Range inclusions and approximate source conditions with general benchmark functions

DANA DÜVELMEYER*, BERND HOFMANN †AND MASAHIRO YAMAMOTO‡

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Abstract

The paper is devoted to the analysis of ill-posed operator equations $Ax = y$ with injective linear operator A and solution x_0 in a Hilbert space setting. We present some new ideas and results for finding convergence rates in Tikhonov regularization based on the concept of approximate source conditions by means of using distance functions with a general benchmark. For the case of compact operator A and benchmark functions of power-type we can show that there is a one-to-one correspondence between the maximal power-type decay rate of the distance function and the best possible Hölder exponent for the noise-free convergence rate in Tikhonov regularization. As is well-known this exponent coincides with the supremum of exponents in power-type source conditions. The main theorem of this paper is devoted to the impact of range inclusions under the smoothness assumption that x_0 is in the range of some positive self-adjoint operator G. It generalizes a convergence rate result proven for compact G in [12] to the case of general operators G with non-closed range.

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1 Introduction

Let X and Y be infinite dimensional separable Hilbert spaces, where the symbol $\|\cdot\|$ denotes the norms in both spaces as well as associated operator norms. Moreover, $\langle \cdot, \cdot \rangle$ designates the inner

*Department of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, GERMANY. Email: dana.duevelmeyer@mathematik.tu-chemnitz.de.

†Department of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, GERMANY. Email: hofmannb@mathematik.tu-chemnitz.de.

‡Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914 JAPAN. Email: myama@ms.u-tokyo.ac.jp.

product in Hilbert space. In this paper, we are going to study *ill-posed* linear operator equations

$$A\,x \,=\, y \qquad\qquad (x \in X,\;\; y \in Y) \tag{1.1}$$

with *injective* and bounded linear operators $A : X \to Y$ having a *non-closed range* $\mathcal{R}(A)$. Then finding the solution $x_0 \in X$ of (1.1), which is uniquely determined in the case $y \in \mathcal{R}(A)$, for given noisy data $y^\delta \in Y$ with $\|y^\delta - y\| \le \delta$ and noise level $\delta > 0$ in a stable manner requires regularization methods. In the sequel we focus on *Tikhonov's method* (see, e.g., [2, 4, 5, 6, 14, 24]) as the most prominent representative of such regularization methods and we are particularly interested in finding convergence rates for regularized solutions based on a new variant of the *distance function* approach. This approach was introduced and exploited in the papers [8, 9] and [12]. The present paper yields an extension of the results in [12] without additional compactness assumptions, and it generalizes the approach in [8] and [9] to general reference functions called *benchmark functions*, which will be used for defining the distance functions under consideration. An application of the distance function technique to general linear regularization methods can be found in the paper [10].

In our paper, we will distinguish regularized solutions

$$x_\alpha \,=\, (A^*A + \alpha I)^{-1}\, A^*\, y$$

with regularization parameter $\alpha > 0$ in the case of noise-free data and

$$x_\alpha^\delta \,=\, (A^*A + \alpha I)^{-1}\, A^* y^\delta$$

in the case of noisy data. Here we focus on the noise-free error function

$$f(\alpha) \,:=\, \|x_\alpha - x_0\| \,=\, \|\alpha\, (A^*A + \alpha I)^{-1}\, x_0\| \qquad\qquad (\alpha > 0) \tag{1.2}$$

for fixed A and x_0. Taking into account the noise level δ this function determines the total regularization error of the Tikhonov regularization

$$e(\alpha,\delta) \,:=\, \|x_\alpha^\delta - x_0\| \le \|x_\alpha - x_0\| + \|x_\alpha^\delta - x_\alpha\|$$

in case of noisy data with the well-known estimate

$$e(\alpha,\delta) \,\le\, f(\alpha) + \|\,(A^*A + \alpha I)^{-1}\, A^*(y^\delta - y)\| \,\le\, f(\alpha) + \frac{\delta}{2\sqrt{\alpha}}. \tag{1.3}$$

The paper is organized as follows. In Section 2 we present the analysis of approximate source conditions and distance functions for general benchmark smoothness with respect to the consequences of this approach for convergence rates in Tikhonov regularization. The power-type-case of benchmark and distance functions and of source conditions is under consideration in Section 3. Here, we show the one-to-one correspondence between the extremal exponents of distance function decay rate and associated Hölder convergence rate of Tikhonov regularization. The main focus of this paper is on Section 4, where we study the impact of range inclusions, which were already studied in [12]. However, in contrast to [12] now we can avoid the undesirable compactness assumption of the self-adjoint operator characterizing the solution smoothness.

2 General and approximate source conditions

In order to obtain convergence rates for the Tikhonov regularization and other linear regularization methods, general source conditions

$$x_0 = \varphi(A^*A)\, w \qquad (w \in X) \tag{2.1}$$

with index functions $\varphi(t)$ $(0 \le t \le a := \|A\|^2)$ are used (see, e.g., [1], [13], [17] – [21], [23]).

Definition 2.1 *We call $\varphi(t)$ $(0 \le t \le \bar{t})$ an index function if this function is continuous and strictly increasing with $\varphi(0) = 0$.*

Such index functions φ were originally introduced to characterize as a subscript elements $X_\varphi(G)$ of *variable Hilbert scales* based on the Hilbert space X and a self-adjoint nonnegative operator G mapping in X (cf. [7]). We will use this notation in Section 4.

