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Regularity conditions via quasi-relative interior in convex programming

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Abstract. We give some new regularity conditions for Fenchel duality in separated locally convex vector spaces, written in terms of the notion of quasi interior and quasi-relative interior, respectively. We provide also an example of a convex optimization problem for which the classical generalized interior-point conditions given so far in the literature cannot be applied, while the one given by us is applicable. Using a technique developed by Magnanti, we derive some duality results for the optimization problem with cone inequality constraints and its Lagrange dual problem and we show that a duality result recently given in the literature for this pair of problems is incorrect.

Key Words. convex programming, Fenchel duality, Lagrange duality, quasi-relative interior

AMS subject classification. 90C25, 46A20, 90C51

1 Introduction

Usually there is a so-called duality gap between the optimal objective values of a primal convex optimization problem and its dual problem. A challenge in convex analysis is to give sufficient conditions which guarantee strong duality, the situation when the optimal objective values of the two problems are equal and the dual problem has an optimal solution. Several generalized interior-point conditions were given in the past in order to eliminate the above mentioned duality gap. Along the classical interior, some generalized interior notions were used, like the core ([14]), the intrinsec core ([9]), or the strong quasi-relative

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interior ([2]), in order to give regularity conditions which guarantee strong duality. For an overview of these conditions we invite the reader to consult [8] and [16] (see also [17] for more on this subject).

Unfortunately, for infinite-dimensional convex optimization problems, also in practice, it can happen that the duality results given in the past cannot be applied because, for instance, the interior of the set involved in the regularity condition is empty. This is the case, for example, when we deal with the positive cones

$$l_+^p = \{x = (x_n)_{n\in\mathbb{N}} \in l^p : x_n \geq 0, \forall n \in \mathbb{N}\}$$

and

$$L_+^p(T, \mu) = \{u \in L^p(T, \mu) : u(t) \geq 0, \text{ a.e.}\}$$

of the spaces l^p and $L^p(T, \mu)$, respectively, where (T, μ) is a σ-finite measure space and $p \in [1, \infty)$. Moreover, also the strong quasi-relative interior (which is the weakest generalized interior notion from the one mentioned above) of this cones is empty. For this reason, for a convex set, Borwein and Lewis introduced the notion of quasi-relative interior ([3]), which generalizes all the above mentioned interior notions. They proved that the quasi-relative interiors of l_+^p and $L_+^p(T, \mu)$ are nonempty.

In this paper, we start by considering the primal optimization problem with the objective function being the sum of two proper convex functions defined on a separated locally convex vector space, to which we attach its Fenchel dual problem, stated in terms of the conjugates of the two functions. We give a new regularity condition for Fenchel duality based on the notion of quasi-relative interior of a convex set using a separation theorem given by Cammaroto and Di Bella in [4]. Further, two stronger regularity conditions are also given. We provide an appropriate example for which our duality results are applicable, unless the other generalized interior-point conditions given in the past, justifying the theory developed in this paper. Then we state duality results for the case when the objective function of the primal problem is the sum of a proper convex function with the composition of another proper convex function with a continuous linear operator. Let us notice that for this case Borwein and Lewis in [3] gave also some conditions by means of the quasi-relative interior, but they considered a more restrictive case, namely that the codomain of the linear operator is finite-dimensional. We consider the more general case, when both of the spaces are infinite-dimensional.

In 1974 Magnanti proved that "Fenchel and Lagrange duality are equivalent" in the sense that the classical Fenchel duality result can be deduced from the classical Lagrange duality result, and viceversa (see [13]). Using this technique we derive some Lagrange duality results for the convex optimization problem with cone inequality constraints, written in terms of the quasi-relative interior. Let us notice that another condition for Lagrange duality, stated also in terms of the quasi-relative interior, was given recently by Cammaroto and Di Bella in [4].

We show that in fact this duality result is vacuous since the hypotheses of the theorem are in contradiction. Let us mention that also in [11] some regularity conditions, in terms of the quasi-relative interior, have been introduced. However, most of these conditions require the interior of a cone to be nonempty, and this fails for many optimization problems as we pointed out above.

The paper is structured as follows. In the next section we give some definitions and results which will be used later in the paper. Section 3 is devoted to the theory of Fenchel duality. We give here the announced regularity conditions written in terms of the quasi-relative interior. Using an idea due to Magnanti we derive in section 4 some duality results for the optimization problem with cone inequality constraints and its Lagrange dual problem.

2 Preliminary notions and results

Consider X a separated locally convex vector space and X^* its continuous dual space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Further, let $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x, \forall x \in X$, be the *identity function* of X. The *indicator function* of $C \subseteq X$, denoted by δ_C, is defined as $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$,

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\mathrm{epi}(f) = \{(x,r) \in X \times \mathbb{R} : f(x) \le r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty, \forall x \in X$. We also denote by $\widehat{\mathrm{epi}}(f) = \{(x,r) \in X \times \mathbb{R} : (x,-r) \in \mathrm{epi}(f)\}$, the symmetric of $\mathrm{epi}(f)$ with respect to the variable $x \in X$. For a given real number α, $f - \alpha : X \to \overline{\mathbb{R}}$ is, as usual, the function defined by $(f-\alpha)(x) = f(x) - \alpha, \forall x \in X$. Given two functions, $f : M_1 \to M_2$ and $g : N_1 \to N_2$, where M_1, M_2, N_1, N_2 are nonempty sets, we define the function $f \times g : M_1 \times N_1 \to M_2 \times N_2$ by $f \times g(m,n) = (f(m), g(n)), \forall (m,n) \in M_1 \times N_1$. The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}, \forall x^* \in X^*.$$

For a subset C of X we denote by $\mathrm{co}\, C$, $\mathrm{aff}\, C$, $\mathrm{cl}\, C$ and $\mathrm{int}\, C$ its *convex hull*, *affine hull*, *closure* and *interior*, respectively. The set $\mathrm{cone}\, C := \bigcup_{\lambda \ge 0} \lambda C$ is the *cone generated by C*. The following property, the proof of which we omit since it presents no difficulty, will be used throughout the paper: if C is convex, then

$$\mathrm{cone}\,\mathrm{co}(C \cup \{0\}) = \mathrm{cone}\, C. \tag{1}$$

The *normal cone* of C at $x \in C$ is defined as $N_C(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0,\ \forall y \in C\}$.

Definition 2.1 ([3]) Let C be a convex subset of X. The *quasi-relative interior* of C is the set

$$\operatorname{qri} C = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) \text{ is a linear subspace of } X\}.$$

We give the following useful characterization of the quasi-relative interior of a convex set.

Proposition 2.2 ([3]) *Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qri} C$ if and only if $N_C(x)$ is a linear subspace of X^*.*

In the following we consider another interior notion for a convex set, which is close to the one of quasi-relative interior.

Definition 2.3 Let C be a convex subset of X. The *quasi interior* of C is the set

$$\operatorname{qi} C = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) = X\}.$$

The following characterization of the quasi interior of a convex set was given in [6], where the space X was considered a reflexive Banach space. One can prove that this property is true even in a separated locally convex vector space.

Proposition 2.4 ([6]) *Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qi} C$ if and only if $N_C(x) = \{0\}$.*

It follows from the definitions above that $\operatorname{qi} C \subseteq \operatorname{qri} C$ and $\operatorname{qri}\{x\} = \{x\}$, $\forall x \in X$. Moreover, if $\operatorname{qi} C \neq \emptyset$, then $\operatorname{qi} C = \operatorname{qri} C$. Although this property is given in [12] in the case of a real normed space, it holds also in an arbitrary separated locally convex vector space, as follows by the properties given above. If X is a finite-dimensional space, then $\operatorname{qi} C = \operatorname{int} C$ (cf. [12]) and $\operatorname{qri} C = \operatorname{ri} C$ (cf. [3]), where $\operatorname{ri} C$ is the *relative interior* of C.

Useful properties of the quasi-relative interior are listed below. For the proof of $(i) - (viii)$ we refer to [1] and [3].

Proposition 2.5 *Let us consider C and D two convex subsets of X, $x \in X$ and $\alpha \in \mathbb{R}$. Then:*

(i) $\operatorname{qri} C + \operatorname{qri} D \subseteq \operatorname{qri}(C + D)$;

(ii) $\operatorname{qri}(C \times D) = \operatorname{qri} C \times \operatorname{qri} D$;

(iii) $\operatorname{qri}(C - x) = \operatorname{qri} C - x$;

(iv) $\operatorname{qri}(\alpha C) = \alpha \operatorname{qri} C$;

(v) $t \operatorname{qri} C + (1-t)C \subseteq \operatorname{qri} C$, $\forall t \in (0,1]$, *hence* $\operatorname{qri} C$ *is a convex set;*

(vi) *if* C *is an affine set then* $\operatorname{qri} C = C$;

(vii) $\operatorname{qri}(\operatorname{qri} C) = \operatorname{qri} C$.

