Orthogonal Symmetric Toeplitz Matrices

Albrecht Böttcher

In Memory of Georgii Litvinchuk (1931 - 2006)

Abstract. We show that the number of orthogonal and symmetric Toeplitz matrices of a given order is finite and determine all these matrices. In this way we also obtain a description of the set of all symmetric Toeplitz matrices whose spectrum is a prescribed doubleton.

Mathematics Subject Classification (2000). Primary 47B35; Secondary 15A18, 15A47.

Keywords. Toeplitz matrix, matrix power, orthogonal matrix, symmetric matrix, circulant, skew circulant, inverse eigenvalue problem.

1. Introduction and main result

Georgii Semyonovich Litvinchuk's favorites included singular integral equations with Carleman shifts, that is, with diffeomorphisms α for which the mth iterate α^m is the identity map. My favorites are Toeplitz matrices and hence, when receiving the invitation to contribute to this volume, I thought it might perhaps be an interesting problem to look for all Toeplitz matrices A whose mth power A^m is equal to the identity matrix I.

The spectrum of an infinite Toeplitz matrix A that generates a bounded operator on ℓ^2 is always connected [6]. Consequently, such a matrix satisfies $A^m = I$ if and only if $A = e^{2\pi ij/m}I$ for some $j \in \{0, 1, \ldots, m-1\}$, and thus the question is not interesting for infinite matrices.

So let us consider finite matrices. We denote by $\mathbb{R}^{n\times n}$ the set of all real $n \times n$ matrices and define O_n, S_n, T_n as the subsets of $\mathbb{R}^{n\times n}$ constituted by the orthogonal, symmetric, and Toeplitz matrices, respectively. We also put

$$OS_n = O_n \cap S_n, \quad OT_n = O_n \cap T_n, \quad ST_n = S_n \cap T_n, \quad OST_n = O_n \cap S_n \cap T_n.$$

The number of elements of a finite set E will be denoted by $|E|$. Throughout the paper we assume that $n \geq 2$. The following three simple propositions will be proved in Section 2.

Proposition 1.1. *For $m \geq 2$, there are uncountably many matrices $A \in T_n$ such that $A^m = I$ and $A^k \neq I$ for $1 \leq k \leq m-1$.*

The infinite set we encounter in Proposition 1.1 seems to have no nice description. Well, letting $\omega_m := e^{2\pi i/m}$ and $\mathbb{T}_m := \{1, \omega_m, \ldots, \omega_m^{m-1}\}$, one could describe the set as the set of all diagonalizable Toeplitz matrices with eigenvalues in $\mathbb{T}_m \setminus \cup_{1 \leq k \leq m-1} \mathbb{T}_k$, but this is not what I understand by nice. However, the problem becomes charming when restricting the search to symmetric matrices.

Proposition 1.2. (a) *For $m \geq 3$, there is no $A \in S_n$ such that $A^m = I$ and $A^k \neq I$ for $1 \leq k \leq m-1$.*

(b) *The set of all $A \in S_n$ for which $A^2 = I$ but $A \neq I$ coincides with $OS_n \setminus \{I\}$.*

This result shows that if $A \in ST_n$, then $A^m = I$ can only happen if A is in OST_n, in which case $A^2 = I$. We are thus led to the set $OST_n = T_n \cap OS_n = ST_n \cap OT_n$.

Proposition 1.3. *The sets T_n, OS_n, ST_n, OT_n are all uncountably infinite.*

In spite of this proposition, the following fact, which is our main result, is quite remarkable.

Theorem 1.4. *The set OST_n is finite and*

$$|OST_n| = \begin{cases} 3 \cdot 2^{n/2} - 2 & \text{if } n \text{ is even,} \\ 2\sqrt{2} \cdot 2^{n/2} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

Our proof of Theorem 1.4 is constructive and will yield all matrices in OST_n. Here is a simple consequence of Theorem 1.4 that concerns the inverse eigenvalue problem for Toeplitz matrices.

Corollary 1.5. *Let α and β be two prescribed distinct real numbers. The number of all matrices in ST_n which have both α and β as eigenvalues and no other eigenvalues is $3 \cdot 2^{n/2} - 4$ if n is even and $2\sqrt{2} \cdot 2^{n/2} - 4$ if n is odd.*

The paper is organized as follows. Section 2 contains the proofs of Propositions 1.1 to 1.3. In Section 3 we show that every matrix in OST_n is a circulant or a skew circulant. This is enough to conclude that OST_n is finite. Theorem 1.4 is proved in Section 4, examples revealing the structure of the matrices in OST_n are given in Section 5, and Corollary 1.5 is the subject of Section 6.

2. Proofs of the surrounding results

We denote by $\operatorname{circ}(a_1, \ldots, a_n)$ the $n \times n$ circulant matrix whose first column is $(\, a_1 \, \ldots \, a_n \,)^\top$ and by F_n the $n \times n$ Fourier matrix,

$$F_n = (\omega_n^{(j-1)(k-1)})_{j,k=1}^n = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix},$$

where $\omega_n := e^{2\pi i/n}$. It is well known [2] that the eigenvalues of the circulant $\operatorname{circ}(a_1, \ldots, a_n)$ are $a(1), a(\omega_n), \ldots, a(\omega_n^{n-1})$ where $a(z) := a_1 + a_2 z + \ldots + a_n z^{n-1}$. The equations $a(\omega_n^{j-1}) = \mu_j$ $(j = 1, \ldots, n)$ may be written in the form

$$F_n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} \tag{1}$$

or, equivalently,

$$n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = F_n^* \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}. \tag{2}$$

