)$ and $x^* \in X^*$ we have

$$f(\overline{x}) + f^*(x^*) \leq \langle x^*, \overline{x}\rangle + \varepsilon \Leftrightarrow x^* \in \partial_\varepsilon f(\overline{x}). \tag{4}$$

Definition 2 *A function $g : Y \to \overline{\mathbb{R}}$ is called K - increasing if for all $x, y \in Y$ fulfilling $x \leq_K y$ the inequality $g(x) \leq g(y)$ holds.*

Definition 3 *The function $h : X \to Y^\bullet$ is called K - convex if for all $x, y \in Y$ and $\lambda \in [0, 1]$ it fulfills the property*

$$h(\lambda x + (1 - \lambda)y) \leq_K \lambda h(x) + (1 - \lambda)h(y).$$

Without fear of confusion we say that the function $h : X \to Y^\bullet$ is *proper* if its *effective domain* $\mathrm{dom}(h) = \{x \in X : h(x) \in Y\}$ is a non - empty set. Moreover, for all $\lambda \in K^*$ the function

$$(\lambda h) : X \to \overline{\mathbb{R}}, \quad (\lambda h)(x) = \begin{cases} \langle \lambda, h(x) \rangle, & x \in \mathrm{dom}(h), \\ +\infty, & \text{otherwise}, \end{cases}$$

is well-defined.

Definition 4 *A function $h : X \to Y^\bullet$ is called star K - lower semicontinuous if the function (λh) is lower semicontinuous for all $\lambda \in K^*$.*

During the last decades various generalizations of the notion of lower semicontinuity have been given (we mention here only two papers, namely [6] and [14], where the authors introduces the K - lower semicontinuous and K - sequentially lower semicontinuous functions). Nevertheless, we prefer to use the star K - lower semicontinuity since it is general enough (one can prove that a K - lower semicontinuous function is a star K - lower semicontinuous function, but the reverse implications fails in general), and, moreover, it is more adequate to our aim.

If $A : X \to Y$ is a linear continuous operator, then $A^* : Y^* \to X^*$ defined such that

$$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle \ \forall x \in X, \ \forall y^* \in Y^*$$

is called its adjoint. We consider the identity function over the space $\mathbb{R}$ defined as follows $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $\mathrm{id}_{\mathbb{R}}(r) = r$ for all $r \in \mathbb{R}$. We define also the function $A^* \times \mathrm{id}_{\mathbb{R}} : Y^* \times \mathbb{R} \to X^* \times \mathbb{R}$, $A^* \times \mathrm{id}_{\mathbb{R}}(y^*, r) = (A^* y^*, r)$ for all $(y^*, r) \in Y^* \times \mathbb{R}$.

We also mention that everywhere in the present paper we write min (max) instead of inf (sup) when the infimum (supremum) is attained.

3 The general case

The functions $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ and $h : X \to Y^\bullet$ are taken such that f is proper, convex and lower semicontinuous, g is proper, convex, lower semicontinuous and K - increasing, while h is proper, K - convex and star K - lower semicontinuous, respectively. The function g will be extended to $Y^\bullet$ by taking $g(\infty_Y) = +\infty$. Moreover, throughout the entire section we assume that the condition $h(\mathrm{dom}(f) + K) \cap \mathrm{dom}(g) \neq \emptyset$ is fulfilled (one can easily prove that this condition secures the properness of the function $f + g \circ h$).

Consider an arbitrary $p^* \in X^*$. We know from the literature (see, for instance, [1]) that for all $\lambda \in K^*$ and for all $x^* \in X^*$ the inequality

$$(f + g \circ h)^*(p^*) \leq g^*(\lambda) + f^*(x^*) + (\lambda h)^*(p^* - x^*) \tag{5}$$

is always fulfilled. Under some circumstances (see also [1] for details) the existence of some $\lambda \in K^*$ and $x^* \in X^*$ such that equality holds in (5) is secured. A necessary and sufficient condition for this is given in the following theorem (see [2] for the proof).

Theorem 1 *The regularity condition*

$$(RC) \qquad \operatorname{epi}(f^*) + \bigcup_{\lambda \in K^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) \textit{ is closed}$$

is fulfilled if and only if

$$(f + g \circ h)^*(p^*) = \min_{\substack{\lambda \in K^*, \\ x^* \in X^*}} \Big\{ g^*(\lambda) + f^*(x^*) + (\lambda h)^*(p^* - x^*) \Big\}, \forall p^* \in X^*. \tag{6}$$

The next theorem gives a general formula for the ε - subdifferential of the function $f + g \circ h$, which holds in case (RC) is fulfilled. We would like to mention that the formula we give is a refinement of the one proved in [16]. Moreover, for $\varepsilon = 0$ we rediscover the subdifferential formula given in [1,2,6].