If $\operatorname{qri} C \neq \emptyset$ *then:*

(viii) $\operatorname{cl} \operatorname{qri} C = \operatorname{cl} C$;

(ix) $\operatorname{cl} \operatorname{cone} \operatorname{qri} C = \operatorname{cl} \operatorname{cone} C$.

Proof. (ix) The inclusion $\operatorname{cl} \operatorname{cone} \operatorname{qri} C \subseteq \operatorname{cl} \operatorname{cone} C$ is obvious. We prove that $\operatorname{cone} C \subseteq \operatorname{cl} \operatorname{cone} \operatorname{qri} C$. Consider $x \in \operatorname{cone} C$ arbitrary. There exist $\lambda \geq 0$ and $c \in C$ such that $x = \lambda c$. Take $x_0 \in \operatorname{qri} C$. Applying the property (v) we get $tx_0 + (1-t)c \in \operatorname{qri} C$ $\forall t \in (0,1]$, so $\lambda t x_0 + (1-t)x = \lambda(tx_0 + (1-t)c) \in \operatorname{cone} \operatorname{qri} C$ $\forall t \in (0,1]$. Passing to the limit as $t \searrow 0$ we obtain $x \in \operatorname{cl} \operatorname{cone} \operatorname{qri} C$ and hence the desired conclusion follows. □

The next lemma plays an important roll in this paper.

Lemma 2.6 *Let* A *and* B *be nonempty convex subsets of* X *such that* $A \cap B \neq \emptyset$. *If* $0 \in \operatorname{qi}(A - A)$ *and* $B \cap \operatorname{qri} A \neq \emptyset$, *then* $0 \in \operatorname{qi}(A - B)$.

Proof. Take $x \in B \cap \operatorname{qri} A$ and let $x^* \in N_{A-B}(0)$ be arbitrary. We get $\langle x^*, a - b \rangle \leq 0, \forall a \in A, \forall b \in B$. This implies

$$\langle x^*, a - x \rangle \leq 0, \forall a \in A, \tag{2}$$

that is $x^* \in N_A(x)$. As $x \in \operatorname{qri} A$, $N_A(x)$ is a linear subspace of X^*, hence $-x^* \in N_A(x)$, which is nothing else than

$$\langle x^*, x - a \rangle \leq 0, \forall a \in A. \tag{3}$$

The relations (2) and (3) give us $\langle x^*, a' - a'' \rangle \leq 0, \forall a', a'' \in A$, so $x^* \in N_{A-A}(0)$. Since $0 \in \operatorname{qi}(A - A)$ we have $N_{A-A}(0) = \{0\}$ (cf. Proposition 2.4) and we get $x^* = 0$. As x^* was arbitrary chosen we obtain $N_{A-B}(0) = \{0\}$ and, using again Proposition 2.4, the conclusion follows. □

Next we give useful separation theorems in terms of the notion of quasi-relative interior.

Theorem 2.7 *Let* C *be a convex subset of* X *and* $x_0 \in C$. *If* $x_0 \notin \operatorname{qri} C$, *then there exists* $x^* \in X^*, x^* \neq 0$ *such that*

$$\langle x^*, x \rangle \leq \langle x^*, x_0 \rangle \ \forall x \in C.$$

Viceversa, if there exists $x^* \in X^*$, $x^* \neq 0$ such that

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle \ \forall x \in C$$

and

$$0 \in \operatorname{qi}(C - C),$$

then $x_0 \notin \operatorname{qri} C$.

Proof. Suppose that $x_0 \notin \operatorname{qri} C$. According to Proposition 2.2, $N_C(x_0)$ is not a linear subspace of X^*, hence there exists $x^* \in N_C(x_0)$, $x^* \neq 0$. Using the definition of the normal cone, we get that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$, $\forall x \in C$.

Conversely, assume that there exists $x^* \in X^*$, $x^* \neq 0$ such that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$, $\forall x \in C$ and $0 \in \operatorname{qi}(C - C)$. We obtain

$$\langle x^*, x - x_0\rangle \leq 0, \forall x \in C, \tag{4}$$

that is $x^* \in N_C(x_0)$. If we suppose that $x_0 \in \operatorname{qri} C$, then $N_C(x_0)$ is a linear subspace of X^*, hence $-x^* \in N_C(x_0)$. Combining this with (4) we get $\langle x^*, x - x_0\rangle = 0$, $\forall x \in C$. The last relation implies $\langle x^*, x\rangle = 0$, $\forall x \in C - C$, and from here one has further that $\langle x^*, x\rangle = 0$, $\forall x \in \operatorname{cl}\operatorname{cone}(C - C) = X$. But, this can be the case just if $x^* = 0$, which is a contradiction. In conclusion, $x_0 \notin \operatorname{qri} C$. □

Remark 2.8 In [5] and [6] a similar separation theorem in case when X is a real normed space is given. For the second part of the above theorem the authors require that the following condition must be fulfilled:

$$\operatorname{cl}(T_C(x_0) - T_C(x_0)) = X,$$

where

$$T_C(x_0) = \big\{y \in X : y = \lim_{n\to\infty} \lambda_n(x_n - x_0), \lambda_n > 0 \ \forall n \in \mathbb{N},$$
$$x_n \in C \ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x_0\big\}$$

is called the *contingent cone* to C at $x_0 \in C$. In general, we have the following inclusion: $T_C(x_0) \subseteq \operatorname{cl}\operatorname{cone}(C - x_0)$. If the set C is convex, then $T_C(x_0) = \operatorname{cl}\operatorname{cone}(C - x_0)$ (cf. [10]). As $\operatorname{cl}(\operatorname{cl} E + \operatorname{cl} F) = \operatorname{cl}(E + F)$, for arbitrary sets E, F in X and $\operatorname{cone} A - \operatorname{cone} A = \operatorname{cone}(A - A)$, if A is a convex subset of X such that $0 \in A$, the condition $\operatorname{cl}(T_C(x_0) - T_C(x_0)) = X$ can be reformulated as follows: $\operatorname{cl}\operatorname{cone}(C - C) = X$ or, equivalently, $0 \in \operatorname{qi}(C - C)$. Indeed, we have

$$\operatorname{cl}[\operatorname{cl}\operatorname{cone}(C - x_0) - \operatorname{cl}\operatorname{cone}(C - x_0)] = X \Leftrightarrow \operatorname{cl}[\operatorname{cone}(C - x_0) - \operatorname{cone}(C - x_0)] = X$$

$$\Leftrightarrow \operatorname{cl}\operatorname{cone}(C - C) = X \Leftrightarrow 0 \in \operatorname{qi}(C - C).$$

This means that Theorem 2.7 is a generalization to the case of separated locally convex vector spaces of the separation theorem given in [5] and [6] in the framework of real normed spaces.

The condition $x_0 \in C$ in Theorem 2.7 is essential (see [6]). However, if x_0 is an arbitrary element in X, we can give also a separation theorem using the following result due to Cammaroto and Di Bella (Theorem 2.1 in [4]). The separation theorem is correct, unlike the duality result Theorem 2.2 given in [4].

Theorem 2.9 ([4]) *Let S and T be nonempty convex subsets of X with $\operatorname{qri} S \neq \emptyset$, $\operatorname{qri} T \neq \emptyset$ and such that $\operatorname{cl}\operatorname{cone}(\operatorname{qri} S - \operatorname{qri} T)$ is not a linear subspace of X. Then, there exists $x^* \in X^*$, $x^* \neq 0$ such that $\langle x^*, s\rangle \leq \langle x^*, t\rangle$ for all $s \in S$, $t \in T$.*

The following result is a direct consequence of Theorem 2.9.

Corollary 2.10 *Let C be a convex subset of X such that $\operatorname{qri} C \neq \emptyset$ and $\operatorname{cl}\operatorname{cone}(C - x_0)$ is not a linear subspace of X, where $x_0 \in X$. Then there exists $x^* \in X^*, x^* \neq 0$ such that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle$ $\forall x \in C$.*

Proof. We take in Theorem 2.9 $S := C$ and $T := \{x_0\}$. Then we apply Proposition 2.5 *(iii)* and *(ix)* to obtain the conclusion. □

3 Fenchel duality

In this section we give some new Fenchel duality results stated in terms of the quasi interior and quasi-relative interior, respectively.