We search for estimates of the noise-free error function of form

$$f(\alpha) \;=\; \|\alpha\,(A^*A+\alpha I)^{-1}\,\varphi(A^*A)\,w\| \;\le\; K\varphi(\alpha)\,\|w\| \qquad (0 < \alpha \le \overline{\alpha}) \tag{2.2}$$

with some constants $1 \le K < \infty$ and $0 < \overline{\alpha} \le a$. From the literature (see [19, 3]) we have the following proposition:

Proposition 2.2 *We assume that (2.1) holds and $\varphi(t)$ $(0 \le t \le a)$ is an index function. If (a) $\varphi(t)/t$ is monotonically decreasing on $(0, a]$, or (b) $\varphi(t)$ is concave on $[0, a]$, then (2.2) holds with $K = 1$. If there exists $\hat{t} \in (0, a)$ such that (c) $\varphi(t)/t$ is monotonically decreasing on $(0, \hat{t}\,]$ or (d) $\varphi(t)$ is concave on $[0, \hat{t}\,]$, then (2.2) is true with $K = \varphi(a)/\varphi(\hat{t}\,)$.*

If an index function φ satisfies one of the requirements (a) – (d) in Proposition 2.2, then φ is a *qualification* of Tikhonov regularization in the sense of the following definition, which is included in the concept of characterizing qualifications of linear regularization methods by means of index functions (see [17] - [19] and [10, 15]).

Definition 2.3 *An index function $\varphi(t)$ $(0 \le t \le a)$ is called a qualification of Tikhonov regularization with constant $1 \le K < \infty$ if for some $0 < \overline{\alpha} \le a$*

$$\sup_{0<t\le a} \frac{\alpha}{t+\alpha}\,\varphi(t) \;\le\; K\,\varphi(\alpha) \qquad (0 < \alpha \le \overline{\alpha})\,. \tag{2.3}$$

Evidently, for all qualifications φ of Tikhonov regularization with constant K the source condition (2.1) implies the error rate (2.2). On the other hand, from an inequality (2.2) we find by (1.3) estimates of form

$$e(\alpha,\delta) \;\le\; K\varphi(\alpha)\,\|w\| + \frac{\delta}{2\sqrt{\alpha}} \qquad (0 < \alpha \le a) \tag{2.4}$$

for the total regularization error. Then by balancing the two terms in the bound of (2.4) for sufficiently small $\overline{\delta} > 0$ we have a constant $\tilde{K} > 0$ such that

$$e(\alpha(\delta), \delta) \le \tilde{K}\, \varphi(\Theta^{-1}(\delta)) \qquad (0 < \delta \le \overline{\delta})\,, \tag{2.5}$$

where with φ also

$$\Theta(\alpha) := \sqrt{\alpha}\, \varphi(\alpha) \qquad (0 < \alpha \le \overline{\alpha})$$

is an index function and the regularization parameter is chosen a priori as $\alpha(\delta) := \Theta^{-1}(\delta)$. Under weak additional assumptions this error rate $\varphi(\Theta^{-1}(\delta))$ is *order optimal*.

Without explicit general source conditions for x_0 this paper presents an alternative approach for finding estimates of the form (2.2) and hence (2.4) and consequently convergence rates for the Tikhonov regularization. In this context, we exploit the fact that any solution x_0 of (1.1) satisfies (2.1) in an approximate manner by considering distance functions

$$d_\varphi(R) \;:=\; \inf\{\|x_0 - \varphi(A^*A)\, w\| \;:\; w \in X,\; \|w\| \le R\} \qquad (R \ge 0) \tag{2.6}$$

that measure the violation of the general source condition (2.1) for x_0. The index function φ has the character of a benchmark function. By general such benchmark functions we extend the corresponding results on this topic, which were published by the authors in the recent papers [8, 9] and [12] with focus on the special case $\varphi(t) = \sqrt{t}$.

Evidently, for every $x_0 \in X$ the nonnegative distance function $d_\varphi(R)$ in (2.6) is well-defined and non-increasing with $\lim\limits_{R\to\infty} d_\varphi(R) \;=\; 0$ as a consequence of the injectivity of $\varphi(A^*A)$ and $\overline{\mathcal{R}(\varphi(A^*A))} = X$. Note that the injectivity of A implies the injectivity of $\varphi(A^*A)$ for any index function φ. The distance function $d_\varphi(R)$ expresses the behaviour of x_0 with respect to the benchmark condition (2.1). There are two cases: Case (a) with $x_0 \notin \mathcal{R}(\varphi(A^*A))$ and $d_\varphi(R) > 0$ for all $R \ge 0$ and case (b) with $x_0 \in \mathcal{R}(\varphi(A^*A))$ implying for some $R_0 > 0$ the situation $d_\varphi(R) > 0$ $(0 \le R < R_0)$ and $d_\varphi(R) = 0$ $(R \ge R_0)$. Only the case (a) is of interest here. For that case, one uses the Lagrange multiplier method (cf. [8, Proof of Lemma 2.5]) to show that $d_\varphi(R)$ is a strictly decreasing function for $R \in (0, \infty)$ and consequently that $d_\varphi(1/t)$ is an index function for $t > 0$. Hence

$$\theta(t) := t\, d_\varphi(1/t) \;\; (t > 0), \qquad \theta(0) := 0 \tag{2.7}$$

is an index function on every interval $[0, \bar{t}\,]$. Note that this function θ is fundamental for the use of approximate source conditions in [10].