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. If $n = 2k+2$, there exists a circulant $C \in \mathbb{R}^{n\times n}$ having the eigenvalues $\alpha, \beta, \lambda_1, \ldots, \lambda_k, \overline{\lambda}_1, \ldots, \overline{\lambda}_k$, and if $n = 2k+1$, there exists a circulant $C \in \mathbb{R}^{n\times n}$ whose eigenvalues are $\alpha, \lambda_1, \ldots, \lambda_k, \overline{\lambda}_1, \ldots, \overline{\lambda}_k$.*

Proof. Let first $n = 2k + 2$. We define $a_1, \ldots, a_n$ by (2) with

$$(\mu_1, \ldots, \mu_n) = (\alpha, \lambda_1, \ldots, \lambda_k, \beta, \overline{\lambda}_1, \ldots, \overline{\lambda}_k).$$

It remains to show that $a_1, \ldots, a_n$ are real. But, for $j = 0, 1, \ldots, n-1$,

$$n a_{j+1} = \alpha + \overline{\omega}_n^{j(k+1)}\beta + \sum_{p=1}^{k} \left(\overline{\omega}_n^{jp}\lambda_p + \overline{\omega}_n^{j(n-p)}\overline{\lambda}_p \right) = \alpha - \beta + \sum_{p=1}^{k} 2\,\mathrm{Re}(\overline{\omega}_n^{jp}\lambda_p)$$

and this is a real number. If $n = 2k + 1$, we take

$$(\mu_1, \ldots, \mu_n) = (\alpha, \lambda_1, \ldots, \lambda_k, \overline{\lambda}_1, \ldots, \overline{\lambda}_k)$$

and define $a_1, \ldots, a_n$ by (2). The same argument as before shows that these are real numbers. □

Proof of Proposition 1.1. If $n \geq 3$, Lemma 2.1 yields the existence of a circulant $C \in \mathbb{R}^{n\times n}$ with the eigenvalues $\omega_m, \overline{\omega}_m, 1, \ldots, 1$. It follows that $C^m = I$ and $C^k \neq I$ for $1 \leq k \leq m-1$. For $\mu \in \mathbb{R} \setminus \{0\}$, put $D_\mu = \operatorname{diag}(1, \mu, \ldots, \mu^{n-1})$. Then $A = D_\mu C D_\mu^{-1}$ is a matrix in T_n for which $A^m = I$ and $A^k \neq I$ for $1 \leq k \leq m-1$.

As different μ's produce different A's we get the assertion. We are left with the case $n = 2$. The eigenvalues of

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in T_2$$

are $a \pm \sqrt{bc}$. Letting a, b, c be any real numbers such that $a = \cos(2\pi/m)$ and $bc = -\sin^2(2\pi/m)$, we obtain uncountably many matrices A with the eigenvalues ω_m and $\overline{\omega}_m$ and thus with the desired property that $A^m = I$ and $A^k \neq I$ for $1 \le k \le m-1$. □

Proof of Proposition 1.2. (a) Let $m \ge 3$ and $A \in S_n$. Then $A = U^\top D U$ with an orthogonal matrix U and a diagonal matrix D which contains all eigenvalues of A. If $A^m = I$ then $\operatorname{sp} A$, the set of the eigenvalues of A, is contained in $\mathbb{T}_m \cap \mathbb{R}$. Since $\mathbb{T}_m \cap \mathbb{R} = \{1\}$ for odd m, we see that $A = I$ in this case, which contradicts the requirement that $A^k \neq I$ for $1 \le k \le m-1$. If m is even, we have $\mathbb{T}_m \cap \mathbb{R} = \{-1, 1\}$ and hence $A^2 = I$. As necessarily $m \ge 4$, this is again a contradiction to the requirement that $A^k \neq I$ for $1 \le k \le m-1$.

(b) Let $A \in S_n \setminus \{I\}$ and $A^2 = I$. Write $A = U^\top D U$ as above. The entries of D are all -1 or 1 and hence D is orthogonal. This implies that A is orthogonal, too. Conversely, if $A \in OS_n \setminus \{I\}$ then $\operatorname{sp} A \subset \mathbb{T} \cap \mathbb{R} = \{-1, 1\}$, which shows that each diagonal entry of D is -1 or 1. It follows that $A^2 = I$. □

Proof of Proposition 1.3. The assertion is trivial for T_n and ST_n. Since each matrix of the form $U^\top D U$ with $U \in O_n$ and a diagonal matrix with $\operatorname{sp} D \subset \{-1, 1\}$ is in OS_n, we see that OS_n is uncountably infinite. Lemma 2.1 tells us that if $\tau \in \mathbb{T}$ is any given number, then there is a circulant $C \in \mathbb{R}^{n \times n}$ $(n \ge 3)$ having the eigenvalues $\tau, \overline{\tau}, 1, \ldots, 1$. Since $C \in OT_n$, we conclude that OT_n is uncountably infinite for $n \ge 3$. Finally, the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in T_2$$

is orthogonal whenever $a^2 + b^2 = 1$. This shows that OT_2 is an uncountable set. □

Remark 2.2. Let UHT_n be the set of all unitary Hermitian Toeplitz matrices in $\mathbb{C}^{n \times n}$. The symmetric Toeplitz matrix B with the top row

$$\begin{pmatrix} \frac{n-2}{n} & -\frac{2}{n} & \ldots & -\frac{2}{n} \end{pmatrix}$$

belongs to OST_n. Let $D_\mu = \operatorname{diag}(1, \mu, \ldots, \mu^{n-1})$. If $\mu \in \mathbb{T}$, then D_μ is unitary and hence $D_\mu B D_\mu^{-1}$ is in UHT_n. This shows that UHT_n is uncountably infinite. The matrices in OST_2 are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and it is easily seen that UHT_2 equals

$$\left\{ \begin{pmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{pmatrix} : \mu \in \mathbb{T} \right\} \cup \left\{ \begin{pmatrix} a & i\sqrt{1-a^2} \\ -i\sqrt{1-a^2} & a \end{pmatrix} : a \in [-1, 1] \setminus \{0\} \right\}.$$

This reveals that not every matrix in UHT_2 is of the form $D_\mu A D_\mu^{-1}$ with $A \in OST_n$ and $\mu \in \mathbb{T}$.