Consider the convex optimization problem

$$(P_F) \ \inf_{x \in X} \{f(x) + g(x)\},$$

where X is a separated locally convex vector space and $f, g : X \to \overline{\mathbb{R}}$ are proper convex functions such that $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$. The Fenchel dual problem to (P_F) is

$$(D_F) \ \sup_{x^* \in X^*} \{-f^*(-x^*) - g^*(x^*)\}.$$

We denote by $v(P_F)$ and $v(D_F)$ the optimal objective values of the primal and the dual problem, respectively. Weak duality always holds, that is $v(D_F) \leq v(P_F)$. For strong duality, the case when $v(P_F) = v(D_F)$ and (D_F) has an optimal solution, several generalized interior-point regularity conditions were given in the literature. In order to recall them we need the following generalized interior notions. For a convex subset C of X we have:

- $\operatorname{core} C := \{x \in C : \operatorname{cone}(C - x) = X\}$, the *core* of C ([14], [17]);
- $\operatorname{icr} C := \{x \in C : \operatorname{cone}(C - x)$ is a linear subspace$\}$, the *intrinsec core* of C ([1], [9], [17]);
- $\operatorname{sqri} C := \{x \in C : \operatorname{cone}(C - x)$ is a closed linear subspace$\}$, the *strong quasi-relative interior* of C ([2], [17]).

We have the following inclusions:

$$\operatorname{core} C \subseteq \operatorname{sqri} C \subseteq \operatorname{qri} C \text{ and } \operatorname{core} C \subseteq \operatorname{qi} C \subseteq \operatorname{qri} C.$$

If X if finite-dimensional then $\operatorname{qri} C = \operatorname{sqri} C = \operatorname{icr} C = \operatorname{ri} C$ ([3], [8]) and $\operatorname{core} C = \operatorname{qi} C = \operatorname{int} C$ ([12], [14]).

Consider now the following regularity conditions:

(*i*) $0 \in \operatorname{int}(\operatorname{dom}(f) - \operatorname{dom}(g))$;

(*ii*) $0 \in \operatorname{core}(\operatorname{dom}(f) - \operatorname{dom}(g))$ (cf. [14]);

(*iii*) $0 \in \operatorname{icr}(\operatorname{dom}(f) - \operatorname{dom}(g))$ and $\operatorname{aff}(\operatorname{dom}(f) - \operatorname{dom}(g))$ is a closed linear subspace (cf. [8]);

(*iv*) $0 \in \operatorname{sqri}(\operatorname{dom}(f) - \operatorname{dom}(g))$ (cf. [15]).

Let us notice that all these conditions guarantee strong duality if we suppose the additional hypotheses that the functions f and g are lower semicontinuous and X is a Fréchet space. Between the above conditions we have the following relation: $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv)$ ([8]).

Trying to give a similar regularity condition for strong duality by means of the notion of quasi-relative interior of a convex set, a natural question arises: is the condition $0 \in \operatorname{qri}(\operatorname{dom}(f) - \operatorname{dom}(g))$ sufficient for strong duality? The following example (which can be found in [8]) gives us a negative answer and this means that we need additional assumptions in order to guarantee Fenchel duality (see Theorem 3.5).

Example 3.1 As in [8], we consider $X = l^2$, the Hilbert space consisting of all sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $\sum\limits_{n=1}^{\infty} x_n^2 < \infty$. Consider also the sets

$$C = \{x \in l^2 : x_{2n-1} + x_{2n} = 0, \forall n \in \mathbb{N}\},$$

$$S = \{x \in l^2 : x_{2n} + x_{2n+1} = 0, \forall n \in \mathbb{N}\}.$$

The sets C and S are closed linear subspaces of l^2 and $C \cap S = \{0\}$. Define the functions $f, g : l^2 \to \overline{\mathbb{R}}$ by $f = \delta_C$ and $g(x) = x_1$ if $x \in S$ and $+\infty$ otherwise. One can see that f and g are proper, convex and lower semicontinuous functions with $\mathrm{dom}(f) = C$ and $\mathrm{dom}(g) = S$. As was shown in [8], $v(P_F) = 0$ and $v(D_F) = -\infty$, so we have a duality gap between the optimal objective values of the primal problem and its Fenchel dual. Moreover, $S - C$ is dense in l^2, thus $\mathrm{cl}\,\mathrm{cone}(\mathrm{dom}(f) - \mathrm{dom}(g)) = \mathrm{cl}(C - S) = l^2$. The last relation implies $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$, hence $0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$.

Let us notice that if $v(P_F) = -\infty$, by the weak duality follows that also strong duality holds. This is the reason why we suppose in the following that $v(P_F) \in \mathbb{R}$.

Lemma 3.2 *The following relation is always true:*

$$0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g)) \Rightarrow (0,1) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))].$$

Proof. One can see that $\widehat{\mathrm{epi}}(g - v(P_F)) = \{(x,r) \in X \times \mathbb{R} : r \leq -g(x) + v(P_F)\}$. Let us prove first that $(0,1) \in \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$. Since $\inf_{x \in X}[f(x) + g(x)] = v(P_F) < v(P_F)+1$, there exists $x' \in X$ such that $f(x') + g(x') < v(P_F)+1$. Then $(0,1) = (x', v(P_F)+1-g(x')) - (x', -g(x')+v(P_F)) \in \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$.

Now let $(x^*, r^*) \in N_{\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))}(0,1)$. We have

$$\langle x^*, x - x' \rangle + r^*(\mu - \mu' - 1) \leq 0, \forall (x,\mu) \in \mathrm{epi}(f), \forall (x', \mu') \in \widehat{\mathrm{epi}}(g - v(P_F)). \quad (5)$$

For $(x,\mu) := (x_0, f(x_0))$ and $(x', \mu') := (x_0, -g(x_0) + v(P_F) - 2)$ in (5), where $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ is fixed, we get $r^*(f(x_0) + g(x_0) - v(P_F) + 1) \leq 0$, hence $r^* \leq 0$. As $\inf_{x \in X}[f(x) + g(x)] = v(P_F) < v(P_F) + 1/2$, there exists $x_1 \in X$ such that $f(x_1) + g(x_1) < v(P_F) + 1/2$. Taking now $(x,\mu) := (x_1, f(x_1))$ and $(x', \mu') := (x_1, -g(x_1) + v(P_F) - 1/2)$ in (5) we obtain $r^*(f(x_1) + g(x_1) - v(P_F) - 1/2) \leq 0$ and so $r^* \geq 0$. Thus $r^* = 0$ and (5) gives: $\langle x^*, x - x' \rangle \leq 0, \forall x \in \mathrm{dom}(f), \forall x' \in \mathrm{dom}(g)$. Hence $x^* \in N_{\mathrm{dom}(f) - \mathrm{dom}(g)}(0)$. Since $N_{\mathrm{dom}(f) - \mathrm{dom}(g)}(0)$ is a linear subspace of X^* (cf. Proposition 2.2), we have $\langle -x^*, x - x' \rangle \leq 0, \forall x \in \mathrm{dom}(f), \forall x' \in \mathrm{dom}(g)$ and so $-(x^*, r^*) = (-x^*, 0) \in N_{\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))}(0,1)$, showing that $N_{\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))}(0,1)$ is a linear subspace of $X^* \times \mathbb{R}$. Hence, applying again Proposition 2.2, we get $(0,1) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. □

Proposition 3.3 *Assume that* $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$. *Then* $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ *is a linear subspace of* $X^* \times \mathbb{R}$ *if and only if* $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0) = \{(0,0)\}$.

Proof. The sufficiency is trivial. Now let us suppose that the set $N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$ is a linear subspace of $X^* \times \mathbb{R}$. Take $(x^*, r^*) \in N_{\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]}(0,0)$. Then

$$\langle x^*, x - x' \rangle + r^*(\mu - \mu') \leq 0, \forall (x,\mu) \in \mathrm{epi}(f), \forall (x', \mu') \in \widehat{\mathrm{epi}}(g - v(P_F)). \quad (6)$$

Let $x_0 \in \operatorname{dom} f \cap \operatorname{dom}(g)$ be fixed. Taking $(x, \mu) := (x_0, f(x_0)) \in \operatorname{epi}(f)$ and $(x', \mu') := (x_0, -g(x_0) + v(P_F) - 1/2) \in \widehat{\operatorname{epi}}(g - v(P_F))$ in the previous inequality we get $r^*(f(x_0) + g(x_0) - v(P_F) + 1/2) \leq 0$, implying $r^* \leq 0$. As $N_{\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]}(0,0)$ is a linear subspace of $X^* \times \mathbb{R}$, the same argument applies also for $(-x^*, -r^*)$, implying $-r^* \leq 0$. In this way we get $r^* = 0$. The inequality (6) and the relation $(-x^*, 0) \in N_{\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]}(0,0)$ imply

$$\langle x^*, x - x' \rangle = 0, \forall (x, \mu) \in \operatorname{epi}(f), \forall (x', \mu') \in \widehat{\operatorname{epi}}(g - v(P_F)),$$

which is nothing else than $\langle x^*, x - x' \rangle = 0, \forall x \in \operatorname{dom}(f), \forall x' \in \operatorname{dom}(g)$, thus $\langle x^*, x \rangle = 0, \forall x \in \operatorname{dom}(f) - \operatorname{dom}(g)$. Since x^* is linear and continuous, the last relation implies $\langle x^*, x \rangle = 0, \forall x \in \operatorname{cl}\operatorname{cone}[(\operatorname{dom}(f) - \operatorname{dom}(g)) - (\operatorname{dom}(f) - \operatorname{dom}(g))] = X$, hence $x^* = 0$ and the conclusion follows. □