Lemma 2.4 *Let $\varphi(t)$ $(0 \le t \le a)$ be an index function which satisfies one of the requirements (a) – (d) in Proposition 2.2 with the corresponding constant $1 \le K < \infty$. Then we obtain the noise-free error estimate for the Tikhonov regularization*

$$f(\alpha) \;=\; \|x_\alpha - x_0\| \;\le\; d_\varphi(R) + K\varphi(\alpha)\, R \tag{2.8}$$

for all $0 < \alpha \le \overline{\alpha}$ and $R \ge 0$.

Proof: For any $w \in X$ with $\|w\| \le R$, based on formula (1.2) and using inequality (2.2) we can estimate for $0 < \alpha \le \overline{\alpha}$ by the triangle inequality as follows:

$$\begin{aligned} f(\alpha) = \|\alpha\,(A^*A+\alpha I)^{-1}\,x_0 - \alpha\,(A^*A+\alpha I)^{-1}\,\varphi(A^*A)w \\ + \alpha\,(A^*A+\alpha I)^{-1}\,\varphi(A^*A)w\| \\ \le \|\alpha\,(A^*A+\alpha I)^{-1}\,(x_0-\varphi(A^*A)w)\| + \|\alpha\,(A^*A+\alpha I)^{-1}\,\varphi(A^*A)w\| \\ \le \alpha\,\|\,(A^*A+\alpha I)^{-1}\,\|\,\|x_0-\varphi(A^*A)w\| + \|\alpha\,(A^*A+\alpha I)^{-1}\,\varphi(A^*A)w\| \\ \le \alpha\,\frac{1}{\alpha}\,\|x_0-\varphi(A^*A)w\| + K\varphi(\alpha)\,\|w\| \le \|x_0-\varphi(A^*A)w\| + K\varphi(\alpha)\,R\,. \end{aligned}$$

Taking the infimum in w with $\|w\| \le R$, we complete the proof of the lemma. □

Note that the assertion of the lemma always holds if φ is a qualification of Tikhonov regularization with constant K, but the sufficient conditions (a) – (d) of Proposition 2.2 are easier to check than the general condition (2.3).

Theorem 2.5 *Let the assumptions of Lemma 2.4 hold. Moreover let*

$$x_0 \notin \mathcal{R}(\varphi(A^*A))\,. \tag{2.9}$$

Then with sufficiently small $\overline{\alpha} > 0$, we have an error estimate

$$f(\alpha) = \|x_\alpha - x_0\| \le (K+1)\,\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))} \qquad (0 < \alpha \le \overline{\alpha}) \tag{2.10}$$

for the Tikhonov regularization and hence a rate $f(\alpha) = \mathcal{O}\left(\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))}\right)$ as $\alpha \to 0$ with the index function θ from (2.7).

Proof: We use the estimate (2.8), which is valid for all $R > 0$, and equate the terms $d_\varphi(R)$ and $R\varphi(\alpha)$. By setting $t := 1/R$ this is equivalent to $\theta(t) = \varphi(\alpha)$. For $\alpha > 0$ small enough there is some $t = t(\alpha) = \theta^{-1}(\varphi(\alpha))$ such that this equation is fulfilled and we find (2.10) from (2.8) taking into account that both φ and θ and also θ^{-1} are index functions. This proves the theorem. □

Remark 2.6 Note that the rate $\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))}$ is slower than the rate $\varphi(\alpha)$ prescribed by the benchmark function, since $\lim_{t\to 0} 1/\theta^{-1}(t) = \infty$. This is a consequence of the assumption (2.9). Moreover, we state that formulae (2.8) and (2.10) remain true if we replace $d_\varphi(R)$, also in formula (2.7), by a majorant which is a strictly decreasing function of R tending to zero as $R \to \infty$.

3 Direct and inverse results for power-type functions and compact operators

According to the relative smoothness of x_0 with respect to the operator A, the distance functions $d_\varphi(R)$ introduced in Section 2 may have very slow (logarithmic) decay rates as $R \to \infty$ as one extremal case, or they may have very fast (exponential) decay rates the second extremal case. The consequences of both extremal situations for convergence rates of Tikhonov regularization were outlined for the benchmark functions $\varphi(t) = \sqrt{t}$ in [8, §2]. Such considerations can be easily applied to a general index function φ. In this section we present a detailed discussion of the moderate case of a decay of $d_\varphi(R)$ characterized by a *power-type rate*.

Theorem 3.1 *As benchmark function φ in (2.6), we choose a power-type function*

$$\varphi(t) \,=\, t^\nu \qquad (0 \le t \le a) \qquad \textit{with exponent} \quad 0 < \nu \le 1\,. \tag{3.1}$$

Moreover, we assume that the solution x_0 of equation (1.1) satisfies the condition (2.9). Then a power-type decay rate of the distance function as

$$d_\varphi(R) \,\le\, \frac{C}{R^{\frac{\eta}{\nu-\eta}}} \qquad (\underline{R} \le R < \infty) \qquad \textit{with} \qquad 0 < \eta < \nu \le 1 \tag{3.2}$$

for some positive constants C and $\underline{R}$, implies an estimate for the regularization error of form

$$f(\alpha) = \|x_\alpha - x_0\| \,\le\, \widehat{C}\,\alpha^\eta \qquad\qquad (0 < \alpha \le \overline{\alpha}) \tag{3.3}$$

with some positive constants $\widehat{C}$ and $\overline{\alpha}$.