3. The set OST_n is finite

We denote by $T(a_1, \ldots, a_n)$ the symmetric Toeplitz matrix whose top row is $(\, a_1 \, \ldots \, a_n \,)$. Let $A = T(a_1, \ldots, a_n)$ be a matrix in OST_n.

Suppose first that $n = 2k+1$ is odd. Consider, for example,

$$T(a,b,c,d,e) = \begin{pmatrix} a & b & c & d & e \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b \\ e & d & c & b & a \end{pmatrix}.$$

Since each row has ℓ^2 norm 1, it follows that $e^2 = b^2$ and $d^2 = c^2$. Thus, $d = \delta c$ and $e = \varepsilon b$ with $\varepsilon, \delta \in \{-1, 1\}$ and therefore

$$T(a,b,c,d,e) = T(a,b,c,\delta c, \varepsilon b).$$

In the general case we see in this way that $T(a_1, \ldots, a_n)$ must be of the form

$$A = T(a_1, \ldots, a_n) = T(a, b_1, \ldots, b_k, \varepsilon_k b_k, \ldots, \varepsilon_1 b_1) \tag{3}$$

with $\varepsilon_j \in \{-1, 1\}$. If $\varepsilon_1 = \ldots = \varepsilon_k = 1$, then (3) is a circulant. For instance,

$$T(a,b,c,c,b) = \begin{pmatrix} a & b & c & c & b \\ b & a & b & c & c \\ c & b & a & b & c \\ c & c & b & a & b \\ b & c & c & b & a \end{pmatrix}$$

is a circulant. If $\varepsilon_j = -1$ and $b_j = 0$, we change ε_j to 1. This does not change A. Thus, we may assume that $b_j \neq 0$ whenever $\varepsilon_j = -1$.

Lemma 3.1. *If $-1 \in \{\varepsilon_1, \ldots, \varepsilon_k\}$, then $b_j = 0$ whenever $\varepsilon_j = 1$.*

Proof. We denote the rows of A by $r_1, \ldots, r_{2k+1}$. Since A is orthogonal, the scalar product (r_i, r_j) is zero for $i \neq j$. Let $\varepsilon_i = -1$ and thus $b_i \neq 0$ for some i.

Step 1. We show that if $i \geq 2$ and $\varepsilon_{i-1} = 1$, then $b_{i-1} = 0$. Let us first consider the example

$$A = T(a,b,c,d,e,f,-f,e,d,c,-b) \tag{4}$$

in which $\varepsilon_1 = -1, \varepsilon_2 = 1$, $\varepsilon_3 = 1$, $\varepsilon_4 = 1$, $\varepsilon_5 = -1$, $f \neq 0$, $b \neq 0$. We have $i = 5$ and we want to show that $e = 0$. The rows 4 to 6 of (4) are

$$\begin{pmatrix} r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} d & c & b & a & b & c & d & e & f & -f & e \\ e & d & c & b & a & b & c & d & e & f & -f \\ f & e & d & c & b & a & b & c & d & e & f \end{pmatrix}.$$

Thus,

$$0 = (r_4, r_5) = (de + \ldots + (-f)f) + e(-f),$$
$$0 = (r_5, r_6) = (de + \ldots + (-f)f) + ef,$$

which gives $e = 0$ as desired because $f \neq 0$. In the general case we have

$$0 = (r_{i-1}, r_i) = \Sigma + b_{i-1}(-b_i), \quad 0 = (r_i, r_{i+1}) = \Sigma + b_{i-1}b_i,$$

which implies that $b_{i-1} = 0$. Analogously one can show that $b_{i+1} = 0$ if $i \leq k-2$ and $\varepsilon_{i+1} = 1$.

Step 2. Suppose $i \geq 3$ and consider b_{i-2}. If $\varepsilon_{i-2} = -1$, we have nothing to prove. So assume that $\varepsilon_{i-2} = 1$. We want to show that $b_{i-2} = 0$. If $\varepsilon_{i-1} = -1$, then $b_{i-2} = 0$ by Step 1. Thus, let $\varepsilon_{i-1} = 1$. From Step 1 we know that $b_{i-1} = 0$. Example (4) for $i = 5$ illustrates just this situation. We have (with $e = 0$)

$$\begin{pmatrix} r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} c & b & a & b & c & d & 0 & f & -f & 0 & d \\ d & c & b & a & b & c & d & 0 & f & -f & 0 \\ 0 & d & c & b & a & b & c & d & 0 & f & -f \\ f & 0 & d & c & b & a & b & c & d & 0 & f \end{pmatrix}$$

and hence

$$0 = (r_3, r_5) = (bd + \ldots + fd) + d(-f),$$
$$0 = (r_4, r_6) = (bd + \ldots + fd) + df,$$

which yields $d = 0$ as desired. In the general case,

$$0 = (r_{i-2}, r_i) = \Sigma + b_{i-2}(-b_i), \quad 0 = (r_{i-1}, r_{i+1}) = \Sigma + b_{i-2}b_i$$

and thus $b_{i-2} = 0$. Similarly one gets $b_{i+2} = 0$ if $i \leq k-3$ and $\varepsilon_{i+2} = 1$.