Theorem 2 *Suppose that the regularity condition (RC) is fulfilled. Then for all $x \in \operatorname{dom}(f + g \circ h)$ and for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon (f + g \circ h)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \Big\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \Big\}. \tag{7}$$

Proof. "$\subseteq$" Let $x \in \operatorname{dom}(f + g \circ h)$ and $\varepsilon \geq 0$ be arbitrary chosen. Take $x^* \in \partial_\varepsilon (f + g \circ h)(x)$. According to relation (4) it holds

$$(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon.$$

Taking into consideration Theorem 1, there exist $\lambda \in K^*$ and $x_1^*, x_2^* \in X^*$, $x_1^* + x_2^* = x^*$, such that

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \leq \langle x_1^* + x_2^*, x\rangle + \varepsilon.$$

The last inequality can be equivalently written as

$$\begin{aligned} &\left[f^*(x_1^*) + f(x) - \langle x_1^*, x\rangle\right] + \left[(\lambda h)^*(x_2^*) + (\lambda h)(x) - \langle x_2^*, x\rangle\right] \\ &\qquad + \left[g^*(\lambda) + g(h(x)) - \langle \lambda, h(x)\rangle\right] \leq \varepsilon. \end{aligned}$$

Further we take $\widetilde{\varepsilon_1} := f^*(x_1^*) + f(x) - \langle x_1^*, x\rangle$, $\overline{\varepsilon_2} := (\lambda h)^*(x_2^*) + (\lambda h)(x) - \langle x_2^*, x\rangle$, and $\overline{\varepsilon_3} := g^*(\lambda) + g(h(x)) - \langle \lambda, h(x)\rangle$. Relation (2) ensures that $\widetilde{\varepsilon_1}$, $\overline{\varepsilon_2}$ and $\overline{\varepsilon_3}$ are non - negative real numbers. Moreover, taking into consideration the previous inequality, one can easily see that $\widetilde{\varepsilon_1} + \overline{\varepsilon_2} + \overline{\varepsilon_3} \leq \varepsilon$. Then for $\overline{\varepsilon_1} := \varepsilon - \overline{\varepsilon_2} - \overline{\varepsilon_3} \geq \widetilde{\varepsilon_1}$ it holds $\overline{\varepsilon_1} + \overline{\varepsilon_2} + \overline{\varepsilon_3} = \varepsilon$ and, moreover, the inequalities $f^*(x_1^*) + f(x) \leq \langle x_1^*, x\rangle + \overline{\varepsilon_1}$, $(\lambda h)^*(x_2^*) + (\lambda h)(x) \leq \langle x_2^*, x\rangle + \overline{\varepsilon_2}$ and $g^*(\lambda) + g(h(x)) \leq \langle \lambda, h(x)\rangle + \overline{\varepsilon_3}$ hold, too. According to (4) we have $x_1^* \in \partial_{\overline{\varepsilon_1}} f(x)$, $x_2^* \in \partial_{\overline{\varepsilon_2}}(\lambda h)(x)$, and $\lambda \in \partial_{\overline{\varepsilon_3}} g(h(x))$. Thus

$$\begin{aligned} x^* &= x_1^* + x_2^* \in \partial_{\overline{\varepsilon_1}} f(x) + \partial_{\overline{\varepsilon_2}}(\lambda h)(x) \\ &\subseteq \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \right\} \end{aligned}$$

and the first part of the proof is finished.

"$\supseteq$" In order to prove the reverse inclusion let

$$x^* \in \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \left\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \right\}$$

be arbitrarily taken. Then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))$, $x_1^* \in \partial_{\varepsilon_1} f(x)$ and $x_2^* \in \partial_{\varepsilon_2}(\lambda h)(x)$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$ and $x^* = x_1^* + x_2^*$. This implies further $f^*(x_1^*) + f(x) \leq \langle x_1^*, x\rangle + \varepsilon_1$, $(\lambda h)^*(x_2^*) + (\lambda h)(x) \leq \langle x_2^*, x\rangle + \varepsilon_2$ and $g^*(\lambda) + g(h(x)) \leq \langle \lambda, h(x)\rangle + \varepsilon_3$. By summing up we acquire

$$\begin{aligned} & f^*(x_1^*) + f(x) + (\lambda h)^*(x_2^*) + (\lambda h)(x) + g^*(\lambda) + g(h(x)) \\ \leq\ & \langle x_1^*, x\rangle + \varepsilon_1 + \langle x_2^*, x\rangle + \varepsilon_2 + \langle \lambda, h(x)\rangle + \varepsilon_3 \\ =\ & \langle \lambda, h(x)\rangle + \langle x_1^* + x_2^*, x\rangle + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \langle \lambda, h(x)\rangle + \langle x^*, x\rangle + \varepsilon. \end{aligned}$$