Remark 3.4 (a) By (1) one can see that $\operatorname{cl}\operatorname{cone}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}] = \operatorname{cl}\operatorname{cone}[\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))]$. Hence one has the following sequence of equivalences: $N_{\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]}(0,0)$ is a linear subspace of $X^* \times \mathbb{R} \Leftrightarrow (0,0) \in \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}] \Leftrightarrow \operatorname{cl}\operatorname{cone}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}]$ is a linear subspace of $X \times \mathbb{R} \Leftrightarrow \operatorname{cl}\operatorname{cone}(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)))$ is a linear subspace of $X \times \mathbb{R}$. The relation $N_{\operatorname{co}[(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))\cup\{(0,0)\}]}(0,0) = \{(0,0)\}$ is equivalent to $(0,0) \in \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}]$ (cf. Proposition 2.4), so in case $0 \in \operatorname{qi}[(\operatorname{dom}(f) - \operatorname{dom}(g)) - (\operatorname{dom}(f) - \operatorname{dom}(g))]$, the conclusion of the previous proposition can be reformulated as follows

$$\operatorname{cl}\operatorname{cone}(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \text{ is a linear subspace of } X \times \mathbb{R} \Leftrightarrow$$

$$(0,0) \in \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}],$$

or, equivalently

$$(0,0) \in \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}] \Leftrightarrow$$

$$(0,0) \in \operatorname{qi}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}].$$

(b) One can prove that the primal problem (P_F) has an optimal solution if and only if $(0,0) \in \operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))$. This means that if we suppose that the primal problem has an optimal solution and $0 \in \operatorname{qi}[(\operatorname{dom}(f) - \operatorname{dom}(g)) - (\operatorname{dom}(f) - \operatorname{dom}(g))]$, then the conclusion of the previous proposition can be rewritten as follows: $N_{(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))}(0,0)$ is a linear subspace of $X^* \times \mathbb{R}$ if and only if $N_{(\operatorname{epi}(f)-\widehat{\operatorname{epi}}(g-v(P_F)))}(0,0) = \{(0,0)\}$ or, equivalently,

$$(0,0) \in \operatorname{qri}[\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))] \Leftrightarrow (0,0) \in \operatorname{qi}[\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))].$$

We give now the first strong duality result for (P_F) and its Fenchel dual (D_F). Let us notice that for the functions f and g we suppose just convexity properties and we do not use any closedness type condition.

Theorem 3.5 *Suppose that* $0 \in \mathrm{qi}[(\mathrm{dom}(f)-\mathrm{dom}(g))-(\mathrm{dom}(f)-\mathrm{dom}(g))]$, $0 \in \mathrm{qri}(\mathrm{dom}(f)-\mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}\, f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$. *Then* $v(P_F) = v(D_F)$ *and* (D_F) *has an optimal solution.*

Proof. Lemma 3.2 ensures that $(0,1) \in \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$, hence $\mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))] \neq \emptyset$. The condition $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}\, f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$, together with the relation $\mathrm{cl}\,\mathrm{cone}\,\mathrm{co}[(\mathrm{epi}\, f - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}] = \mathrm{cl}\,\mathrm{cone}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$, imply that $\mathrm{cl}\,\mathrm{cone}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$ is not a linear subspace of $X \times \mathbb{R}$. We apply Corollary 2.10 with $C := \mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))$ and $x_0 = (0,0)$. Thus there exists $(x^*, \lambda) \in X^* \times \mathbb{R}$, $(x^*, \lambda) \neq (0,0)$ such that

$$\langle x^*, x\rangle + \lambda\mu \geq \langle x^*, x'\rangle + \lambda\mu', \forall (x,\mu) \in \widehat{\mathrm{epi}}(g - v(P_F)), \forall (x',\mu') \in \mathrm{epi}(f). \quad (7)$$

We claim that $\lambda \leq 0$. Indeed, if $\lambda > 0$, then for $(x,\mu) := (x_0, -g(x_0) + v(P_F))$ and $(x',\mu') := (x_0, f(x_0) + n), n \in \mathbb{N}$, where $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ is fixed, we obtain from (7): $\langle x^*, x_0\rangle + \lambda(-g(x_0) + v(P_F)) \geq \langle x^*, x_0\rangle + \lambda(f(x_0) + n), \forall n \in \mathbb{N}$. Passing to the limit as $n \to +\infty$ we obtain a contradiction. Next we prove that $\lambda < 0$. Suppose that $\lambda = 0$. Then from (7) we have $\langle x^*, x\rangle \geq \langle x^*, x'\rangle, \forall x \in \mathrm{dom}(g), \forall x' \in \mathrm{dom}(f)$, hence $\langle x^*, x\rangle \leq 0, \forall x \in \mathrm{dom}(f) - \mathrm{dom}(g)$. Using the second part of Theorem 2.7, we obtain $0 \notin \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$, which contradicts the hypothesis. Thus we must have $\lambda < 0$ and so we obtain from (7):

$$\langle \frac{1}{\lambda}x^*, x\rangle + \mu \leq \langle \frac{1}{\lambda}x^*, x'\rangle + \mu', \forall (x,\mu) \in \widehat{\mathrm{epi}}(g - v(P_F)), \forall (x',\mu') \in \mathrm{epi}(f).$$

Let be $r \in \mathbb{R}$ such that

$$\mu' + \langle x_0^*, x'\rangle \geq r \geq \mu + \langle x_0^*, x\rangle, \forall (x,\mu) \in \widehat{\mathrm{epi}}(g - v(P_F)), \forall (x',\mu') \in \mathrm{epi}(f),$$

where $x_0^* := \frac{1}{\lambda}x^*$. The first inequality shows that $f(x) \geq \langle -x_0^*, x\rangle + r, \forall x \in X$, that is $f^*(-x_0^*) \leq -r$. The second one gives us $-g(x) + v(P_F) + \langle x_0^*, x\rangle \leq r, \forall x \in X$, hence $g^*(x_0^*) \leq r - v(P_F)$ and so we have $-f^*(-x_0^*) - g^*(x_0^*) \geq r + v(P_F) - r = v(P_F)$. This implies that $v(D_F) \geq v(P_F)$. As the opposite inequality is always true, we get $v(P_F) = v(D_F)$ and x_0^* is an optimal solution of the problem (D_F). □

The above theorem combined with Remark 3.4(b) gives us the following result.

Corollary 3.6 *Suppose that the primal problem* (P_F) *has an optimal solution,* $0 \in \mathrm{qi}[(\mathrm{dom}(f)-\mathrm{dom}(g))-(\mathrm{dom}(f)-\mathrm{dom}(g))]$, $0 \in \mathrm{qri}(\mathrm{dom}(f)-\mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. *Then* $v(P_F) = v(D_F)$ *and* (D_F) *has an* *optimal solution.*

Remark 3.7 The condition $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$ implies

$$0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g)) \Leftrightarrow 0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g)).$$

Indeed, denote by $C := \mathrm{dom}(f) - \mathrm{dom}(g)$. Obviously $0 \in \mathrm{qi}\, C$ implies $0 \in \mathrm{qri}\, C$. Suppose now that $0 \in \mathrm{qri}\, C$ and let $x^* \in N_C(0)$ be arbitrary. We have $\langle x^*, x\rangle \leq 0, \forall x \in C$. Since $N_C(0)$ is a linear subspace of X^*, we obtain $\langle x^*, x\rangle = 0, \forall x \in C$. We get further $\langle x^*, x\rangle = 0, \forall x \in \mathrm{cl}\,\mathrm{cone}(C - C) = X$, which implies that $x^* = 0$. Thus $N_C(0) = \{0\}$ and the conclusion follows.

Some stronger versions of Theorem 3.5 and Corollary 3.6, respectively, follows.