Step 3. Continuing as above we see that $b_{i\pm\ell} = 0$ whenever $i \pm \ell \in \{1, \ldots, k\}$ and $\varepsilon_{i\pm\ell} = -1$. $\square$

Lemma 3.1 implies that if at least one of the numbers $\varepsilon_1, \ldots, \varepsilon_k$ is -1, then A is a skew circulant, that is, a matrix that results from a circulant by multiplying all entries below the main diagonal by -1. For example,

$$T(a, 0, c, -c, 0) = \begin{pmatrix} a & 0 & c & -c & 0 \\ 0 & a & 0 & c & -c \\ c & 0 & a & 0 & c \\ -c & c & 0 & a & 0 \\ 0 & -c & c & 0 & a \end{pmatrix}$$

is a skew circulant. Let $\operatorname{scirc}(a_1, \ldots, a_n)$ be the skew circulant whose first column is $(\, a_1 \ \ldots \ a_n \,)^\top$. Thus,

$$\operatorname{scirc}(a_1, a_2, a_3) = \begin{pmatrix} a_1 & -a_3 & -a_2 \\ a_2 & a_1 & -a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}.$$

The eigenvalues of $\mathrm{scirc}\,(a_1, \ldots, a_n)$ are $a(\sigma_n)$, $a(\sigma_n\omega_n)$, $\ldots$, $a(\sigma_n\omega_n^{n-1})$ where $\sigma_n = e^{\pi i/n}$ and ω_n and $a(z)$ are as above (see [2]). The equations $a(\sigma_n\omega_n^{j-1}) = \mu_j$ $(j = 1, \ldots, n)$ now take the form

$$F_n \begin{pmatrix} a_1 \\ \sigma_n a_2 \\ \vdots \\ \sigma_n^{n-1} a_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}, \tag{5}$$

which is equivalent to

$$n \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = F_n^* \begin{pmatrix} \mu_1 \\ \overline{\sigma}_n \mu_2 \\ \vdots \\ \overline{\sigma}_n^{n-1} \mu_n \end{pmatrix}. \tag{6}$$

At this point we are in a position to prove the following weakened version of Theorem 1.4.

Proposition 3.2. *If n is odd, then OST_n is a finite set and $|OST_n| \le 2^{n+1} - 2$.*

Proof. We have seen that a matrix A in OST_n is a circulant or a skew circulant. The eigenvalues of A belong to $\{-1, 1\}$. Thus, if $A = \mathrm{circ}\,(a_1, \ldots, a_n)$, then $a_1, \ldots, a_n$ are given by (2) with each μ_j being -1 or 1, and if $A = \mathrm{scirc}\,(a_1, \ldots, a_n)$, then $a_1, \ldots, a_n$ are determined by (6) with each μ_j in $\{-1, 1\}$. The two matrices I and $-I$ are counted both as a circulant and as a skew circulant. This gives at most $2 \cdot 2^n - 2$ solutions. (That $|OST_n|$ will actually turn out to be much smaller than $2^{n+1} - 2$ is due to the circumstance that not every right-hand side of (2) and (6) with $\mu_j \in \{-1, 1\}$ gives a left-hand side with real numbers.) $\square$

Now suppose $n = 2k + 2 \ge 2$ is even. The case $k = 0$ was disposed of in Remark 2.2, where we observed that OST_2 consists of exactly four matrices. So let $k \ge 1$. We denote the rows of A by $r_1, \ldots, r_{2k+2}$. Consideration of the scalar products $(r_j, r_j) = 1$ shows that A is of the form

$$A = T(a_1, \ldots, a_n) = T(a, b_1, \ldots, b_k, c, b_k, \ldots, b_1)$$

with $\varepsilon_j \in \{-1, 1\}$. Again we will assume without loss of generality that $\varepsilon_j = 1$ if $b_j = 0$. If all ε_j are 1, then A is a circulant. For example,

$$T(a, b, d, c, d, b) = \begin{pmatrix} a & b & d & c & d & b \\ b & a & b & d & c & d \\ d & b & a & b & d & c \\ c & d & b & a & b & d \\ d & c & d & b & a & b \\ b & d & c & d & b & a \end{pmatrix}$$

is a circulant. Thus, assume there is a -1 among $\varepsilon_1, \ldots, \varepsilon_k$.

Lemma 3.3. *Let $-1 \in \{\varepsilon_1, \ldots, \varepsilon_k\}$. Then $c = 0$ and $b_j = 0$ whenever $\varepsilon_j = 1$.*

Proof. It can be shown as in the proof of Lemma 3.1 that $b_j = 0$ if $\varepsilon_j = 1$. Let us prove that $c = 0$. Again we first do an example. If

$$A = T(a, 0, d, c, -d, 0) = \begin{pmatrix} a & 0 & d & c & -d & 0 \\ 0 & a & 0 & d & c & -d \\ d & 0 & a & 0 & d & c \\ c & d & 0 & a & 0 & d \\ -d & c & d & 0 & a & 0 \\ b & -d & c & d & 0 & a \end{pmatrix},$$

with $d \neq 0$, then $0 = (r_3, r_4) = 2cd = 0$ gives $c = 0$. In the general case the argument is as follows. If $\varepsilon_k = -1$ and hence $b_k \neq 0$, then

$$(r_k, r_{k+1}) = (r_{k+1}, r_{k+2}) - 2cb_k,$$

and since $(r_k, r_{k+1}) = (r_{k+1}, r_{k+2}) = 0$, it follows that $c = 0$. So let $\varepsilon_k = 1$ and thus $b_k = 0$. If $\varepsilon_{k-1} = -1$, then $b_{k-1} \neq 0$ and