Thus we get

$$g^*(\lambda) + f^*(x_1^*) + \big(\lambda h\big)^*(x_2^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon,$$

and using (5) follows

$$(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x \rangle + \varepsilon.$$

But this implies $x^* \in \partial_\varepsilon (f + g \circ h)(x)$ and the proof of the theorem is complete. □

In the following we prove that (RC) is not just a sufficient, but also a necessary condition for having (7) fulfilled for all $x \in \operatorname{dom}(f + g \circ h)$ and all $\varepsilon \geq 0$.

Theorem 3 *If relation* (7) *holds for all* $x \in \operatorname{dom}(f + g \circ h)$ *and all* $\varepsilon \geq 0$, *then the regularity condition* (RC) *is fulfilled.*

Proof. We actually prove that

$$\operatorname{epi}(f^*) + \bigcup_{\lambda \in K^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) = \operatorname{epi}((f + g \circ h)^*)$$

and, since the later set is a closed set, the conclusion follows automatically. To this aim take first an arbitrary

$$(x^*, r) \in \operatorname{epi}(f^*) + \bigcup_{\lambda \in K^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big).$$

Then there exist $\lambda \in K^*$ and the tuples $(x_1^*, r_1) \in \operatorname{epi}(f^*)$ and $(x_2^*, r_2) \in \operatorname{epi}((\lambda h)^*)$ such that

$$(x^*, r) = (x_1^*, r_1) + (x_2^*, r_2) + (0, g^*(\lambda)).$$

This equality implies $x^* = x_1^* + x_2^*$ and, taking into consideration the properties of the epigraph, we acquire $r = r_1 + r_2 + g^*(\lambda) \geq g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*)$. Since the inequality (5) is always satisfied, we get $(f + g \circ h)^*(x^*) \leq r$ and so $(x^*, r) \in \operatorname{epi}((f + g \circ h)^*)$. As no additional assumptions are imposed regarding the tuple (x^*, r) we conclude that

$$\operatorname{epi}(f^*) + \bigcup_{\lambda \in K^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) \subseteq \operatorname{epi}((f + g \circ h)^*). \tag{8}$$

Let us prove now that relation (7) secures the reverse inclusion in (8). Take an arbitrary tuple $(x^*, r) \in \operatorname{epi}((f + g \circ h)^*)$. Then $(f + g \circ h)^*(x^*) \leq r$ and for some arbitrary $x \in \operatorname{dom}((f + g \circ h))$ we get

$$(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x\rangle + [r + (f + g \circ h)(x) - \langle x^*, x\rangle].$$

Further we take

$$\varepsilon := r + (f + g \circ h)(x) - \langle x^*, x\rangle.$$

Using the Fenchel - Young inequality one can see that since $(f+g\circ h)^*(x^*) \leq r$ one has $\varepsilon \geq r - (f + g \circ h)^*(x^*) \geq 0$. Moreover, as $(f + g \circ h)^*(x^*) + (f + g \circ h)(x) \leq \langle x^*, x\rangle + \varepsilon$, relation (4) implies $x^* \in \partial_\varepsilon(f + g \circ h)(x)$. According to relation (7) there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x))$, $x_1^* \in \partial_{\varepsilon_1} f(x)$ and $x_2^* \in \partial_{\varepsilon_2}(\lambda h)(x)$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$ and $x_1^* + x_2^* = x^*$. Making use of relation (4) we get $f^*(x_1^*) + f(x) \leq \langle x_1^*, x\rangle + \varepsilon_1$, $(\lambda h)^*(x_2^*) + (\lambda h)(x) \leq \langle x_2^*, x\rangle + \varepsilon_2$ and $g^*(\lambda) + g(h(x)) \leq \langle \lambda, h(x)\rangle + \varepsilon_3$. By summing up the three inequalities given above we get

$$\begin{aligned} &f^*(x_1^*) + f(x) + (\lambda h)^*(x_2^*) + (\lambda h)(x) + g^*(\lambda) + g(h(x)) \\ &\leq \langle x_1^*, x\rangle + \varepsilon_1 + \langle x_2^*, x\rangle + \varepsilon_2 + \langle \lambda, h(x)\rangle + \varepsilon_3 = \langle x^*, x\rangle + \langle \lambda, h(x)\rangle + \varepsilon, \end{aligned}$$

which implies

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \leq \langle x^*, x\rangle + \varepsilon.$$