Proof: Noting that φ is concave, we can immediately apply Theorem 2.5, where in the sense of Remark 2.6 $d_\varphi(R)$ is replaced by its majorant $C\,R^{-\frac{\eta}{\nu-\eta}}$. This yields $\theta(t) = C\,t^{\frac{\nu}{\nu-\eta}}$, for sufficiently small positive t. Therefore $\theta^{-1}(t) = \widetilde{C}\,t^{\frac{\nu-\eta}{\nu}}$, and for sufficiently small $\alpha > 0$ we have $\theta^{-1}(\varphi(\alpha)) = \widetilde{C}\,\alpha^{\nu-\eta}$, that is, $\varphi(\alpha)/\theta^{-1}(\varphi(\alpha)) = \widehat{C}\,\alpha^\eta$ with corresponding constants $\widetilde{C}$ and $\widehat{C}$. This proves the theorem. □

We give some comments on the above theorem. Firstly we remark that the exponent $\frac{\eta}{\nu-\eta}$ in formula (3.2) attains all positive values if η varies through the whole the open interval $(0,\nu)$. Secondly, we note that it is evident from Proposition 2.2 that an error estimate $f(\alpha) = \mathcal{O}(\alpha^\eta)$ as obtained with formula (3.3) in Theorem 3.1 also occurs if x_0 satisfies a general source condition (2.1) for power-type source function

$$\psi(t) \,=\, t^\eta \qquad (0 \le t \le a) \qquad \text{with exponent} \quad 0 < \eta < \nu\,. \tag{3.4}$$

So it seems to be of some interest to answer the question whether $x_0 \in \mathcal{R}((A^*A)^\eta)$ also implies a decay rate of form (3.2) for the distance function. We prove such a vice versa result for the case of compact operators A.

Theorem 3.2 *Let a benchmark function φ of form (3.1) be given. Moreover, we suppose that the operator A is compact and that the smoothness of the solution x_0 of equation (1.1) is characterized by the conditions*

$$x_0 \notin \mathcal{R}((A^*A)^\nu), \qquad x_0 \in \mathcal{R}((A^*A)^\eta) \qquad \text{with} \quad 0 < \eta < \nu \le 1. \tag{3.5}$$

Then we have a decay rate (3.2) for the distance function d_φ.

Proof: By (3.5), we set $x_0 = (A^*A)^\eta\, w$ with $w \in X$. Let us suppose that the compact operator A has the ordered singular values $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_{i-1} \ge \sigma_i \ge ...$, where $\lim_{i\to\infty} \sigma_i = 0$ and $\{u_i\}_{i=1}^\infty \subset X$ is a complete orthonormal system of eigenelements such that $A^*A\, u_i = \sigma_i^2\, u_i$ $(i = 1, 2, ...)$. Now we can directly see that $\sum\limits_{i=1}^{\infty} \frac{\langle x_0, u_i\rangle^2}{\sigma_i^{4\eta}} = \widetilde{K} < \infty$. Then by (3.1) and (3.5) we can use the Lagrange multiplier method for finding an explicit expression for $d_\varphi(R)$ as

$$d_\varphi(R) = \|\lambda\, \left[(A^*A)^{2\nu} + \lambda\, I\right]^{-1} (A^*A)^\eta\, w\| \le \|\lambda\, \left[(A^*A)^{2\nu} + \lambda\, I\right]^{-1} (A^*A)^\eta\|\, \|w\|, \tag{3.6}$$

where $\lambda = \lambda(R)$ is the uniquely determined root of equation

$$R^2 = \|\, \left[(A^*A)^{2\nu} + \lambda I\,\right]^{-1} (A^*A)^\nu\, x_0\|^2\,. \tag{3.7}$$

Moreover by the spectral thoery (e.g., [25]), we have

$$\|\lambda\, \left[(A^*A)^{2\nu} + \lambda\, I\right]^{-1} (A^*A)^\eta\| = \sup_{0<s\le a} \frac{\lambda\, s^\eta}{s^{2\nu} + \lambda} \le \sup_{0<s\le a} \left(\frac{\lambda^{(1-\frac{\eta}{2\nu})}\, \left[s^{2\nu}\right]^{\frac{\eta}{2\nu}}}{s^{2\nu} + \lambda} \right) \lambda^{\frac{\eta}{2\nu}} \le \lambda^{\frac{\eta}{2\nu}}\,,$$

since $\frac{\lambda^{(1-\frac{\eta}{2\nu})}\, [s^{2\nu}]^{\frac{\eta}{2\nu}}}{s^{2\nu}+\lambda} \le 1$ as a consequence of Young's inequality. Therefore $d_\varphi(R) \le \lambda^{\frac{\eta}{2\nu}}\, \|w\|$. In order to derive (3.2), we use a majorant for the right-hand side of (3.7). Indeed we obtain

$$R^2 = \sum_{i=1}^{\infty} \frac{(\sigma_i^2)^{2\nu}\, \langle x_0,\, u_i\rangle^2}{[(\sigma_i^2)^{2\nu} + \lambda]^2} = \sum_{i=1}^{\infty} \frac{\langle x_0,\, u_i\rangle^2}{\sigma_i^{4\eta}} \left(\frac{\lambda^{(1-\frac{\eta}{\nu})}\, [(\sigma_i^2)^{2\nu}]^{(1+\frac{\eta}{\nu})}}{[(\sigma_i^2)^{2\nu} + \lambda]^2} \right) \lambda^{(\frac{\eta}{\nu}-1)} \le \widetilde{K}\, \lambda^{(\frac{\eta}{\nu}-1)}\,,$$

since we again have $\frac{\lambda^{(1-\frac{\eta}{\nu})}\, [(\sigma_i^2)^{2\nu}]^{(1+\frac{\eta}{\nu})}}{[(\sigma_i^2)^{2\nu}+\lambda]^2} \le 1$ due to Young's inequality. Then for all $R > 0$, we have $\lambda \le \tilde{\lambda}$ with λ solving (3.7) and $\tilde{\lambda}$ solving the equation $R^2 = \widetilde{K}\, \lambda^{(\frac{\eta}{\nu}-1)}$, which yields $\tilde{\lambda} = \widehat{K}\, R^{\frac{2\nu}{\eta-\nu}}$ with some positive constant $\widehat{K}$. If we exploit this result for further estimation from above of $d_\varphi(R)$ based on (3.6) we find $d_\varphi(R) \le C\, R^{\frac{\eta}{\eta-\nu}}$ for some constant $K > 0$. This, however, can be rewritten as (3.2) and proves the theorem. $\square$