$$(r_{k-1}, r_{k+1}) = (r_k, r_{k+2}) - 2cb_{k-1},$$

which gives $c = 0$ as before. If $\varepsilon_k = 1$, $b_k = 0$, $\varepsilon_{k-1} = 1$, $b_{k-1} = 0$, $\varepsilon_{k-2} = -1$, $b_{k-2} \neq 0$ we have

$$(r_{k-2}.r_{k+1}) = (r_{k-1}, r_{k+2}) - 2cb_{k-2}$$

and so on. Eventually we get $c = 0$. □

Lemma 3.3 reveals that if $-1 \in \{\varepsilon_1, \dots, \varepsilon_k\}$, then A is a skew circulant. For example,

$$T(a, b, d, 0, -d, -b) = \begin{pmatrix} a & b & d & 0 & -d & -b \\ b & a & b & d & 0 & -d \\ d & b & a & b & d & 0 \\ 0 & d & b & a & b & d \\ -d & 0 & d & b & a & b \\ -b & -d & 0 & d & b & a \end{pmatrix}$$

is a skew circulant.

Proposition 3.4. *If n is even, then OST_n is finite and $|OST_n| \leq 2^{n+1} - 2$.*

Proof. Proceed as in the proof of Proposition 3.2. □

4. The matrices in OST_n

In this section we prove Theorem 1.4.

Let first $n = 2k + 1 \geq 3$ be odd. By the results of Section 3, a matrix A is in OST_n if and only if $\operatorname{sp} A \in \{-1, 1\}$ and A is a circulant of the form

$$A = \operatorname{circ}(a_1, \dots, a_n) = \operatorname{circ}(a, b_1, \dots, b_k, b_k, \dots, b_1) \tag{7}$$

or a skew circulant of the form

$$A = \operatorname{scirc}(a_1, \dots, a_n) = \operatorname{scirc}(a, b_1, \dots, b_k, -b_k, \dots, -b_1) \tag{8}$$

Suppose A is the circulant (7). We denote the right-hand side of (1) by

$$(\,\gamma\,\varepsilon_1\ \dots\ \varepsilon_k\,\delta_k\ \dots\ \delta_1\,)^\top.$$

The numbers $\gamma, \varepsilon_j, \delta_j$ are all -1 or 1. Abbreviating ω_n to ω we then get from (1) that

$$\varepsilon_j = a + \sum_{p=1}^{k} b_p \left(\omega^{pj} + \omega^{(n-p)j}\right) = a + \sum_{k=1}^{p} b_p \left(\omega^{pj} + \omega^{-pj}\right),$$

$$\delta_j = a + \sum_{p=1}^{k} b_p \left(\omega^{p(n-j)} + \omega^{(n-p)(n-j)}\right) = a + \sum_{k=1}^{p} b_p \left(\omega^{-pj} + \omega^{pj}\right),$$

whence $\varepsilon_j = \delta_j$ for $1 \le j \le k$. Conversely, choose $\gamma, \varepsilon_1, \dots, \varepsilon_k$ in $\{-1, 1\}$, insert

$$(\mu_1, \dots, \mu_n) = (\gamma, \varepsilon_1, \dots, \varepsilon_k, \varepsilon_k, \dots, \varepsilon_1)$$

in (2), denote the column on the left of (2) by

$$(\,a\,b_1\ \dots\ b_k\,c_k\ \dots c_1\,)^\top,$$

and pass to complex conjugates. Clearly, $a = \gamma + 2\sum \varepsilon_j$ is real. Furthermore,

$$n\overline{b}_j = \gamma + \sum_{p=1}^{k} \varepsilon_p \left(\omega^{pj} + \omega^{j(n-p)}\right) = \gamma + \sum_{p=1}^{k} \varepsilon_p \left(\omega^{pj} + \omega^{-pj}\right),$$

$$n\overline{c}_j = \gamma + \sum_{p=1}^{k} \varepsilon_p \left(\omega^{p(n-j)} + \omega^{(n-j)(n-p)}\right) = \gamma + \sum_{p=1}^{k} \varepsilon_p \left(\omega^{-pj} + \omega^{pj}\right).$$

It follows that $b_j = c_j$ for $1 \le j \le k$, and since $\omega^{pj} + \omega^{-pj}$ is real, the numbers b_j are also real. As there are exactly 2^{k+1} choices of $\gamma, \varepsilon_1, \dots, \varepsilon_k$ in $\{-1, 1\}^k$ and as different right-hand sides of (1) give different $(a_1, \dots, a_n)$, we conclude that OST_n contains exactly 2^{k+1} matrices of the form (7).

Now suppose A is of the form (8). Put $\sigma := \sigma_n$ and notice that

$$(1, \sigma, \dots, \sigma^n) = (1, \sigma, \dots, \sigma^k, -\sigma^{k+1}, \dots, -\sigma^{2k}).$$

We denote the columns on the left and right of (5) by

$$(\,a\ \ \sigma b_1\ \dots\ \sigma^k b_k\ \ -\sigma^{k+1} b_k\ \dots\ -\sigma^{2k} b_1\,)^\top \quad \text{and} \quad (\,\varepsilon_1\ \dots \varepsilon_k\,\gamma\,\delta_k\ \dots\ \delta_1\,)^\top,$$

respectively. The numbers $\gamma, \varepsilon_j, \delta_j$ are all in $\{-1, 1\}$. From (5) we infer that

$$\varepsilon_j = a + \sum_{p=1}^{k} b_p \left(\omega^{(j-1)p}\sigma^p - \omega^{(j-1)(n-p)}\sigma^{2k+1-p}\right),$$

$$\delta_j = a + \sum_{p=1}^{k} b_p \left(\omega^{(n-j)p}\sigma^p - \omega^{(n-j)(n-p)}\sigma^{2k+1-p}\right).$$

