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Preprint 2007-1

Fakultät für Mathematik

Preprintreihe der Fakultät für Mathematik
ISSN 1614-8835

A Generalized Bivariate Ornstein-Uhlenbeck Model for Financial Assets

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17th January 2007

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Abstract

In this paper, we study mathematical properties of a generalized bivariate Ornstein-Uhlenbeck model for financial assets. Originally introduced by Lo and Wang, this model possesses a stochastic drift term which influences the statistical properties of the asset in the real (observable) world. Furthermore, we generalize the model with respect to a time-dependent (but still non-random) volatility function.

Although it is well-known, that drift terms – under weak regularity conditions – do not affect the behaviour of the asset in the risk-neutral world and consequently the Black-Scholes option pricing formula holds true, it makes sense to point out that these regularity conditions are fulfilled in the present model and that option pricing can be treated in analogy to the Black-Scholes case.

MSC2000 classification scheme numbers: 60G15, 60G44, 91B28

Keywords: generalized Ornstein-Uhlenbeck process, option pricing, Black-Scholes formula

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1 Introduction

We consider the price P_t of a financial asset during the time interval $[0, T]$. By p_t the logarithm of the asset price is denoted, $p_t = \ln P_t$. The basis for the model which is analysed in this paper forms the Bivariate Trending Ornstein-Uhlenbeck model of Lo and Wang, introduced in [3]. The logarithm of the asset price p_t is assumed to have a linear deterministic trend μt. Then it is convenient to introduce the process

$$q_t := p_t - \mu t, \tag{1.1}$$

and to consider the stochastic properties of the centered (detrended) log-price process q_t.

Uncertainty is modelled by means of a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Additionally, we consider a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions (see for instance [4]). The Bivariate Trending Ornstein-Uhlenbeck model of Lo and Wang assumes that q_t satisfies the following pair of stochastic differential equations,

$$\begin{aligned} dq_t &= -\left(\gamma q_t - \lambda X_t\right) dt + \sigma\, dW_t^q \\ dX_t &= -\beta X_t\, dt + \sigma_X\, dW_t^X, \end{aligned} \tag{1.2}$$

where $\gamma \geq 0$, $\lambda \geq 0$, $\beta \geq 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ and $\sigma_X > 0$ are real-valued parameters, the initial conditions $q_0 = c_q$, $X_0 = c_X$ hold and W^q and W^X are correlated Wiener processes with correlation coefficient $\varkappa$, i. e. $\mathbb{E}\left(W_t^q W_t^X\right) = \varkappa t$.

As a motivation for considering this model Lo and Wang argue that empirical observations have indicated that the returns $r_t^\tau = \ln\left(\frac{P_t}{P_{t-\tau}}\right)$ show certain correlation patterns, which means that the classical Black-Scholes model is inappropriate for describing the price process of these assets. For a detailed discussion and further properties it is referred to [3]. In this paper, the aim consists only in the description of the mathematical properties, namely in the explicit solution of the defining stochastic differential equations and in the problem of pricing European call options written on a corresponding asset.

The process X_t which influences the stochastic drift component of q_t is some underlying process, which may also be relevant for other assets. It should be noted, that if one is not able to observe it, one could also consider a scaled version $\hat{X}_t := \frac{1}{\sigma_X} X_t$. This process $\hat{X}_t$ satisfies the stochastic differential equation

$$d\hat{X}_t = -\beta \hat{X}_t\, dt + 1\, dW_t^X$$

with initial condition $\hat{X}_0 = \hat{c}_X := \frac{1}{\sigma_X} c_X$. Thus, setting $\hat{\lambda} = \lambda \sigma_X$ the process q_t could also be described by

$$\begin{aligned} dq_t &= -\left(\gamma q_t - \hat{\lambda} \hat{X}_t\right) dt + \sigma\, dW_t^q \\ d\hat{X}_t &= -\beta \hat{X}_t dt + 1\, dW_t^X \end{aligned} \tag{1.3}$$

with initial conditions $q_0 = c_q$ and $\hat{X}_0 = \hat{c}_X$.