As a consequence of both theorems we can formulate a corollary that makes an implication from a given decay rate $d_\varphi(R) \to 0$ as $R \to \infty$ of distance function to a corresponding smoothness (source condition) of solution x_0. This is an inverse theorem in the sense of approximation theory (see [16]).

Corollary 3.3 *For a power-type benchmark function φ from (3.1) with $0 < \nu \le 1$ and for a solution x_0 of equation (1.1) satisfying (2.9) let exist some value $\eta \in (0,\nu)$ such that*

$$d_\varphi(R) \le \frac{C}{R^{\frac{\eta}{\nu-\eta}}} \qquad (\underline{R} \le R < \infty)$$

with positive constants C and $\underline{R}$. Then we have

$$x_0 \in \mathcal{R}((A^*A)^\mu) \qquad \text{for all} \qquad 0 < \mu < \eta. \tag{3.8}$$

Moreover there exists

$$\eta_{max} := \max\{\eta > 0: \ d_\varphi(R) = \mathcal{O}(R^{-\eta/(\nu-\eta)}) \ \text{ as } \ R \to \infty\} \in (0,\nu),$$

and for compact operator A we even have the equality

$$\eta_{max} \ = \ \sup\{\mu > 0: \ x_0 \in \mathcal{R}((A^*A)^\mu)\}, \tag{3.9}$$

i.e. in case of power-type functions there is a one-to-one correspondence between the highest decay rate of distance function and the best possible rate of source condition.

Proof: Under the assumptions stated in the corollary the existence of $\eta_{max} \in (0,\nu)$ is evident. From Theorem 3.1 with $d_\varphi(R) = \mathcal{O}(R^{-\eta/(\nu-\eta)})$ we obtain a convergence rate $f(\alpha) = \mathcal{O}(\alpha^\eta)$ of Tikhonov regularization. A well-known converse theorem of regularization theory (see [22] or [10, Theorem 3.7 and Example 3.10]) then implies (3.8). If A is moreover compact, then Theorem 3.2 immediately provides us with the equality (3.9). □

4 Solution smoothness and range inclusions

Provided that the chosen benchmark function φ is a qualification of Tikhonov regularization, in particular if one the sufficient conditions (a) – (d) in Proposition 2.2 is satisfied, by Lemma 2.4 and Theorem 2.5, we can find convergence rates of Tikhonov regularization for solution x_0 if majorants of the distance function $d_\varphi(R)$ are available. Let $G: X \to X$ be a given positive self-adjoint bounded linear operator with non-closed range $\mathcal{R}(G)$ whose spectrum is in an interval $[0,b]$.

For an index function ρ on $[0,b]$, we will define a Hilbert space $X_\rho(G)$ as introduced in [7] (see also [17]) which is generated by an injective bounded positive self-adjoint linear operator G in X. First we note that there exists a partition of unity $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ such that

(i) $E(\lambda)$, $\lambda \in \mathbb{R}$, is an orthogonal projection (i.e., $E(\lambda)^2 = E(\lambda)$ and $E(\lambda)^* = E(\lambda)$).

(ii) $E(\lambda)E(\mu) = E(\min\{\lambda,\mu\})$.

(iii) $\lim_{\varepsilon\downarrow 0} E(\lambda+\varepsilon)x = E(\lambda)x \ \ (x \in X)$.

(iv) $E(\lambda) = I$ for $\lambda > b$ and $E(\lambda) = 0$ for $\lambda < 0$.

(v) $Gx = \int_0^b \lambda dE(\lambda)x \;\; (x \in X)$.

(vi) For $h \in C^1(0,b)$, we have

$$h(G)\,x = \int_0^b h(\lambda)dE(\lambda)x \quad \text{and} \quad \|h(G)\,x\|^2 = \int_0^b |h(\lambda)|^2 d(E(\lambda)x,x)$$

if $\int_0^b |h(\lambda)|^2 d(E(\lambda)x,x) < \infty$.

Let ρ be an index function on $[0,b]$. Then the Hilbert space $X_\rho(G)$ is the completion of

$$\{x - E(t)x : \; 0 < t < b, \; x \in X\}$$

with respect to the norm

$$\|x - E(t)x\|^2_{X_\rho(G)} = \int_0^b \frac{1}{\rho(\lambda)^2} d(E(\lambda)(x - E(t)x), x - E(t)x)$$
$$= \int_t^b \frac{1}{\rho(\lambda)^2} d(E(\lambda) - E(t))x, x) = \int_t^b \frac{1}{\rho(\lambda)^2} d(E(\lambda)x, x).$$

Note that we can also write $X_\rho(G) = \mathcal{R}(\rho(G))$. Namely, the Hilbert space $X_\rho(G)$ contains just those elements of X which belong to the range of the operator $\rho(G)$ defined by

$$\rho(G)\,x = \int_0^b \rho(\lambda)dE(\lambda)x \qquad (x \in \mathrm{dom}(\rho(G))).$$