Theorem 3.8 *Let us suppose that* $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$. *Then* $v(P_F) = v(D_F)$ *and* (D_F) *has an optimal solution.*

Proof. We have $\mathrm{dom}(f) - \mathrm{dom}(g) \subseteq (\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))$, so the condition $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$ implies $0 \in \mathrm{qi}[(\mathrm{dom}(f) - \mathrm{dom}(g)) - (\mathrm{dom}(f) - \mathrm{dom}(g))]$ and $0 \in \mathrm{qri}(\mathrm{dom}(f) - \mathrm{dom}(g))$. Then we apply Theorem 3.5 to obtain the conclusion. □

Corollary 3.9 *Suppose that the primal problem* (P_F) *has an optimal solution,* $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. *Then* $v(P_F) = v(D_F)$ *and* (D_F) *has an optimal solution.*

Theorem 3.10 *Suppose that* $\mathrm{dom}(f) \cap \mathrm{qri}\,\mathrm{dom}(g) \neq \emptyset$, $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))) \cup \{(0,0)\}]$. *Then* $v(P_F) = v(D_F)$ *and* (D_F) *has an optimal solution.*

Proof. We apply Lemma 2.6 with $A := \mathrm{dom}(g)$ and $B := \mathrm{dom}(f)$. We get $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(f))$ or, equivalently, $0 \in \mathrm{qi}(\mathrm{dom}(f) - \mathrm{dom}(g))$. We obtain the result by applying Theorem 3.8. □

Corollary 3.11 *Suppose that the primal problem* (P_F) *has an optimal solution,* $\mathrm{dom}(f) \cap \mathrm{qri}\,\mathrm{dom}(g) \neq \emptyset$, $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}[\mathrm{epi}(f) - \widehat{\mathrm{epi}}(g - v(P_F))]$. *Then* $v(P_F) = v(D_F)$ *and* (D_F) *has an optimal solution.*

Remark 3.12 (a) We introduced above three new regularity conditions for Fenchel duality. As one can easily see from the proof of these results, the relation between these conditions is the following one: the regularity condition given in

Theorem 3.10 (Corollary 3.11) implies the one given in Theorem 3.8 (Corollary 3.9), which implies the one given in Theorem 3.5 (Corollary 3.6).

(b) If we renounce the condition $(0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F))) \cup \{(0,0)\}]$, or, respectively, $(0,0) \notin \operatorname{qri}(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)))$, in the case when the primal problem has an optimal solution, then the duality results given above may fail. By using again Example 3.1 we show that these conditions are essential in our theory. Let us notice that for the problem in Example 3.1 the conditions $0 \in \operatorname{qi}[(\operatorname{dom}(f) - \operatorname{dom}(g)) - (\operatorname{dom}(f) - \operatorname{dom}(g))]$ and $0 \in \operatorname{qri}(\operatorname{dom}(f) - \operatorname{dom}(g))$ are fulfilled. We prove in the following that in the aforementioned example we have $(0,0) \in \operatorname{qri}(\operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)))$. Note that the scalar product on l^2, $\langle \cdot, \cdot \rangle : l^2 \times l^2 \to \mathbb{R}$ is given by $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$, $\forall x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in l^2$. Also, for $k \in \mathbb{N}$, we denote by $e^{(k)}$ the element in l^2 which has on the k-th position 1 and on the other positions 0, that is $e_n^{(k)} = 1$, if $n = k$ and $e_n^{(k)} = 0$, $\forall n \in \mathbb{N}\backslash\{k\}$. We have $\operatorname{epi}(f) = C \times [0, \infty)$. Further, $\widehat{\operatorname{epi}}(g - v(P_F)) = \{(x,r) \in l^2 \times \mathbb{R} : r \leq -g(x)\} = \{(x,r) \in l^2 \times \mathbb{R} : x = (x_n)_{n\in\mathbb{N}} \in S, r \leq -x_1\} = \{(x, -x_1 - \varepsilon) \in l^2 \times \mathbb{R} : x = (x_n)_{n\in\mathbb{N}} \in S, \varepsilon \geq 0\}$. Then $A := \operatorname{epi}(f) - \widehat{\operatorname{epi}}(g - v(P_F)) = \{(x - x', x_1' + \varepsilon) : x \in C, x' = (x_n')_{n\in\mathbb{N}} \in S, \varepsilon \geq 0\}$. Take $(x^*, r) \in N_A(0,0)$, where $x^* = (x_n^*)_{n\in\mathbb{N}}$. We have

$$\langle x^*, x - x' \rangle + r(x_1' + \varepsilon) \leq 0, \forall x \in C, \forall x' = (x_n')_{n\in\mathbb{N}} \in S, \forall \varepsilon \geq 0. \tag{8}$$

Taking in (8) $x' = 0$ and $\varepsilon = 0$ we get $\langle x^*, x \rangle \leq 0, \forall x \in C$. As C is a linear subspace of X we have

$$\langle x^*, x \rangle = 0, \forall x \in C. \tag{9}$$

Since $e^{(2k-1)} - e^{(2k)} \in C$, $\forall k \in \mathbb{N}$, the relation (9) implies

$$x_{2k-1}^* - x_{2k}^* = 0, \forall k \in \mathbb{N}. \tag{10}$$

From (8) and (9) we obtain

$$\langle -x^*, x' \rangle + r(x_1' + \varepsilon) \leq 0, \forall x' = (x_n')_{n\in\mathbb{N}} \in S, \forall \varepsilon \geq 0. \tag{11}$$

Taking $\varepsilon = 0$ and $x' := me^1 \in S$ in (11), where $m \in \mathbb{Z}$ is arbitrary, we get $m(-x_1^* + r) \leq 0, \forall m \in \mathbb{Z}$, thus $r = x_1^*$. For $\varepsilon = 0$ in (11) we obtain $-\sum_{n=1}^{\infty} x_n^* x_n' + r x_1' \leq 0, \forall x' \in S$. Taking into account that $r = x_1^*$, we get $-\sum_{n=2}^{\infty} x_n^* x_n' \leq 0$, $\forall x' \in S$. As S is a linear subspace of X it follows $\sum_{n=2}^{\infty} x_n^* x_n' = 0$, $\forall x' \in S$, but, since $e^{(2k)} - e^{(2k+1)} \in S$, $\forall k \in \mathbb{N}$, the above relation shows that

$$x_{2k}^* - x_{2k+1}^* = 0, \forall k \in \mathbb{N}. \tag{12}$$

Combining (10) with (12) we get $x^* = 0$ (since $x^* \in l^2$). Because $r = x_1^*$, we have also $r = 0$. Thus $N_A(0,0) = \{(0,0)\}$ and Proposition 2.4 gives us the desired conclusion.

(c) Since in all the strong duality results given above, the relation $0 \in \text{qi}[(\text{dom}(f) - \text{dom}(g)) - (\text{dom}(f) - \text{dom}(g))]$ must be fulfilled, in view of Remark 3.4, the condition $(0,0) \notin \text{qri}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}]$ (respectively, $(0,0) \notin \text{qri}[\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))]$) is equivalent to $(0,0) \notin \text{qi}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}]$, (respectively $(0,0) \notin \text{qi}[\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))]$).

(d) We have the following relation

$$(0,0) \in \text{qi}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}] \Rightarrow 0 \in \text{qi}(\text{dom}(f) - \text{dom}(g)).$$

Indeed, $(0,0) \in \text{qi}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}] \Leftrightarrow \text{cl}\,\text{cone}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}] = X \times \mathbb{R}$, hence $\text{cl}\,\text{cone}(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) = X \times \mathbb{R}$. Since $\text{cl}\,\text{cone}(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \subseteq \text{cl}\,\text{cone}(\text{dom}(f) - \text{dom}(g)) \times \mathbb{R}$, this implies $\text{cl}\,\text{cone}(\text{dom}(f) - \text{dom}(g)) = X$, that is $0 \in \text{qi}(\text{dom}(f) - \text{dom}(g))$. Hence

$$0 \notin \text{qi}(\text{dom}(f) - \text{dom}(g)) \Rightarrow (0,0) \notin \text{qi}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}].$$

Nevertheless, in the regularity conditions given above one cannot substitute the condition $(0,0) \notin \text{qri}\,\text{co}[(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \cup \{(0,0)\}]$ by the "nice-looking" one $0 \notin \text{qi}(\text{dom}(f) - \text{dom}(g))$, since in all three strong duality theorems the other hypotheses we consider imply $0 \in \text{qi}(\text{dom}(f) - \text{dom}(g))$ (cf. Remark 3.7).