Taking into consideration the way the constant ε was defined we obtain

$$\begin{aligned} &g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + (f + g \circ h)(x) \\ &\leq \langle x^*, x\rangle + r + (f + g \circ h)(x) - \langle x^*, x\rangle = r + (f + g \circ h)(x) \end{aligned}$$

and from here the inequality

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) \leq r$$

can be easily deduced. Let $t \geq 0$ be such that

$$g^*(\lambda) + f^*(x_1^*) + (\lambda h)^*(x_2^*) + t = r.$$

Then

$$\begin{aligned} (x^*, r) &= (x_1^*, f^*(x_1^*) + t) + (x_2^*, (\lambda h)^*(x_2^*)) + (0, g^*(\lambda)) \\ &\in \operatorname{epi}(f^*) + \bigcup_{\lambda \in K^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda))\Big) \end{aligned}$$

and the proof is over. $\square$

Let us consider the optimization problem

$$(P) \qquad \inf_{x \in X} \left\{ f(x) + (g \circ h)(x) \right\}.$$

For ε $geq0$ we say that $\bar{x} \in X$ is an ε - optimal solution to (P) if one has that

$$f(\bar{x}) + (g \circ h)(\bar{x}) \leq \inf_{x \in X} \left\{ f(x) + (g \circ h)(x) \right\} + \varepsilon.$$

One can easily see that $\overline{x}$ is an ε - optimal solution to (P) if and only if $0 \in \partial_\varepsilon (f + g \circ h)(\overline{x})$. The results presented above allow us to provide necessary and sufficient optimality conditions for ε - optimal solutions of the problem (P). The next theorem is devoted to that matter.

Theorem 4 *(a) Suppose that the condition (RC) is fulfilled. If $\overline{x} \in X$ is an ε - optimal solution of the problem (P) for some $\varepsilon \geq 0$, then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in K^*$ and $\overline{x^*} \in X^*$ such that*

(i) $0 \leq g^*(\overline{\lambda}) + g(h(\overline{x})) - \langle \overline{\lambda}, h(\overline{x}) \rangle \leq \varepsilon_3$;

(ii) $0 \leq f^*(\overline{x^*}) + f(\overline{x}) - \langle \overline{x^*}, \overline{x} \rangle \leq \varepsilon_1$;

(iii) $0 \leq (\overline{\lambda} h)^*(-\overline{x^*}) + (\overline{\lambda} h)(\overline{x}) + \langle \overline{x^*}, \overline{x} \rangle \leq \varepsilon_2$;

(iv) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$.

(b) If there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in K^*$ and $\overline{x^*} \in X^*$ such that the relations $(i) - (iv)$ hold for some $\overline{x} \in X$, then $\overline{x}$ is an ε - optimal solution of the problem (P).

Proof. (a) As $\overline{x}$ is an ε - optimal solution of the problem (P) we know that $0 \in \partial_\varepsilon (f + g \circ h)(\overline{x})$. By relation (7) there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ and $\overline{\lambda} \in K^*$, such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$, $\overline{\lambda} \in K^* \cap \partial_{\varepsilon_3} g(h(\overline{x}))$ and $0 \in \partial_{\varepsilon_1} f(\overline{x}) + \partial_{\varepsilon_2} (\overline{\lambda} h)(\overline{x})$. As $\overline{\lambda} \in \partial_{\varepsilon_3} g(h(\overline{x}))$ the assertion (i) is a direct consequence of (4). Moreover, there exists some $\overline{x^*} \in X^*$ such that $\overline{x^*} \in \partial_{\varepsilon_1} f(\overline{x})$ and $-\overline{x^*} \in \partial_{\varepsilon_2} (\overline{\lambda} h)(\overline{x})$ and, using once more relation (4), the assertions (ii) and, respectively, (iii), can be easily deduced.