However, in this paper we consider the model in the form (1.2). We restrict our considerations to the case of independent Wiener processes W^q and W^X, i.e. $\varkappa = 0$. On the other hand with respect to several effects in option pricing we are interested in a more general behaviour of the asset prices with respect to the risk neutral measure as the constant volatility coefficient σ would admit. Therefore we generalise this model inasmuch as we allow the volatility σ to be time-dependent (but still non-random).

Furthermore, in order to allow a scaling of the prices (which plays the role of adjusting the monetary unit) we introduce an additive constant d to q_t. Obviously, this leads to a multiplication of the asset prices by $\exp(d)$.

Summarising, the model which is in the focus of this paper is described by

Model 1.1 We assume that the detrended log-price process

$$q_t := \ln P_t - \mu t - d \tag{1.4}$$

of a tradable financial asset satisfies

$$\begin{aligned} dq_t &= -\left(\gamma q_t - \lambda X_t\right) dt + \sigma(t)\, dW_t^q \\ dX_t &= -\beta X_t dt + \sigma_X\, dW_t^X . \end{aligned} \tag{1.5}$$

with $\mu, d \in \mathbb{R}$, $\gamma \geq 0$, $\lambda \geq 0$, $\beta \geq 0$ and a time-dependent, continuous volatility function $\sigma(t)$ with $\sigma(t) > 0, 0 \leq t \leq T$. The initial values $q_0 = c_q$ and $X_0 = c_X$ are assumed to be stochastic variables with finite second order moments, i. e.,

$$\mathbb{E}c_q^2 < \infty \quad \text{and} \quad \mathbb{E}c_X^2 < \infty .$$

Furthermore, we assume the vector $(c_q\ c_X)^T$, which contains the initial values of the processes q_t and X_t, and the Wiener processes W^q, W^X to be mutually independent.

Using Itôs Lemma it can be easily shown that under the assumptions of Model 1.1 the price process itself satisfies the stochastic differential equation

$$dP_t = \left(-\gamma \ln P_t + \gamma\, \mu\, t + \gamma d + \lambda\, X_t + \mu + \frac{\sigma^2(t)}{2}\right) P_t\, dt + \sigma(t) P_t\, dW_t^q , \tag{1.6}$$

with initial condition $P_0 = \exp(d + c_q)$.

2 Solution of the stochastic differential equation

As model 1.1 leads to a system of linear stochastic differential equations, the solution can easily be derived. This will be done in this section, we will prove the existence and uniqueness of the solutions q_t and X_t, $t \in [0, T]$ of (1.5). To do this, we combine the processes q_t and X_t into an $\mathbb{R}^2$-dimensional random process Y_t and the independent Wiener processes W^q and W^X into a 2-dimensional Wiener process W, i. e., we set

$$Y_t := \begin{pmatrix} q_t \\ X_t \end{pmatrix} \quad \text{and} \quad W_t = \begin{pmatrix} W_t^q \\ W_t^X \end{pmatrix} .$$

Thus, the system (1.5) attains the form

$$dY_t = \begin{pmatrix} -\gamma & \lambda \\ 0 & -\beta \end{pmatrix} Y_t\, dt + \begin{pmatrix} \sigma(t) & 0 \\ 0 & \sigma_X \end{pmatrix} dW_t \tag{2.1}$$

with initial value $Y_0 = c := (c_q\ c_X)^T$, which is a linear stochastic equation in the narrow sense (cf. [1][p. 128 ff.]). Note, that the matrix-valued function $B : [0, T] \to \mathbb{R}^{2\times 2}$, defined by

$$B(t) := \begin{pmatrix} \sigma(t) & 0 \\ 0 & \sigma_X \end{pmatrix} ,$$

is measurable and bounded on $[0,T]$. Thus, Theorem 8.1.5 of [1] implies that for every initial value c there exists a unique solution Y_t of (2.1).