Example 3.13 Consider again the space $X = l^2$ equipped with the norm $\|\cdot\| : l^2 \to \mathbb{R}$, $\|x\|^2 = \sum_{n=1}^{\infty} x_n^2$, $\forall x = (x_n)_{n \in \mathbb{N}} \in l^2$. We define the functions $f, g : l^2 \to \overline{\mathbb{R}}$ by

$$f(x) = \begin{cases} \|x\|, & \text{if } x \in x_0 - l^2_+, \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \langle c, x \rangle, & \text{if } x \in l^2_+, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $l^2_+ = \{(x_n)_{n \in \mathbb{N}} \in l^2 : x_n \geq 0, \forall n \in \mathbb{N}\}$ is the positive cone, $x_0 = (\frac{1}{n})_{n \in \mathbb{N}}$ and $c = (\frac{1}{2^n})_{n \in \mathbb{N}}$. Note that $v(P_F) = \inf_{x \in l^2} \{f(x) + g(x)\} = 0$ and the infimum is attained at $x = 0$. We have $\text{dom}(f) = x_0 - l^2_+ = \{(x_n)_{n \in \mathbb{N}} \in l^2 : x_n \leq x_0, \forall n \in \mathbb{N}\}$ and $\text{dom}(g) = l^2_+$. Since $\text{qri}\, l^2_+ = \{(x_n)_{n \in \mathbb{N}} \in l^2 : x_n > 0, \forall n \in \mathbb{N}\}$ (cf. [3]), we get $\text{dom}(f) \cap \text{qri}\,\text{dom}(g) = \{(x_n)_{n \in \mathbb{N}} \in l^2 : 0 < x_n \leq x_0, \forall n \in \mathbb{N}\} \neq \emptyset$. Also, $\text{cl}\,\text{cone}(\text{dom}(g) - \text{dom}(g)) = l^2$, so $0 \in \text{qi}(\text{dom}(g) - \text{dom}(g))$. Further, $\text{epi}(f) = \{(x,r) \in l^2 \times \mathbb{R} : x \in x_0 - l^2_+, \|x\| \leq r\} = \{(x, \|x\| + \varepsilon) \in l^2 \times \mathbb{R} : x \in x_0 - l^2_+, \varepsilon \geq 0\}$ and $\widehat{\text{epi}}(g - v(P_F)) = \{(x,r) \in l^2 \times \mathbb{R} : r \leq -g(x)\} = \{(x,r) \in l^2 \times \mathbb{R} : r \leq -\langle c, x \rangle, x \in l^2_+\} = \{(x, -\langle c, x \rangle - \varepsilon) : x \in l^2_+, \varepsilon \geq 0\}$. We get $\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F)) = \{(x - x', \|x\| + \varepsilon + \langle c, x'\rangle + \varepsilon') : x \in x_0 - l^2_+, x' \in l^2_+, \varepsilon, \varepsilon' \geq 0\} = \{(x - x', \|x\| + \langle c, x'\rangle + \varepsilon) : x \in x_0 - l^2_+, x' \in l^2_+, \varepsilon \geq 0\}$. We claim that $(0,0) \notin \text{qri}(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F)))$. Indeed, if we suppose that $(0,0) \in \text{qri}(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F)))$ then we obtain $(0,0) \in \text{qi}(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F)))$ (see Remark 3.4 and Remark 3.12(c)), thus $l^2 \times \mathbb{R} = \text{cl}\,\text{cone}(\text{epi}(f) - \widehat{\text{epi}}(g - v(P_F))) \subseteq l^2 \times \text{cl}\,\text{cone}\{\|x\| + \langle c, x'\rangle + \varepsilon : x \in x_0 - l^2_+, x' \in l^2_+, \varepsilon \geq 0\}$. The last relation implies $\text{cl}\,\text{cone}\{\|x\| + \langle c, x'\rangle + \varepsilon : x \in x_0 - l^2_+, x' \in l^2_+, \varepsilon \geq 0\} = \mathbb{R}$. Let us notice that for a subset A of $\mathbb{R}$ the relation $\text{cl}\,\text{cone}\,A = \mathbb{R}$ is fulfilled if and only if $\exists(a_1, a_2) \in (A \cap (-\infty, 0)) \times (A \cap (0, +\infty))$. Thus there must exist $x_1 \in x_0 - l^2_+, x'_1 \in l^2_+$ and $\varepsilon \geq 0$ such that $\|x_1\| + \langle c, x'_1\rangle + \varepsilon < 0$ and this is a contradiction. Hence the conditions of Corollary 3.11 are fulfilled, thus strong duality holds. Let us notice that the regularity conditions given in Corollary 3.6 and Corollary 3.9 are also fulfilled (see Remark 12(a)). Moreover, one can see that $x^* = 0$ is an optimal solution of the dual problem.

On the other hand, l^2 is a Fréchet space (being a Hilbert space), the functions f and g are lower semicontinuous and, as $\text{sqri}(\text{dom}(f) - \text{dom}(g)) = \text{sqri}(x_0 - l^2_+) = \emptyset$, none of the constraint qualifications $(i) - (iv)$ presented in the beginning of this section can be applied for this optimization problem.

In the following, by using the results introduced above, we give regularity conditions for the following convex optimization problem

$$(P_A) \inf_{x \in X} \{f(x) + (g \circ A)(x)\},$$

where X and Y are separated locally convex vector spaces with their continuous dual spaces X^* and Y^*, respectively, $A : X \to Y$ is a linear continuous mapping, $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ are proper convex functions such that $A(\text{dom}(f)) \cap \text{dom}(g) \neq \emptyset$. The Fenchel dual problem to (P_A) is (cf. [17])

$$(D_A) \sup_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\},$$

where $A^* : Y^* \to X^*$ is the *adjoint operator* of A, defined in the usual way: $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle, \forall (y^*, x) \in Y^* \times X$. In the following theorem the set

$$A \times \text{id}_{\mathbb{R}}(\text{epi}(f)) = \{(Ax, r) \in Y \times \mathbb{R} : f(x) \leq r\}$$

is the image of $\text{epi}(f)$ through the operator $A \times \text{id}_{\mathbb{R}}$.

Theorem 3.14 *Suppose that* $0 \in \text{qi}[(A(\text{dom}(f)) - \text{dom}(g)) - (A(\text{dom}(f)) - \text{dom}(g))]$, $0 \in \text{qri}(A(\text{dom}(f)) - \text{dom}(g))$ *and* $(0,0) \notin \text{qri}\,\text{co}[(A \times \text{id}_{\mathbb{R}}(\text{epi}(f)) - \widehat{\text{epi}}(g - v(P_A))) \cup \{(0,0)\}]$. *Then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

Proof. Let us introduce the following functions: $F, G : X \times Y \to \overline{\mathbb{R}}$, $F(x, y) = f(x) + \delta_{\{x \in X : Ax = y\}}(x)$ and $G(x, y) = g(y)$. The functions F and G are proper and convex and $\inf\limits_{(x,y) \in X \times Y} [F(x, y) + G(x, y)] = \inf\limits_{x \in X} \{f(x) + (g \circ A)(x)\} = v(P_A)$. Moreover, $\text{dom}(F) = \text{dom}(f) \times A(\text{dom}(f))$ and $\text{dom}(G) = X \times \text{dom}(g)$, so $\text{dom}(F) \cap \text{dom}(G) \neq \emptyset$. Further,

$$\text{dom}(F) - \text{dom}(G) = X \times (A(\text{dom}(f)) - \text{dom}(g)).$$

Combining the last relation with the hypotheses, we obtain $(0, 0) \in \text{qi}[(\text{dom}(F) - \text{dom}(G)) - (\text{dom}(F) - \text{dom}(G))]$ and $(0, 0) \in \text{qri}(\text{dom}(F) - \text{dom}(G))$. Since $\text{epi}(F) = \{(x, Ax, r) : f(x) \leq r\}$ and $\widehat{\text{epi}}(G - v(P_A)) = \{(x, y, r) : r \leq -G(x, y) + v(P_A)\} = X \times \widehat{\text{epi}}(g - v(P_A))$, we obtain

$$\text{epi}(F) - \widehat{\text{epi}}(G - v(P_A)) = X \times (A \times \text{id}_{\mathbb{R}}(\text{epi}(f)) - \widehat{\text{epi}}(g - v(P_A))),$$

and this means that $(0, 0, 0) \notin \text{qri}\,\text{co}[(\text{epi}(F) - \widehat{\text{epi}}(G - v(P_A))) \cup \{(0, 0, 0)\}]$. Theorem 3.5 yields for F and G:

$$\inf_{(x,y) \in X \times Y} [F(x, y) + G(x, y)] = \max_{(x^*, y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\}.$$

On the other hand, $F^*(x^*, y^*) = f^*(x^* + A^*y^*), \forall (x^*, y^*) \in X^* \times Y^*$ and

$$G^*(x^*, y^*) = \begin{cases} g^*(y^*), & \text{if } x^* = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore, $\max\limits_{(x^*, y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\} = \max\limits_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\}$ and the conclusion follows. □

Corollary 3.15 *Suppose that the primal problem (P_A) has an optimal solution,* $0 \in \text{qi}[(A(\text{dom}(f)) - \text{dom}(g)) - (A(\text{dom}(f)) - \text{dom}(g))]$, $0 \in \text{qri}(A(\text{dom}(f)) - \text{dom}(g))$ *and* $(0, 0) \notin \text{qri}[A \times \text{id}_{\mathbb{R}}(\text{epi}(f)) - \widehat{\text{epi}}(g - v(P_A))]$. *Then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