(*b*) By summing up the relations $(i) - (iii)$ and taking into consideration (iv) we acquire

$$\begin{aligned} &g^*(\overline{\lambda}) + g(h(\overline{x})) - \langle \overline{\lambda}, h(\overline{x}) \rangle + f^*(\overline{x^*}) + f(\overline{x}) - \langle \overline{x^*}, \overline{x} \rangle \\ &+ (\overline{\lambda} h)^*(-\overline{x^*}) + (\overline{\lambda} h)(\overline{x}) + \langle \overline{x^*}, \overline{x} \rangle \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon. \end{aligned}$$

By (5) we get

$$(f + g \circ h)^*(0) + (f + g \circ h)(\overline{x}) \leq \varepsilon$$

and this is nothing else that $0 \in \partial_\varepsilon (f + g \circ h)(\overline{x})$. Thus $\overline{x}$ is an ε - optimal solution of (P) and the proof is complete. □

Remark 1 (*a*) *To the optimization problem* (P) *one can introduce the following conjugate dual problem (see [1], [2])*

$$(D) \qquad \sup_{\lambda \in K^*, x^* \in X^*} \Big\{ - g^*(\lambda) - f^*(x^*) - (\lambda h)^*(-x^*) \Big\}.$$

By Theorem 1 (taking in (6) $p^* = 0$*) follows that* (RC) *is a sufficient condition for strong duality between* (P) *and* (D)*. In the hypotheses of Theorem 4, having that* $(i) - (iv)$ *are fulfilled, it holds*

$$\sup_{\lambda \in K^*, x^* \in X^*} \Big\{ -g^*(\lambda) - f^*(x^*) - (\lambda h)^*(-x^*) \Big\} - \varepsilon = \inf_{x \in X} \Big\{ f(x) + (g \circ h)(x) \Big\} - \varepsilon$$

$$= f(\bar{x}) + (g \circ h)(\bar{x}) - \varepsilon \leq -g^*(\bar{\lambda}) - f^*(\bar{x}^*) - (\bar{\lambda} h)^*(-\bar{x}^*).$$

This means that $(\bar{\lambda}, \bar{x}^*)$ *is an* ε*-optimal solution for the dual problem* (D)*.*

(*b*) *In case* $\varepsilon = 0$ *by means of* $(i) - (iii)$ *we rediscover the optimality conditions given in the past for characterizing the (exact) optimal solutions of the primal problem* (P) *and its dual problem* (D) *(see, for example, [3])*

(*i*) $g^*(\overline{\lambda}) + g(h(\overline{x})) - \langle \overline{\lambda}, h(\overline{x}) \rangle = 0$;

(*ii*) $f^*(\overline{x^*}) + f(\overline{x}) - \langle \overline{x^*}, \overline{x} \rangle = 0$;

(*iii*) $(\overline{\lambda} h)^*(-\overline{x^*}) + (\overline{\lambda} h)(\overline{x}) + \langle \overline{x^*}, \overline{x} \rangle = 0$.

Remark 2 *If the variable* x *does not cover the whole space* X*, but a non - empty closed convex subset* $C \subseteq X$*, then some similar results can be easily provided if we replace the function* h *with the function*

$$h_C : X \to Y^\bullet, h_C(x) = \begin{cases} h(x), & x \in \mathrm{dom}(h) \cap C, \\ +\infty_Y, & \textit{otherwise}. \end{cases}$$

Before going further we would like to mention that the regularity condition (RC) is weaker than the conditions imposed in [6–8] for composed convex optimization problems (see [1] for an elaborate discussion on this topic).

4 Special cases

4.1 Composition with a linear operator

Let us consider $A : X \to Y$ a linear continuous operator and take

$$h : X \to Y, \quad h(x) = Ax, \ \forall x \in X.$$

Taking $K = \{0\} \subset Y$ one has that h and g are K - convex and K - increasing, respectively. Moreover, one can easily prove that for all $\lambda \in Y^* = K^*$ we have

$$(\lambda h)^*(x^*) = \begin{cases} 0, & x^* = A^*\lambda, \\ +\infty, & \text{otherwise.} \end{cases}$$

For this choice formula (6) becomes

$$(f + g \circ A)^*(p^*) = \min_{\substack{\lambda \in K^*, \\ x^* \in X^*}} \left\{ g^*(\lambda) + f^*(x^*) + (\lambda h)^*(p^* - x^*) \right\}$$

$$= \min_{\substack{\lambda \in Y^*, \\ x^* \in X^*, \\ p^* - x^* = A^*\lambda}} \left\{ g^*(\lambda) + f^*(x^*) \right\} = \min_{\lambda \in Y^*} \left\{ g^*(\lambda) + f^*(p^* - A^*\lambda) \right\}, \forall p^* \in X^*. \quad (9)$$

Moreover, using only the definition of the epigraph and the special form of the function $(\lambda h)^*$ the equality

$$\bigcup_{\lambda \in Y^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) = A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*))$$

can be easily proved (see also [1, 4]). Thus the regularity condition (RC) becomes in this special case

$$(RC^A) \qquad \operatorname{epi}(f^*) + A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(g^*)) \text{ is closed}$$

and according to Theorem 1 it is fulfilled if and only if relation (9) holds.