Furthermore, Corollary 8.2.4 of [1] states that this solution is given by

$$Y_t = e^{tA}c + \int_0^t e^{(t-s)A}B(s)\,dW_s\,, \tag{2.2}$$

where we have introduced the notation

$$A := \begin{pmatrix} -\gamma & \lambda \\ 0 & -\beta \end{pmatrix}.$$

In order to compute the exponential of the matrix A it is necessary to consider the two cases $\gamma \neq \beta$ and $\gamma = \beta$ separately.

1. For $\gamma \neq \beta$ the matrix A can be diagonalised, i. e., there exists an invertible matrix P such that $D = P^{-1}\,A\,P$ is a diagonal matrix. Indeed, the matrix

$$P = \begin{pmatrix} 1 & \lambda \\ 0 & \gamma-\beta \end{pmatrix} \quad \text{has the inverse} \quad P^{-1} = \begin{pmatrix} 1 & \frac{\lambda}{\beta-\gamma} \\ 0 & \frac{1}{\gamma-\beta} \end{pmatrix}$$

and it holds

$$D = P^{-1}\,A\,P = \begin{pmatrix} -\gamma & 0 \\ 0 & -\beta \end{pmatrix}.$$

Thus, e^{tA} can be computed by $e^{tA} = P\,e^{tD}\,P^{-1}$ and we obtain

$$e^{tA} = \begin{pmatrix} e^{-\gamma t} & \frac{\lambda}{\gamma-\beta}\left(e^{-\beta t} - e^{-\gamma t}\right) \\ 0 & e^{-\beta t} \end{pmatrix}.$$

Therefore, the solution Y_t of (2.1) is given by

$$\begin{aligned} Y_t = &\begin{pmatrix} e^{-\gamma t} & \frac{\lambda}{\gamma-\beta}\left(e^{-\beta t} - e^{-\gamma t}\right) \\ 0 & e^{-\beta t} \end{pmatrix} c \\ &+ \int_0^t \begin{pmatrix} e^{-\gamma(t-s)} & \frac{\lambda}{\gamma-\beta}\left(e^{-\beta(t-s)} - e^{-\gamma(t-s)}\right) \\ 0 & e^{-\beta(t-s)} \end{pmatrix} \begin{pmatrix} \sigma(s) & 0 \\ 0 & \sigma_X \end{pmatrix} dW_s\,. \end{aligned}$$

Thus, the processes q_t and X_t are given by

$$\begin{aligned} q_t =& e^{-\gamma t}c_q + \frac{\lambda}{\gamma-\beta}\left(e^{-\beta t} - e^{-\gamma t}\right)c_X + \frac{\lambda\sigma_X}{\gamma-\beta}\int_0^t \left[e^{-\beta(t-s)} - e^{-\gamma(t-s)}\right] dW_s^X \\ &+ \int_0^t \sigma(s)e^{-\gamma(t-s)}\,dW_s^q \\ X_t =& e^{-\beta t}c_X + \sigma_X\int_0^t e^{-\beta(t-s)}\,dW_s^X\,. \end{aligned}$$

2. For the situation $\gamma = \beta$ the matrix A is a multiple of a 2×2 Jordan block. Thus, the exponential can be easily computed and it holds

$$e^{tA} = \begin{pmatrix} e^{-\gamma t} & \lambda t\,e^{-\gamma t} \\ 0 & e^{-\gamma t} \end{pmatrix}.$$

Therefore, in this situation the solution Y_t of (2.1) is given by

$$Y_t = \begin{pmatrix} e^{-\gamma t} & \lambda t e^{-\gamma t} \\ 0 & e^{-\gamma t} \end{pmatrix} c + \int_0^t \begin{pmatrix} e^{-\gamma(t-s)} & \lambda(t-s)e^{-\gamma(t-s)} \\ 0 & e^{-\gamma(t-s)} \end{pmatrix} \begin{pmatrix} \sigma(s) & 0 \\ 0 & \sigma_X \end{pmatrix} dW_s$$