We assume for index functions ρ_1, ρ_2 defined on $[0,b]$ the *range inclusion*

$$\mathcal{R}(\rho_1(G)) \subset \mathcal{R}(\varphi(A^*A))\,, \tag{4.1}$$

the *smoothness condition*

$$x_0 = \rho_2(G)\,v \quad (v \in X) \tag{4.2}$$

and that there is some $0 < \varepsilon \le b$ such that

$$q(0) := 0, \quad q(t) := \left(\frac{\rho_1}{\rho_2}\right)(t) \quad (0 < t \le \varepsilon) \qquad \text{is an index function on } [0,\varepsilon]\,. \tag{4.3}$$

We note that under the assumptions stated above, in particular due to the continuity of the quotient function $q(t)$ in (4.3) which is positive for $t > 0$, there exists some constant $C_1 \ge 1$ such that

$$\sup_{\varepsilon \le t \le b} \left(\frac{\rho_2}{\rho_1}\right)(t) \le C_1 \left(\frac{\rho_2}{\rho_1}\right)(\varepsilon)\,. \tag{4.4}$$

The study of this section is an extension of the recent results of [12] in two points. First, in contrast to [12] we use general benchmark functions φ being a qualification of Tikhonov regularization, noting that $\varphi(A^*A)$ is a self-adjoint bounded linear operator with non-closed range for any index function φ whenever A is so. Second we include the situation of non-compact operators G in this study. However, we remark that the consequences of the assumptions (4.1) with $\varphi(t) = \sqrt{t}$ and (4.2) for convergence rates of Tikhonov regularization were also discussed in [3].

Here we assume that $x_0 \notin \mathcal{R}(\varphi(A^*A))$, in particular $x_0 \neq 0$. Because, if $x_0 \in \mathcal{R}(\varphi(A^*A))$, then the distance function d_φ degenerates and we have $d_\varphi(R) = 0$ for sufficiently large $R > 0$, so that a convergence rate $f(\alpha) = \mathcal{O}(\varphi(\alpha))$ follows directly from Proposition 2.2. We again note that the assertion of this proposition can be extended to all qualifications φ of Tikhonov regularization.

It is evident that conditions (4.1) – (4.4) represent the counterpart of the standing assumption in [12] with respect to our extension. Note that the range $\mathcal{R}(\varphi(A^*A))$ is 'large' if the decay rate of the index function $\varphi(t) \to 0$ as $t \to 0$ is 'slow' and vice versa the range is 'small' if the decay rate is 'fast'. Hence under all qualifications φ of Tikhonov regularization, the range is 'smallest' if $\varphi(t) = t$.

First we reformulate Lemma 2 from [12], where the operator A in the original lemma is replaced by the self-adjoint operator $\varphi(A^*A)$ in our context. Taking into account that zero is an accumulation point of the spectrum of G the proof can be done without the compactness of G.

Lemma 4.1 *There exists some constant $C_2 > 0$ such that the inclusion*

$$\{\rho_1(G)w : \, w \in X, \, \|w\| \leq C_2 R\} \subset \{\varphi(A^*A)w : \, w \in X, \, \|w\| \leq R\} \tag{4.5}$$

is valid for all $R > 0$.

Proof: Henceforth we set

$$\|x\|_{X_{\rho_1(G)}} = \|w\|$$

for $x = \rho_1(G)w$. By assumption (4.1), we have

$$\{x : \, \|x\|_{X_{\rho_1}(G)} \leq 1\} = \bigcup_{n=1}^{\infty} \{\varphi(A^*A)g : \, \|g\| \leq n\} \cap \{x : \, \|x\|_{X_{\rho_1}(G)} \leq 1\}$$

$$\subset \bigcup_{n=1}^{\infty} \overline{\{\varphi(A^*A)g : \, \|g\| \leq n\} \cap \{x : \, \|x\|_{X_{\rho_1}(G)} \leq 1\}}^{X_{\rho_1}(G)}.$$

In contrast to the closure $\overline{\{\cdot\}}$ with respect to the norm in X, we denote by $\overline{\{\cdot\}}^{X_{\rho_1}(G)}$ the closure with respect to the norm in $X_{\rho_1}(G)$. Then by means of Baire's category theorem (see, e.g., [25]), there exist $x_0 \in X_{\rho_1}(G)$, $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\{x : \, \|x - x_0\|_{X_{\rho_1}(G)} \leq \varepsilon_0\} \\ &\subset \overline{\{\varphi(A^*A)g : \, \|g\| \leq n_0\} \cap \{x : \, \|x\|_{X_{\rho_1}(G)} \leq 1\}}^{X_{\rho_1}(G)} \\ &\subset \overline{\{\varphi(A^*A)g : \, \|g\| \leq n_0\}}. \end{aligned} \tag{4.6}$$

Here we used that $x_k \to x$ in $X_{\rho_1}(G)$ implies $x_k \to x$ in X. In fact,

$$\|x_k - x\|^2_{X_{\rho_1}(G)} = \lim_{t\to 0}\int_t^b \frac{1}{\rho_1(\lambda)^2} d((E(\lambda)-E(t))(x_k-x), x_k-x).$$

Then we have

$$\begin{aligned}
&\int_t^b \frac{1}{\rho_1(\lambda)^2} d((E(\lambda)-E(t))(x_k-x), x_k-x) \ge C_3\int_t^b d((E(\lambda)-E(t))(x_k-x), x_k-x)\\
= \quad & C_3\left(\int_0^b - \int_0^t\right) d((E(\lambda)-E(t))(x_k-x), x_k-x)\\
= \quad & C_3\int_0^b d(E(\lambda)(x_k-x), x_k-x) - C_3\int_0^t d(E(\lambda)(x_k-x), x_k-x)\\
\ge \quad & C_3\|x_k-x\|^2 - C_3\int_0^t d(E(\lambda)(x_k-x), x_k-x) \ge C_3\|x_k-x\|^2 - C_3\sqrt{t}\|x_k-x\|,
\end{aligned}$$

that is,

$$\|x_k - x\|^2_{X_{\rho_1}(G)} \ge C_3\|x_k-x\|^2.$$

Thus the last inclusion in (4.6) is seen.