Now let $\varepsilon \geq 0$ be arbitrarily taken. According to relation (4) we have $x^* \in \partial_\varepsilon(\lambda h)(x)$ if and only if $(\lambda h)^*(x^*) + (\lambda h)(x) \leq \langle x^*, x\rangle + \varepsilon$. Because of the special form of the function $(\lambda h)^*$ it is binding to have $x^* = A^*\lambda$. Thus $x^* \in \partial_\varepsilon(\lambda h)(x)$ if and only if $x^* = A^*\lambda$ and $\langle \lambda, Ax\rangle \leq \langle A^*\lambda, x\rangle + \varepsilon$. As the last inequality is always fulfilled (see the definition of the adjoint operator), we get $\partial_\varepsilon(\lambda h)(x) = \{A^*\lambda\}$. Relation (7) becomes

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3 \geq 0,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon}} \left\{\partial_{\varepsilon_1} f(x) + A^*\lambda : \lambda \in \partial_{\varepsilon_3} g(Ax)\right\}$$

and taking into consideration Theorem 2 and Theorem 3 the following result can be easily proved.

Theorem 5 *The regularity condition* (RC^A) *is fulfilled if and only if for all* $x \in \operatorname{dom}(f + g \circ A)$ *and for all* $\varepsilon \geq 0$ *we have*

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1,\varepsilon_3 \geq 0,\\ \varepsilon_1+\varepsilon_3=\varepsilon}} \left\{\partial_{\varepsilon_1} f(x) + A^*\partial_{\varepsilon_3} g(Ax)\right\}. \tag{10}$$

Before going further we would like to mention that in this special case the formulae (9) and (10) coincide with the ones given in [16] (see also [4] for A the identity operator of the space X).

Further we consider the optimization problem

$$(P^A) \qquad \inf_{x \in X} (f + g \circ A)(x).$$

The following theorem, which provides necessary and sufficient optimality conditions for ε - optimal solutions of the problem (P^A), is a consequence of Theorem 4.

Theorem 6 (*a*) *Suppose that* (RC^A) *holds. If* $\overline{x} \in X$ *is an* ε *- optimal solution of the problem* (P^A) *for some* $\varepsilon \geq 0$, *then there exist* $\varepsilon_1, \varepsilon_3 \geq 0$ *and* $\overline{\lambda} \in Y^*$ *such that*

(i^A) $0 \leq g^*(\overline{\lambda}) + g(A\overline{x}) - \langle \overline{\lambda}, A\overline{x}\rangle \leq \varepsilon_3$;

(ii^A) $0 \leq f^*(-A^*\overline{\lambda}) + f(\overline{x}) + \langle A^*\overline{\lambda}, \overline{x}\rangle \leq \varepsilon_1$;

(iii^A) $\varepsilon_1 + \varepsilon_3 = \varepsilon$.

(b) If there exist $\varepsilon_1, \varepsilon_3 \geq 0$ and $\overline{\lambda} \in Y^*$ such that the relations $(i^A) - (iii^A)$ hold for some $\overline{x} \in X$, then $\overline{x}$ is an ε - optimal solution of the problem (P^A).

Proof. (a) Since the hypotheses of Theorem 4 are fulfilled there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in Y^*$ and $\overline{x^*} \in X^*$ such that the inequalities $(i) - (iv)$ hold. It is easy to see that in this case the assertions (i) and (iv) are equivalent to the assertions (i^A) and (iii^A), respectively. Since $(\overline{\lambda}h)^*(-\overline{x^*}) = +\infty$ for all $-\overline{x^*} \neq A^*\overline{\lambda}$, relation (iii) implies $\overline{x^*} = -A^*\overline{\lambda}$ and now it is easy to see that the assertion (ii^A) is equivalent to (ii).