Since $\omega^n = 1$, $\sigma^{2k+1} = -1$, $\omega = \sigma^2$, we have

$$\omega^{(j-1)p}\sigma^p = \sigma^{2jp-p}, \quad -\omega^{(j-1)(n-p)}\sigma^{2k+1-p} = \omega^{-p(j-1)}\sigma^{-p} = \sigma^{p-2jp},$$
$$\omega^{(n-j)p}\sigma^p = \omega^{-jp}\sigma^p = \sigma^{p-2jp}, \quad -\omega^{(n-j)(n-p)}\sigma^{2k+1-p} = \omega^{jp}\sigma^{-p} = \sigma^{2jp-p},$$

which shows that $\varepsilon_j = \delta_j$ for $1 \le j \le k$. Conversely, choose $\gamma, \varepsilon_1, \dots, \varepsilon_k$ in $\{-1,1\}$, insert

$$(\mu_1, \dots, \mu_n) = (\varepsilon_1, \dots, \varepsilon_k, \gamma, \varepsilon_k, \dots, \varepsilon_1),$$

in (6), write $(a_1, \dots, a_n)$ as $(a, b_1, \dots, b_k, c_k, \dots, c_1)$, and pass to complex conjugates. It follows that $a = \gamma + 2\sum \varepsilon_k$ is real and that

$$n\overline{b}_j = \sigma^j\omega^{jk}\gamma + \sum_{p=1}^{k} \varepsilon_p \left(\sigma^j\omega^{j(p-1)} + \sigma^j\omega^{j(n-p)}\right),$$

$$n\overline{c}_j = \sigma^{n-j}\omega^{(n-j)k}\gamma + \sum_{p=1}^{k} \varepsilon_p \left(\sigma^{n-j}\omega^{(n-j)(p-1)} + \sigma^{n-j}\omega^{(n-j)(n-p)}\right).$$

Taking into account that $\omega^n = 1$, $\sigma^{2k+1} = -1$, $\omega = \sigma^2$, we see as above that $c_j = -b_j$ for $1 \le j \le k$ and that $b_1, \dots, b_k$ are real. Thus, what we obtained is a real matrix of the form (8). This proves that there are exactly 2^{k+1} matrices of the form (8) in OST_n.

The matrix A is both of the form (7) and of the form (8) if and only if $b_1 = \ldots = b_k = 0$, that is, if and only if $A = I$ or $A = -I$. Thus, in summary the number of matrices in OST_n is

$$2 \cdot 2^{k+1} - 2 = 2 \cdot 2^{(n+1)/2} - 2 = 2\sqrt{2} \cdot 2^{n/2} - 2.$$

In addition, we have proved the following.

Theorem 4.1. *Let $n = 2k+1 \ge 3$. The set OST_n consists of the 2^{k+1} circulants $T(a_1, \dots, a_n)$ which are given by* (2) *with $(\mu_1, \dots, \mu_n) = (\gamma, \varepsilon_1, \dots, \varepsilon_k, \varepsilon_k, \dots, \varepsilon_1)$ and*

$$(\gamma, \varepsilon_1, \dots, \varepsilon_k) \in \{-1,1\}^{k+1}$$

and the $2^{k+1}-2$ skew circulants $T(a_1, \dots, a_n)$ which result from (6) *with the choice $(\mu_1, \dots, \mu_n) = (\varepsilon_1, \dots, \varepsilon_k, \gamma, \varepsilon_k, \dots, \varepsilon_1)$ and*

$$(\varepsilon_1, \dots, \varepsilon_k, \gamma) \in \{-1,1\}^{k+1} \setminus \{(1,1,\dots,1), (-1,-1,\dots,-1)\}.$$

Now suppose $n = 2k+2 \ge 4$ is even. We know from Section 3 that A belongs to OST_n if and only if $\operatorname{sp} A \subset \{-1,1\}$ and A is a circulant of the form

$$A = \operatorname{circ}(a_1, \dots, a_n) = \operatorname{circ}(a, b_1, \dots, b_k, c, b_k, \dots, b_1) \tag{9}$$

or a skew circulant of the form

$$A = \operatorname{scirc}(a_1, \dots, a_n) = \operatorname{scirc}(a, b_1, \dots, b_k, 0, -b_k, \dots, -b_1) \tag{10}$$

Proceeding as in the case where n is odd, we see that if A is given by (9), then the right-hand side of (1) is

$$(\mu_1, \dots, \mu_n) = (\gamma, \varepsilon_1, \dots, \varepsilon_k, \beta, \varepsilon_k, \dots, \varepsilon_1)$$

with $\gamma, \beta, \varepsilon_j \in \{-1, 1\}$ and that different such right-hand sides produce different real matrices of the form (9) via (2). Analogously, a matrix of the form (10) yields

$$(\mu_1, \dots, \mu_n) = (\varepsilon_1, \dots, \varepsilon_k, \gamma, \beta, \varepsilon_k, \dots, \varepsilon_1) \tag{11}$$

with $\gamma, \beta, \varepsilon_j \in \{-1, 1\}$ in (5). If we insert such a right-hand side in (6), we obtain a left-hand side with

$$(a_1, \dots, a_n) = (a, b_1, \dots, b_k, d, c_k, \dots, c_1).$$

The numbers $b_1, \dots, b_k$ and $c_1, \dots, c_k$ are real if and only if $\gamma = \beta$. (This is not obvious but requires a computation, which, however, is straightforward.) If $\gamma = \beta$, then $c_j = -b_j$ for all j and $d = 0$, that is, we get indeed a real matrix of the form (10). Different choices of (11) with $\gamma = \beta$ lead to different left-hand sides of (9), and $A = I$ and $A = -I$ are the only matrices that are both of the form (9) and the form (10). Thus, we arrive at the conclusion that the number of matrices in OST_n is

$$2^{k+2} + 2^{k+1} - 2 = 3 \cdot 2^{k+1} - 2 = 3 \cdot 2^{n/2} - 2$$

and that, moreover, OST_n can be described as follows.