Further we can prove that

$$\{x:\ \|x\|_{X_{\rho_1}(G)} \le \varepsilon_0\} \subset \overline{\{\varphi(A^*A)g:\ \|g\| \le 2n_0\}}. \tag{4.7}$$

In fact, since $x_0 \in \overline{\{\varphi(A^*A)g:\ \|g\| \le n_0\}}$ by (4.6), there exist g_m ($m \in \mathbb{N}$) such that $\|g_m\| \le n_0$ and $\lim_{m\to\infty}\|\varphi(A^*A)g_m - x_0\| = 0$. Let $v \in X_{\rho_1}(G)$ be an arbitrary element satisfying $\|v\|_{X_{\rho_1}(G)} \le \varepsilon_0$. Therefore by (4.6), we can choose $\widetilde{g_m}$ ($m \in \mathbb{N}$), such that $\|\widetilde{g_m}\| \le n_0$ and $\lim_{m\to\infty}\|\varphi(A^*A)\widetilde{g_m} - (x_0+v)\|_X = 0$. Therefore we have chosen $z_m = \widetilde{g_m} - g_m$ ($m \in \mathbb{N}$), such that $\lim_{m\to\infty}\|\varphi(A^*A)z_m - v\| = 0$ and $\|z_m\| \le \|\widetilde{g_m}\| + \|g_m\| \le 2n_0$. This means that $v \in \overline{\{\varphi(A^*A)g:\ \|g\| \le 2n_0\}}$. Since $v \in \{x:\ \|x\|_{X_{\rho_1}(G)} \le \varepsilon_0\}$ is arbitrary, inclusion (4.7) is valid.

In order to complete the proof of Lemma 4.1, we set $C_2 = \frac{\varepsilon_0}{2n_0}$. Let $\|x\|_{X_{\rho_1}(G)} \le C_2 R$. For $\widetilde{x} = \frac{\varepsilon_0}{C_2 R}x$, we then have $\|\widetilde{x}\|_{X_{\rho_1}(G)} \le \varepsilon_0$. Hence (4.7) yields

$$\widetilde{x} = \frac{\varepsilon_0}{C_2 R}x \in \overline{\{\varphi(A^*A)g:\ \|g\| \le 2n_0\}},$$

that is,

$$x \in \overline{\left\{\varphi(A^*A)\left(\frac{C_2 R}{\varepsilon_0}g\right);\ \|g\| \le 2n_0\right\}} = \overline{\{\varphi(A^*A)h:\ \|h\| \le R\}}.$$

Thus the proof of Lemma 4.1 is completed. □

Then we can make explicit upper bounds for the distance function $d_\varphi(R)$ by the following lemma. This is the basis for the application of the ideas of Theorem 2.5 under the assumptions (4.1) – (4.3).

Lemma 4.2 *There exists some* $\underline{R} > 0$ *such that*

$$d_\varphi(R) \le \rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\|x_0\|_{X_{\rho_2}(G)}}\right)\right)\|x_0\|_{X_{\rho_2}(G)} \qquad (R > \underline{R})\,.$$

Proof: Let $t \in (0,b)$ be sufficiently small such that $0 < t < \varepsilon$. We set $\widetilde{x} = (I - E(t))x_0$. Then, by (4.3) and the definition of the norm in $X_{\rho_1}(G)$, we have

$$\begin{aligned}
\|\widetilde{x}\|^2_{X_{\rho_1}(G)} &= \int_0^b \frac{1}{\rho_1(\lambda)^2} d(E(\lambda)(I-E(t))x_0,(I-E(t))x_0)\\
&= \int_t^b \frac{1}{\rho_1(\lambda)^2} d((E(\lambda)-E(t))x_0,x_0)\\
&= \int_t^b \left(\frac{\rho_2(\lambda)}{\rho_1(\lambda)}\right)^2 \frac{1}{\rho_2(\lambda)^2} d((E(\lambda)-E(t))x_0,x_0)\\
&\le C_1^2\left(\frac{\rho_2(t)}{\rho_1(t)}\right)^2 \int_t^b \frac{1}{\rho_2(\lambda)^2} d(E(\lambda)x_0,x_0) \le C_1^2\left(\frac{\rho_2(t)}{\rho_1(t)}\right)^2 \int_0^b \frac{1}{\rho_2(\lambda)^2} d(E(\lambda)x_0,x_0)\\
&\le C_1^2\left(\frac{\rho_2(t)}{\rho_1(t)}\right)^2 \|x_0\|^2_{X_{\rho_2}(G)}.
\end{aligned}$$

Since ρ_2 is increasing, we see that

$$\|x_0 - \widetilde{x}\|^2 = \int_0^b d(E(\lambda)E(t)x_0,E(t)x_0) = \int_0^t d(E(\lambda)x_0,x_0) = \int_0^t \rho_2(\lambda)^2 \frac{1}{\rho_2(\lambda)^2} d(E(\lambda)x_0,x_0)$$

$\le \rho_2(t)^2 \int_0^t \frac{1}{\rho_2(\lambda)^2} d(E(\lambda)x_0,x_0) \le \rho_2(t)^2\|x_0\|^2_{X_{\rho_2}(G)}$.