(b) For $\varepsilon_2 = 0$ and $\overline{x^*} = -A^*\overline{\lambda}$ it can be easily proved that the assertions $(i) - (iv)$ of Theorem 4 are fulfilled. □

4.2 The case $f \equiv 0$

Consider the function $f : X \to \mathbb{R}$ fulfilling $f(x) = 0$ for all $x \in X$. Thus

$$f^*(x^*) = \begin{cases} 0, & \text{if } x^* = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case relation (6) becomes

$$(g \circ h)^*(p^*) = \min_{\lambda \in K^*} \Big\{ g^*(\lambda) + (\lambda h)^*(p^*) \Big\}, \forall p^* \in X^*. \tag{11}$$

As $\text{epi}(f^*) = \{0\} \times \mathbb{R}_+$ we get

$$\begin{aligned} & \text{epi}(f^*) + \bigcup_{\lambda \in K^*} \Big(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) \\ = \ & \{0\} \times \mathbb{R}_+ + \bigcup_{\lambda \in K^*} \Big(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) \\ = \ & \bigcup_{\lambda \in K^*} \Big(\text{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) \end{aligned}$$

and the regularity condition (RC) becomes

$$(RC_0) \qquad \bigcup_{\lambda \in K^*} \Big(\operatorname{epi}((\lambda h)^*) + (0, g^*(\lambda)) \Big) \text{ is closed.}$$

Moreover, according to Theorem 1 the condition (RC_0) is fulfilled if and only if relation (11) holds.

Taking into consideration relation (4) and the form of the function f^* it is not hard to prove that $\partial_\varepsilon f(x) = \{0\} \subset X^*$ for all $x \in X$ and all $\varepsilon \geq 0$. The next result, which provides a formula for the ε - subdifferential of the function $g \circ h$, is a straightforward consequence of Theorem 2 and Theorem 3.

Theorem 7 *The regularity condition (RC_0) is fulfilled if and only if for all $x \in \operatorname{dom}(g \circ h)$ and for all $\varepsilon \geq 0$ we have*

$$\partial_\varepsilon (g \circ h)(x) = \bigcup_{\substack{\varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_2 + \varepsilon_3 = \varepsilon}} \Big\{ \partial_{\varepsilon_2} (\lambda h)(x) : \lambda \in K^* \cap \partial_{\varepsilon_3} g(h(x)) \Big\}. \tag{12}$$

Consider the optimization problem

$$(P_0) \qquad \inf_{x \in X} (g \circ h)(x).$$

The next theorem provides necessary and sufficient optimality conditions for the ε - optimal solutions of the problem (P_0).

Theorem 8 *(a) Suppose that (RC_0) is fulfilled. If $\overline{x} \in X$ is an ε - optimal solution of the problem (P_0) for some $\varepsilon \geq 0$, then there exist $\varepsilon_2, \varepsilon_3 \geq 0$ and $\overline{\lambda} \in K^*$ such that*

(i_0) $0 \leq g^*(\overline{\lambda}) + g(h(\overline{x})) - \langle \overline{\lambda}, h(\overline{x}) \rangle \leq \varepsilon_3$;

(ii_0) $0 \leq (\overline{\lambda} h)^*(0) + (\overline{\lambda} h)(\overline{x}) \leq \varepsilon_2$;

(iii_0) $\varepsilon_2 + \varepsilon_3 = \varepsilon$.

(b) If there exist $\varepsilon_2, \varepsilon_3 \geq 0$ and $\overline{\lambda} \in K^*$ such that the relations $(i_0) - (iii_0)$ hold for some $\overline{x} \in X$, then $\overline{x}$ is an ε - optimal solution of the problem (P_0).

Proof. (a) By Theorem 4 there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$, $\overline{\lambda} \in K^*$ and $\overline{x^*} \in X^*$ such that the assertions $(i) - (iv)$ are fulfilled. Obviously (i_0) and (iii_0) are implied by (i) and (iv), respectively. Moreover, since the relation (ii) implies $\overline{x^*} = 0$, (ii_0) can be easily derived from (iii).

(b) In this special case the assertions $(i) - (iv)$ of Theorem 4 are fulfilled for $\varepsilon_1 = 0$ and $\overline{x^*} = 0$. $\square$

4.3 The ordinary convex optimization problem

Consider the function

$$g : Y \to \overline{\mathbb{R}}, \quad g(y) = \delta_{-K}(y) = \begin{cases} 0, & \text{if } y \in -K, \\ +\infty, & \text{otherwise.} \end{cases}$$

One can prove that the function g is convex and K - increasing and that $g^* = \delta_{K^*}$. In this case relation (6) is nothing else than

$$(f + \delta_{-K} \circ h)^*(p^*) = \min_{\substack{\lambda \in K^*, \\ x^* \in X^*}} \left\{ f^*(x^*) + (\lambda h)^*(p^* - x^*) \right\}, \forall p^* \in X^*. \tag{13}$$

Moreover, according to Theorem 1 the relation (13) holds if and only if the regularity condition

$$(RC^O) \qquad \operatorname{epi}(f^*) + \bigcup_{\lambda \in K^*} \operatorname{epi}((\lambda h)^*) \text{ is closed}$$

is fulfilled.