Theorem 4.2. *Let $n = 2k + 2 \geq 4$. The set OST_n consists of the 2^{k+2} circulants $T(a_1, \dots, a_n)$ which result from (2) with $(\mu_1, \dots, \mu_n) = (\gamma, \varepsilon_1, \dots, \varepsilon_k, \beta, \varepsilon_k, \dots, \varepsilon_1)$ and*

$$(\gamma, \varepsilon_1, \dots, \varepsilon_k, \beta) \in \{-1, 1\}^{k+2}$$

and the $2^{k+1} - 2$ skew circulants $T(a_1, \dots, a_n)$ which are obtained from (6) with $(\mu_1, \dots, \mu_n) = (\varepsilon_1, \dots, \varepsilon_k, \beta, \beta, \varepsilon_k, \dots, \varepsilon_1)$ and

$$(\varepsilon_1, \dots, \varepsilon_k, \beta) \in \{-1, 1\}^{k+1} \setminus \{(1, 1, \dots, 1), (-1, -1, \dots, -1)\}.$$

5. Examples

***OST_7*.** By Theorem 1.4, $|OST_7| = 30$. Theorems 4.1 and 4.2 in conjunction with some obvious simplifications provide us with all matrices in OST_7. Formula (2) with

$$(\mu_1, \dots, \mu_7) = (\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3, \varepsilon_2, \varepsilon_1)$$

and $(\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{-1, 1\}^4$ yields the 16 matrices

$$\operatorname{circ}(a, b_1, b_2, b_3, b_3, b_2, b_1) = T(a, b_1, b_2, b_3, b_3, b_2, b_1)$$

with

$$\begin{aligned}
7a &= \gamma + 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3, \\
7b_1 &= \gamma + 2\varepsilon_1 \cos\frac{2\pi}{7} + 2\varepsilon_2 \cos\frac{4\pi}{7} + 2\varepsilon_3 \cos\frac{6\pi}{7}, \\
7b_2 &= \gamma + 2\varepsilon_1 \cos\frac{4\pi}{7} + 2\varepsilon_2 \cos\frac{6\pi}{7} + 2\varepsilon_3 \cos\frac{2\pi}{7}, \\
7b_3 &= \gamma + 2\varepsilon_1 \cos\frac{6\pi}{7} + 2\varepsilon_2 \cos\frac{2\pi}{7} + 2\varepsilon_3 \cos\frac{4\pi}{7}.
\end{aligned}$$

The remaining 14 matrices are delivered by (6) with

$$(\mu_1, \ldots, \mu_7) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma, \varepsilon_3, \varepsilon_2, \varepsilon_1)$$

and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma) \in \{-1, 1\}^4 \setminus \{(1, 1, 1, 1), (-1, -1, -1, -1)\}$. These matrices are

$$\operatorname{scirc}(a, b_1, b_2, b_3, -b_3, -b_2, -b_1) = T(a, b_1, b_2, b_3, -b_3, -b_2, -b_1)$$

with

$$\begin{aligned}
7a &= \gamma + 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3,\\
7b_1 &= -\gamma + 2\varepsilon_1 \cos\frac{\pi}{7} + 2\varepsilon_2 \cos\frac{3\pi}{7} - 2\varepsilon_3 \cos\frac{2\pi}{7},\\
7b_2 &= \gamma + 2\varepsilon_1 \cos\frac{2\pi}{7} - 2\varepsilon_2 \cos\frac{\pi}{7} - 2\varepsilon_3 \cos\frac{3\pi}{7},\\
7b_3 &= -\gamma + 2\varepsilon_1 \cos\frac{3\pi}{7} - 2\varepsilon_2 \cos\frac{2\pi}{7} + 2\varepsilon_3 \cos\frac{\pi}{7}.
\end{aligned}$$

This "parametrization" of OST_7 indicates that our approach perfectly fits with the problem and that any attempt to tackle the problem straightforwardly seems to be a hopeless venture. For instance, a matrix A of the form $T(a, b, c, d, d, c, b)$ is orthogonal if and only if

$$\begin{aligned}
a^2 + 2b^2 + 2c^2 + 2d^2 &= 1,\\
2ab + 2bc + 2cd + d^2 &= 0,\\
2ac + b^2 + 2bd + 2cd &= 0,\\
2ad + 2bc + 2bd + c^2 &= 0.
\end{aligned}$$

(Incidentally, the same system is produced when applying the results of Gu and Patton [4] to the equation $A^\top A - I \cdot I = 0$.) Thus, we arrived at four equations with four unknowns. This makes us expect that the set of solutions is finite. However, proving that the system has indeed only finitely many solution and even finding the solutions is another story. The skeptical reader is invited to try and solve this system "directly" and to find the 16 solutions listed above.

***OST*₂.** We have $|OST_2| = 4$ and $OST_2 = \{T(\varepsilon, 0), T(0, \varepsilon) : \varepsilon \in \{-1, 1\}\}$.

***OST*₃.** The 6 matrices in the set OST_3 are $T(\varepsilon, 0, 0)$ and $T\left(\frac{\varepsilon}{3}, \frac{2\delta}{3}, -\frac{2\varepsilon}{3}\right)$ with $\varepsilon, \delta \in \{-1, 1\}$.