Hence

$$\begin{aligned}
\inf\left\{\|x_0 - \rho_1(G)w\| : w \in X, \|w\| \le C_1\left(\frac{\rho_2}{\rho_1}\right)(t)\|x_0\|_{X_{\rho_2}(G)}\right\} &\le \|x_0 - \widetilde{x}\| \qquad (4.8)\\
&\le \rho_2(t)\|x_0\|_{X_{\rho_2}(G)}.
\end{aligned}$$

Now we set $R := \left(\frac{\rho_2}{\rho_1}\right)(t)$ and let $R_1 > 0$ be fixed. Due to a one-to-one correspondence between the interval $(0,\varepsilon]$ for t and some interval $[R_1,\infty)$ for R, by (4.3) and (4.8) we have

$$\inf\{\|x_0-\rho_1(G)w\| : w \in X,\ \|w\| \le C_1 R\|x_0\|_{X_{\rho_2}(G)}\} \le \rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}(R)\right)\|x_0\|_{X_{\rho_2}(G)} \ \ (R \ge R_1)\,.$$

Then by Lemma 4.1, for $R \ge \underline{R} := \frac{C_1\|x_0\|_{X_{\rho_2}(G)}R_1}{C_2} > 0$ we further estimate

$$d_\varphi(R) = \inf\{\|x_0-\varphi(A^*A)w\| : w \in X,\ \|w\| \le R\} \le \rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\|x_0\|_{X_{\rho_2}(G)}}\right)\right)\|x_0\|_{X_{\rho_2}(G)}.$$

Hence, the proof is complete. □

Similarly to Theorem 2.5, we can prove the main theorem of this paper:

Theorem 4.3 *Let the assumptions (4.1) – (4.3) hold and let $x_0 \notin \mathcal{R}(\varphi(A^*A))$, where φ is a qualification of Tikhonov regularization with constant $1 \le K < \infty$. Then with $\widetilde{\alpha} > 0$ sufficiently small we have the estimate*

$$f(\alpha) \le (K+1)\,\max\left\{\frac{C_1}{C_2},\,1\right\}\,\rho_2\left(\rho_1^{-1}\left(\varphi(\alpha)\right)\right)\,\|x_0\|_{X_{\rho_2}(G)} \qquad (0 < \alpha \le \widetilde{\alpha})$$

for the noise-free error of Tikhonov regularization.

Proof: We set $\gamma = \|x_0\|_{X_{\rho_2}(G)}$. By Lemma 2.4, which is in general valid if φ is a qualification of Tikhonov regularization with constant $1 \le K < \infty$, and by Lemma 4.2 we obtain

$$f(\alpha) \le (K+1)\left\{\gamma\rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right)\right) + \varphi(\alpha)R\right\} \quad (R \ge \underline{R} > 0,\ 0 < \alpha \le \overline{\alpha}).$$

Provided that $\alpha > 0$ is sufficiently small we find $R > 0$ such that

$$\gamma\rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right)\right) = \varphi(\alpha)R,$$

that is,

$$\frac{\gamma}{R}\rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right)\right) = \varphi(\alpha).$$

This is a consequence of the fact that $q(t) = \left(\frac{\rho_1}{\rho_2}\right)(t)$ $(0 \le t \le \varepsilon)$ is an index function (see (4.3)).

Since we assume that $\alpha > 0$ is sufficiently small, we can see that $R > 0$ is sufficiently large, in particular, $R \ge \underline{R}$. On the other hand, we can directly verify that

$$\frac{\gamma}{R}\rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right)\right) = \frac{C_2}{C_1}\rho_1\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right)\right).$$

Hence

$$\frac{C_2}{C_1}\rho_1\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right)\right) = \varphi(\alpha),$$

that is,

$$\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right) = \rho_1^{-1}\left(\frac{C_1\varphi(\alpha)}{C_2}\right). \tag{4.9}$$

Because of $\lim_{R\to\infty}\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\gamma}\right) = 0$, equation (4.9) possesses a root $R \ge \underline{R}$ for sufficiently small $\alpha > 0$, say for $0 < \alpha \le \widetilde{\alpha} \le \overline{\alpha}$. Then this R gives the bound

$$f(\alpha) \le (K+1)\gamma\rho_2\left(\rho_1^{-1}\left(\frac{C_1}{C_2}\varphi(\alpha)\right)\right)$$

for $0 < \alpha \le \widetilde{\alpha}$. Using the same technique as in the proof of Theorem 3 in [12] we finally get

$$f(\alpha) \le (K+1)\ \max\left\{\frac{C_1}{C_2}, 1\right\}\ \gamma\rho_2\left(\rho_1^{-1}\left(\varphi(\alpha)\right)\right),$$

since on the one hand with ρ_1 and ρ_2 also $\rho_2(\rho_1^{-1}(t))$ is an index function for sufficiently small $t > 0$, and on the other hand the quotient function $\frac{\rho_2(\rho_1^{-1}(t))}{t}$ is non-increasing for small $t > 0$ as a consequence of (4.3). This proves the theorem. □

Remark 4.4 Along the lines of [10] (see also [11]) convergence rates results of this paper can be generalized from the specific case of Tikhonov regularization to wider classes of general linear regularization methods, where additional and different constants occur.

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