Taking into consideration the way g^* looks like and relation (4) one can easily show that for all $y \in -K$ we have $\lambda \in \partial_\varepsilon \delta_{-K}(y)$ if and only if $\lambda \in K^*$ and $0 \leq \langle \lambda, y \rangle + \varepsilon$. Relation (7) states in this case

$$\partial_\varepsilon (f + \delta_{-K} \circ h)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \Big\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (\lambda h)(x) : \lambda \in K^*,$$

$$0 \leq \langle \lambda, h(x) \rangle + \varepsilon_3 \Big\}.$$

Making use of Theorem 2 and Theorem 3 the subsequent result can be easily proved.

Theorem 9 *The regularity condition* (RC^O) *is fulfilled if and only if for all* $x \in \operatorname{dom}(f + \delta_{-K} \circ h)$ *and for all* $\varepsilon \geq 0$ *we have*

$$\partial_\varepsilon (f + \delta_{-K} \circ h)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 - \varepsilon \leq \langle \lambda, h(x) \rangle}} \Big\{ \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (\lambda h)(x) : \lambda \in K^* \Big\}. \quad (14)$$

Before going further we would like to mention that the set given in (RC^O) appears also in the so-called closed cone constraint qualification given in [10] (for an equivalent formulation see [5]).

Consider the optimization problem

$$(P^O) \qquad \inf_{\substack{x \in X, \\ h(x) \leqq_K 0}} f(x),$$

which is nothing else than the classical convex optimization problem with geometric and cone constraints. It can be rewritten as

$$(P^O) \qquad \inf_{x \in X} (f + \delta_{-K} \circ h)(x).$$

Necessary and sufficient conditions for ε - optimal solutions of the problem (P^O) can be derived from (14).

Theorem 10 (a) *Suppose that* (RC^O) *is fulfilled. If* $\overline{x} \in X$ *is an* ε *- optimal solution of the problem* (P^O) *for some* $\varepsilon \geq 0$*, then there exist* $\varepsilon_1, \varepsilon_2 \geq 0$*,* $\overline{\lambda} \in K^*$ *and* $\overline{x^*} \in X^*$ *such that*

(i^O) $0 \leq f^*(\overline{x^*}) + f(\overline{x}) - \langle \overline{x^*}, \overline{x} \rangle \leq \varepsilon_1$;

(ii^O) $0 \leq (\overline{\lambda} h)^*(-\overline{x^*}) + (\overline{\lambda} h)(\overline{x}) + \langle \overline{x^*}, \overline{x} \rangle \leq \varepsilon_2$;

(iii^O) $0 \leq -\langle \overline{\lambda}, h(\bar{x}) \rangle \leq \varepsilon - \varepsilon_1 - \varepsilon_2$.

(b) *If there exist* $\varepsilon_1, \varepsilon_2 \geq 0$*,* $\overline{\lambda} \in K^*$ *and* $\overline{x^*} \in X^*$ *such that the relations* $(i^O) - (iii^O)$ *hold for some* $\overline{x} \in X$*, then* $\overline{x}$ *is an* ε *- optimal solution of the problem* (P^O).

Proof. (a) Once again we apply Theorem 4. Thus there exist $\overline{\lambda} \in K^*$ and $\overline{x^*} \in X^*$ such that the relations $(i) - (iv)$ are fulfilled. Since (i^O) and (ii^O) are direct consequences of the assertions (ii) and (iii), respectively, it remains to prove (iii^O). As assertion (i) implies $0 \leq -\langle\overline{\lambda}, h(\bar{x})\rangle \leq \varepsilon_3$, the last inequality and (iv) are enough to ensure the desired conclusion.

(b) It is straightforward to see that for $\varepsilon_3 = \varepsilon - \varepsilon_1 - \varepsilon_2$ the assertions $(i) - (iv)$ of Theorem 4 are fulfilled. □

5 Conclusions

In this paper we give a new formula for the ε - subdifferential of the sum of a function and the composition of another convex function which is K - increasing with a K - convex function (we suppose that K is a closed convex cone). Using the epigraphs of the conjugates of the functions involved we give a closedness type regularity condition which turns out to be equivalent to the formula mentioned above. Using the strong connection between the ε - subdifferential of a function and its conjugate function we provide necessary and sufficient optimality conditions for composed convex optimization problems. Moreover, some special cases are considered, rediscovering some results already given in the literature.

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