Theorem 3.16 *Suppose that* $0 \in \text{qi}(A(\text{dom}(f)) - \text{dom}(g))$ *and* $(0, 0) \notin \text{qri}\,\text{co}[(A \times \text{id}_{\mathbb{R}}(\text{epi}(f)) - \widehat{\text{epi}}(g - v(P_A))) \cup \{(0, 0)\}]$. *Then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

Proof. Considering the functions F and G from the proof of Theorem 3.14, we have $\text{cl}\,\text{cone}(\text{dom}(F) - \text{dom}(G)) = X \times \text{cl}\,\text{cone}(A(\text{dom}(f)) - \text{dom}(g)) = X \times Y$, thus $(0, 0) \in \text{qi}(\text{dom}(F) - \text{dom}(G))$. Also we have $(0, 0, 0) \notin \text{qri}\,\text{co}[(\text{epi}(F) - \widehat{\text{epi}}(G - v(P_A))) \cup \{(0, 0, 0)\}]$. Theorem 3.8 yields for F and G:

$$\inf_{(x,y) \in X \times Y} [F(x, y) + G(x, y)] = \max_{(x^*, y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\}$$

and the conclusion follows. $\square$

Corollary 3.17 *Suppose that the primal problem* (P_A) *has an optimal solution,* $0 \in \mathrm{qi}(A(\mathrm{dom}(f)) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}[A \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f)) - \widehat{\mathrm{epi}}(g - v(P_A))]$. *Then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

Theorem 3.18 *Suppose that* $A(\mathrm{dom}(f)) \cap \mathrm{qri}\,\mathrm{dom}(g) \neq \emptyset$, $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}\,\mathrm{co}[(A \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f)) - \widehat{\mathrm{epi}}(g - v(P_A))) \cup \{(0,0)\}]$. *Then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

Proof. Consider again the functions F and G defined as in the proof of Theorem 3.14. We have $\mathrm{dom}(F) \cap \mathrm{qri}\,\mathrm{dom}(G) = (\mathrm{dom}(f) \times (A(\mathrm{dom}(f))) \cap (X \times \mathrm{qri}\,\mathrm{dom}(g))) = \mathrm{dom}(f) \times (A(\mathrm{dom}(f)) \cap \mathrm{qri}\,\mathrm{dom}(g)) \neq \emptyset$. Also, $\mathrm{cl}\,\mathrm{cone}(\mathrm{dom}(G) - \mathrm{dom}(G)) = X \times \mathrm{cl}\,\mathrm{cone}(\mathrm{dom}(g) - \mathrm{dom}(g)) = X \times Y$, hence $(0,0) \in \mathrm{qi}(\mathrm{dom}(G) - \mathrm{dom}(G))$. Moreover, $(0,0,0) \notin \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}(F) - \widehat{\mathrm{epi}}(G - v(P_A))) \cup \{(0,0,0)\}]$. Theorem 3.10 yields for F and G:

$$\inf_{(x,y) \in X \times Y} [F(x,y) + G(x,y)] = \max_{(x^*,y^*) \in X^* \times Y^*} \{-F^*(-x^*, -y^*) - G^*(x^*, y^*)\}$$

and the conclusion follows. $\square$

Corollary 3.19 *Suppose that the primal problem* (P_A) *has an optimal solution,* $A(\mathrm{dom}(f)) \cap \mathrm{qri}\,\mathrm{dom}(g) \neq \emptyset$, $0 \in \mathrm{qi}(\mathrm{dom}(g) - \mathrm{dom}(g))$ *and* $(0,0) \notin \mathrm{qri}[A \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f)) - \widehat{\mathrm{epi}}(g - v(P_A))]$. *Then* $v(P_A) = v(D_A)$ *and* (D_A) *has an optimal solution.*

4 Lagrange duality

Using an approach due to Magnanti (cf. [13]), in this section we derive from the results we got in the previous section some duality results concerning the Lagrange dual problem. We work in the following setting. Let X be a real linear topological space and S a non-empty subset of X. Let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function $(f,g) : S \to \mathbb{R} \times Y$, defined by $(f,g)(x) = (f(x), g(x)), \forall x \in S$, is convex-like with respect to the cone $\mathbb{R}_+ \times C \subseteq \mathbb{R} \times Y$, that is the set $(f,g)(S) + \mathbb{R}_+ \times C$ is convex. Let us notice that this property implies that the sets $f(S) + [0, \infty)$ and $g(S) + C$ are convex (the reverse implication does not always hold). Consider the optimization problem

$$(P_L) \quad \inf_{\substack{x \in S \\ g(x) \in -C}} f(x),$$

where the constraint set $T = \{x \in S : g(x) \in -C\}$ is assumed to be nonempty. The Lagrange dual problem associated to (P_L) is

$$(D_L) \quad \sup_{\lambda \in C^*} \inf_{x \in S} [f(x) + \langle \lambda, g(x) \rangle],$$

where $C^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0, \forall x \in C\}$ is the *dual cone* of C. Let us denote by $v(P_L)$ and $v(D_L)$ the optimal objective values of the primal and the dual problem, respectively. A regularity condition for strong duality between (P_L) and (D_L) was proposed in [4]. We show first that the main theorem there is incorrect. To this end we prove the following lemma.

Lemma 4.1 *Suppose that* $\text{cl}(C-C) = Y$ *and* $\exists \overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *Then the following assertions are true:*

(a) $0 \in \operatorname{qi}(g(S) + C)$;

(b) $\operatorname{cl} \operatorname{cone}[\operatorname{qri}(g(S) + C)]$ *is a linear subspace of* Y.

Proof. (a) We apply Lemma 2.6 with $A := -C$ and $B := g(S) + C$. We have $0 \in A \cap B$. The condition $\text{cl}(C - C) = Y$ implies $0 \in \operatorname{qi}(A - A)$. The Slater-type condition implies $g(\overline{x}) \in B \cap \operatorname{qri} A$. Hence, by Lemma 2.6 we obtain $0 \in \operatorname{qi}(A - B)$, that is $0 \in \operatorname{qi}(-g(S) - C)$, which is nothing else than $0 \in \operatorname{qi}(g(S) + C)$.

(b) From (a) it follows that $0 \in \operatorname{qri}(g(S) + C)$. Applying Proposition 2.5 (vii) we get $0 \in \operatorname{qri}(\operatorname{qri}(g(S) + C))$, which is nothing else than $\operatorname{cl} \operatorname{cone}[\operatorname{qri}(g(S) + C)]$ is a linear subspace of Y. □

In order to get strong duality between (P_L) and (D_L) in Theorem 2.2 in [4] the authors ask that the following hypotheses are fulfilled: $\text{cl}(C - C) = Y$, $\exists \overline{x} \in S$ such that $g(\overline{x}) \in -\operatorname{qri} C$, $\operatorname{qri}(g(S) + C) \neq \emptyset$ and $\operatorname{cl} \operatorname{cone}[\operatorname{qri}(g(S) + C)]$ is not a linear subspace of Y. The previous lemma proves that this result due to Cammaroto and Di Bella is completely useless, since the hypotheses of this theorem are in contradiction.

Next we prove some Lagrange duality results written in terms of the quasi interior and quasi-relative interior, respectively.

Consider the following convex set

$$\mathcal{E}_{v(P_L)} = \{(f(x) + \alpha - v(P_L), g(x) + y) : x \in S, \alpha \geq 0, y \in C\} \subseteq \mathbb{R} \times Y.$$

Let us notice that the set $-\mathcal{E}_{v(P_L)}$ is in analogy with the *conic extension*, a notion used by Giannessi in the theory of image space analysis (see [7]). One can easily prove that the primal problem (P_L) has an optimal solution if and only if $(0, 0) \in \mathcal{E}_{v(P_L)}$. Let us introduce the functions $f_1, f_2 : \mathbb{R} \times Y \to \overline{\mathbb{R}}$,

$$f_1(y_0, y_1) = \begin{cases} y_0, & \text{if } (y_0, y_1) \in \mathcal{E}_{v(P_L)} + (v(P_L), 0), \\ +\infty, & \text{otherwise.} \end{cases}$$

and $f_2 = \delta_{\mathbb{R}\times(-C)}$. It holds

$$\mathrm{dom}(f_1) - \mathrm{dom}(f_2) = \mathbb{R} \times (g(S) + C). \tag{13}$$

Moreover, as pointed out by Magnanti (cf. [13]), we have

$$\inf_{(y_0,y_1)\in\mathbb{R}\times Y}\{f_1(y_0,y_1) + f_2(y_0,y_1)\} = \inf_{\substack{x\in S\\ g(x)\in -C}} f(x) = v(P_L) \tag{14}$$

and

$$\sup_{(y_0^*,y_1^*)\in\mathbb{R}\times Y^*}\{-f_1^*(-y_0^*,-y_1^*) - f_2^*(y_0^*,y_1^*)\} = \sup_{\lambda\in C^*}\inf_{x\in S}[f(x) + \langle\lambda, g(x)\rangle] = v(D_L). \tag{15}$$

By this approach, we can derive from the strong duality results given for Fenchel duality corresponding strong duality results for Lagrange duality.