***OST*₄.** The set OST_4 consists of the 10 matrices

$$T(\varepsilon, 0, 0, 0), \quad T(0, 0, \varepsilon, 0), \quad T\left(\frac{\varepsilon}{2}, \frac{\delta}{2}, -\frac{\varepsilon}{2}, \frac{\delta}{2}\right), \quad T\left(0, \frac{\varepsilon}{\sqrt{2}}, 0, -\frac{\varepsilon}{\sqrt{2}}\right)$$

with $\varepsilon, \delta \in \{-1, 1\}$.

***OST*₅.** A list of the 14 matrices in OST_5 is

$$T(\varepsilon,0,0,0,0), \quad T\left(\frac{3\varepsilon}{5},\frac{2\delta}{5},-\frac{2\varepsilon}{5},\frac{2\delta}{5},-\frac{2\varepsilon}{5}\right),$$
$$T\left(\frac{1}{5}\varepsilon,\frac{1+\gamma\sqrt{5}}{5}\delta,\frac{1-\gamma\sqrt{5}}{5}\varepsilon,\frac{1-\gamma\sqrt{5}}{5}\delta,\frac{1+\gamma\sqrt{5}}{5}\varepsilon\right)$$

with $\varepsilon,\delta,\gamma \in \{-1,1\}$.

***OST*₆.** The set OST_6 is constituted by the 22 matrices

$$T(\varepsilon,0,0,0,0,0), \quad T(0,0,0,\varepsilon,0,0),$$
$$T\left(0,-\frac{2\varepsilon}{3},0,\frac{\varepsilon}{3},0,-\frac{2\varepsilon}{3}\right), \quad T\left(\frac{\varepsilon}{3},0,\frac{2\delta}{3},0,-\frac{2\varepsilon}{3},0\right),$$
$$T\left(\frac{\varepsilon}{3},\frac{\delta}{3},\frac{\varepsilon}{3},-\frac{2\delta}{3},\frac{\varepsilon}{3},\frac{\delta}{3}\right), \quad T\left(-\frac{2\varepsilon}{3},\frac{\delta}{3},\frac{\varepsilon}{3},\frac{\delta}{3},\frac{\varepsilon}{3},\frac{\delta}{3}\right),$$
$$T\left(\frac{\varepsilon}{3},\frac{\delta}{\sqrt{3}},-\frac{\varepsilon}{3},0,\frac{\varepsilon}{3},-\frac{\delta}{\sqrt{3}}\right)$$

with $\varepsilon,\delta \in \{-1,1\}$.

6. An inverse eigenvalue problem

The inverse eigenvalue problem for symmetric Toeplitz matrices consists in finding all $A \in ST_n$ for which $\operatorname{sp} A$ is a prescribed set of at most n points on $\mathbb{R}$. The pioneering work on this problem is due to Delsarte and Genin [3], who in particular solved the problem for $n \le 4$. Henry Landau [5] proved that if $E \subset \mathbb{R}$ is any set with $|E| \le n$, then there exists a matrix $A \in ST_n$ with $\operatorname{sp} A = E$. In connection with the problem studied in the present paper, we mention the paper [1] by Chu and Erbrecht, which deals with the question of finding all matrices in ST_n that have two prescribed pairs of eigenvalues and eigenvectors. The following result in conjunction with Theorems 4.1 and 4.2 provides us with all matrices $A \in ST_n$ for which $\operatorname{sp} A$ is a given doubleton.

Proposition. *Let $\alpha < \beta$ be two real numbers. The set of all matrices $A \in ST_n$ with* $\operatorname{sp} A = \{\alpha,\beta\}$ *is*

$$\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}\, OST_n',$$

where $OST_n' := OST_n \setminus \{I,-I\}$.

Proof. Let $f(\lambda) := \frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}\lambda$ and $f^{-1}(\lambda) := \frac{2}{\beta-\alpha}\left(\lambda - \frac{\alpha+\beta}{2}\right)$. If $A \in ST_n$ and $\operatorname{sp} A = \{\alpha,\beta\}$, then $f^{-1}(A) \in ST_n$ and $\operatorname{sp} f^{-1}(A) = \{f^{-1}(\alpha), f^{-1}(\beta)\} = \{-1,1\}$. It follows that $f^{-1}(A) \in OST_n'$ and hence $A \in f(OST_n')$. Conversely, if A is in $f(OST_n')$ then A is in ST_n and $\operatorname{sp} A = \{f(-1), f(1)\} = \{\alpha,\beta\}$. □

Clearly this proposition and Theorem 1.4 yield Corollary 1.5.

References

[1] M. T. Chu and M. A. Erbrecht, *Symmetric Toeplitz matrices with two prescribed eigenpairs,* SIAM J. Matrix Anal. Appl. **15** (1994), 623–635.

[2] P. J. Davis, *Circulant Matrices,* John Wiley & Sons, New York, Chichester, Brisbane 1979.

[3] P. Delsarte and Y. Genin, *Spectral properties of finite Toeplitz matrices,* In: Mathematical Theory of Networks and Systems (Beer Sheva, 1983), Lecture Notes in Control and Inform. Sci. **58**, Springer, London 1984, pp. 194–213,

[4] C. Gu and L. Patton, *Commutation relations for Toeplitz and Hankel matrices,* SIAM J. Matrix Anal. Appl. **24** (2003), 728–746.

[5] H. J. Landau, *The inverse eigenvalue problem for real symmetric Toeplitz matrices,* J. Amer. Math. Soc. **7** (1994), 749–767.

[6] H. Widom, *On the spectrum of a Toeplitz operator,* Pacific. J. Math. **14** (1964), 365–375.

Albrecht Böttcher
Fakultät für Mathematik
TU Chemnitz
D-09107 Chemnitz
Germany
e-mail: `aboettch@mathematik.tu-chemnitz.de`