Theorem 4.2 *Suppose that* $0 \in \mathrm{qi}[(g(S)+C)-(g(S)+C)]$, $0 \in \mathrm{qri}(g(S)+C)$ *and* $(0,0) \notin \mathrm{qri}\,\mathrm{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$. *Then* $v(P_L) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Proof. The hypotheses of the theorem and (13) imply that the conditions $(0,0) \in \mathrm{qi}[(\mathrm{dom}(f_1)-\mathrm{dom}(f_2))-(\mathrm{dom}(f_1)-\mathrm{dom}(f_2))]$ and $(0,0) \in \mathrm{qri}(\mathrm{dom}(f_1)-\mathrm{dom}(f_2))$ are fulfilled. Further, $\mathrm{epi}(f_1) = \{(y_0,y_1,r) \in \mathbb{R}\times Y\times\mathbb{R} : (y_0,y_1) \in \mathcal{E}_{v(P_L)} + (v(P_L),0), y_0 \leq r\} = \{(f(x)+\alpha, g(x)+y, r) : x \in S, \alpha \geq 0, y \in C, f(x)+\alpha \leq r\}$ and $\widehat{\mathrm{epi}}(f_2-v(P_L)) = \{(y_0,y_1,r) \in \mathbb{R}\times Y\times\mathbb{R} : r \leq -f_2(y_0,y_1)+v(P_L)\} = \{(y_0,y_1,r) \in \mathbb{R}\times Y\times\mathbb{R} : y_0 \in \mathbb{R}, y_1 \in -C, r \leq v(P_L)\} = \mathbb{R}\times(-C)\times(-\infty, v(P_L)]$. Thus $\mathrm{epi}(f_1) - \widehat{\mathrm{epi}}(f_2 - v(P_L)) = \mathrm{epi}(f_1) + \mathbb{R}\times C\times[-v(P_L),+\infty) = \{(f(x)+\alpha+a, g(x)+y, r-v(P_L)+\varepsilon) : x \in S, \alpha \geq 0, a \in \mathbb{R}, y \in C, \varepsilon \geq 0, f(x)+\alpha \leq r\} = \{(f(x)+\alpha+a, g(x)+y, f(x)+\alpha+\varepsilon-v(P_L)) : x \in S, \alpha \geq 0, a \in \mathbb{R}, y \in C, \varepsilon \geq 0\}$ and this means that

$$\mathrm{epi}(f_1)-\widehat{\mathrm{epi}}(f_2-v(P_L)) = \mathbb{R}\times\{(g(x)+y, f(x)+\alpha-v(P_L)) : x \in S, \alpha \geq 0, y \in C\}.$$

Thus $(0,0,0) \in \mathrm{qri}\,\mathrm{co}[(\mathrm{epi}(f_1)-\widehat{\mathrm{epi}}(f_2-v(P_L)))\cup\{(0,0,0)\}]$ if and only if $(0,0) \in \mathrm{qri}\,\mathrm{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$. Now we can apply Theorem 3.5 for f_1 and f_2 and we obtain

$$\inf_{(y_0,y_1)\in\mathbb{R}\times Y}\{f_1(y_0,y_1) + f_2(y_0,y_1)\} = \max_{(y_0^*,y_1^*)\in\mathbb{R}\times Y^*}\{-f_1^*(-y_0^*,-y_1^*) - f_2^*(y_0^*,y_1^*)\}.$$

By (14) and (15) the conclusion follows. □

Corollary 4.3 *Suppose that the primal problem* (P_L) *has an optimal solution,* $0 \in \mathrm{qi}[(g(S)+C)-(g(S)+C)]$, $0 \in \mathrm{qri}(g(S)+C)$ *and* $(0,0) \notin \mathrm{qri}\,\mathcal{E}_{v(P_L)}$. *Then*

$v(P_L) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Further, like for Fenchel duality, other Lagrange duality results can be stated.

Theorem 4.4 *Suppose that* $0 \in \operatorname{qi}(g(S) + C)$ *and* $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$. *Then* $v(P_L) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Proof. This is a direct consequence of the previous theorem since $g(S) + C \subseteq (g(S) + C) - (g(S) + C)$ and so the condition $0 \in \operatorname{qi}(g(S) + C)$ implies $0 \in \operatorname{qi}[(g(S) + C) - (g(S) + C)]$ and $0 \in \operatorname{qri}(g(S) + C)$. □

Corollary 4.5 *Suppose that the primal problem* (P_L) *has an optimal solution,* $0 \in \operatorname{qi}(g(S) + C)$ *and* $(0,0) \notin \operatorname{qri}\mathcal{E}_{v(P_L)}$. *Then* $v(P_L) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Theorem 4.6 *Suppose that* $\operatorname{cl}(C - C) = Y$ *and* $\exists \overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *If* $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$, *then* $v(P_L) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Proof. The condition $(0,0) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(P_L)} \cup \{(0,0)\})$ implies that $(0,0,0) \notin \operatorname{qri}\operatorname{co}[(\operatorname{epi}(f_1) - \widehat{\operatorname{epi}}(f_2 - v(P_L))) \cup \{(0,0,0)\}]$ (cf. the proof of Theorem 4.2). Further, we have $\operatorname{dom}(f_1) \cap \operatorname{qri}\operatorname{dom}(f_2) = [\mathcal{E}_{v(P_L)} + (v(P_L), 0)] \cap \operatorname{qri}(\mathbb{R} \times (-C)) = [\mathcal{E}_{v(P_L)} + (v(P_L), 0)] \cap [\mathbb{R} \times (-\operatorname{qri} C)]$. From the Slater-type condition we get that $(f(\overline{x}), g(\overline{x})) \in [\mathcal{E}_{v(P_L)} + (v(P_L), 0)] \cap [\mathbb{R} \times (-\operatorname{qri} C)]$ hence $\operatorname{dom}(f_1) \cap \operatorname{qri}\operatorname{dom}(f_2) \neq \emptyset$. Moreover, $\operatorname{cl}\operatorname{cone}(\operatorname{dom}(f_2) - \operatorname{dom}(f_2)) = \operatorname{cl}\operatorname{cone}[\mathbb{R} \times (C - C)] = \mathbb{R} \times \operatorname{cl}(C - C) = \mathbb{R} \times Y$, hence $(0,0) \in \operatorname{qi}(\operatorname{dom}(f_2) - \operatorname{dom}(f_2))$. By Theorem 3.10 for f_1 and f_2 we obtain

$$\inf_{(y_0,y_1) \in \mathbb{R} \times Y} \{f_1(y_0, y_1) + f_2(y_0, y_1)\} = \max_{(y_0^*, y_1^*) \in \mathbb{R} \times Y^*} \{-f_1^*(-y_0^*, -y_1^*) - f_2^*(y_0^*, y_1^*)\},$$

and using again (14) and (15) the conclusion follows. □

Corollary 4.7 *Suppose that the primal problem* (P_L) *has an optimal solution,* $\operatorname{cl}(C - C) = Y$ *and* $\exists \overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *If* $(0,0) \notin \operatorname{qri}\mathcal{E}_{v(P_L)}$, *then* $v(P_L) = v(D_L)$ *and* (D_L) *has an optimal solution.*

Remark 4.8 Let us notice that from the above results one can derive duality theorems for the case when in the set of constraints one has also equalities defined by affine functions. Indeed, consider the optimization problem

$$(P_L^{aff}) \quad \inf_{\substack{x \in S \\ g(x) \in -C \\ h(x) = 0}} f(x)$$

where $h : X \to Z$ is an affine mapping and Z is a real normed space (the hypotheses regarding the functions f and g remain the same as in the beginning of this section). The Lagrange dual problem associated to (P_L^{aff}) is

$$(D_L^{aff}) \quad \sup_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} [f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle],$$

where Z^* is the topological dual space of Z.

Using Theorem 4.2 and Theorem 4.4 one can formulate Lagrange duality theorems for (P_L^{aff}) and (D_L^{aff}) by noticing that the primal problem can be reformulated as

$$\inf_{\substack{x \in S \\ g(x) \in -C \\ h(x)=0}} f(x) = \inf_{\substack{x \in S \\ u(x) \in -(C \times \{0\})}} f(x),$$

where $u : S \to Y \times Z$, $u(x) = (g(x), h(x))$. For the optimization problem with equality and cone inequality constraints some regularity conditions have been given in [5] by using the notion of quasi-relative interior. Unfortunately, the strong duality theorem in [5] is not correct. The approach proposed above is offering an alternative for dealing with Lagrange duality for this class of optimization problems.

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