Thus, the processes q_t and X_t are given by

$$\begin{aligned} q_t =& e^{-\gamma t} c_q + \lambda t e^{-\gamma t} c_X + \lambda \sigma_X \int_0^t (t-s) e^{-\gamma(t-s)} \, dW_s^X + \int_0^t \sigma(s) \, e^{-\gamma(t-s)} \, dW_s^q \\ X_t =& e^{-\gamma t} c_X + \sigma_X \int_0^t e^{-\gamma(t-s)} \, dW_s^X \, . \end{aligned}$$

Clearly, if $c = (c_q \; c_X)^T$ is normally distributed (or non-random) the vector process $(q_t \; X_t)^T$ is Gaussian.

3 Pricing of European Call Options

We consider now an European vanilla call option C with strike K and expiry T, i. e., $C := \max\{P_T - K, 0\}$. We are interested in the fair price of this option at the time point $t \in [0, T]$. As in the Black-Scholes model, besides the tradable asset of Model 1.1 a bond, whose price process is given by $B_t = \exp(rt)$ shall exist.

We introduce the Black-Scholes function U_{BS} as follows.

Definition 3.1
For parameters $\tilde{P} > 0$, $K > 0$, $r \geq 0$, $\tau \geq 0$ and $s \geq 0$ the Black-Scholes function *is defined as*

$$U_{BS}(\tilde{P}, K, r, \tau, s) := \begin{cases} \tilde{P}\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) & \text{if } s > 0 \\ \max\left(\tilde{P} - Ke^{-r\tau}, 0\right) & \text{if } s = 0 \end{cases} \tag{3.1}$$

with

$$d_1 := \frac{\ln\left(\frac{\tilde{P}}{K}\right) + r\tau + \frac{s}{2}}{\sqrt{s}}, \qquad d_2 := d_1 - \sqrt{s} \, .$$

In (3.1) Φ denotes the distribution function of the standard normal distribution.

Furthermore we set

$$S(t) := \int_0^t \sigma^2(u) \, du \qquad t \in [0, T] \, .$$

It is well-known that the option price formula derived from the Black-Scholes model is unaffected by the drift term of the underlying asset. As long as the logarithm of the price process of the underlying satisfies the stochastic differential equation

$$d \ln P_t = \mu(\cdot) \, dt + \sigma(t) \, dW_t \, ,$$

the fair price of a European call option with payoff $\max\{P_T - K, 0\}$ at maturity T at a time point $t \in [0,T]$ is given by

$$C(t, P_t) = U_{BS}\left(P_t, K, r, T-t, S(T)-S(t)\right) . \tag{3.2}$$

While in the Black-Scholes model μ is assumed to be a constant it is well-known that $\mu(\cdot)$ can be a stochastic process, depending on P_t itself as well as on other stochastic influences, which fulfil mild regularity conditions. In the remaining part of this section we show that in the considered Model 1.1 the fair option price is indeed given by (3.2).

In general, the initial values $q_0 = c_q$ and $X_0 = c_X$ are assumed to be stochastic variables. For the sake of simplicity here we restrict to the case where c_q and c_X are deterministic quantities, which are chosen to be zero. It should be mentioned, that as X_t is in general not observable the assumption about X_0 is a slight restriction of the model. However, the following considerations can be generalised straightforwardly to the case of stochastic initial conditions. Moreover, we concentrate to the fair option price at time point $t = 0$. The generalization to times $t \in [0,T]$ is also straightforward.

In a first step, we show that there exists an admissible self-financing strategy duplicating the call option. The existence of such a strategy can be shown under very general assumptions, for instance as long as the asset price is modelled by

$$dP_t = \sigma(t) P_t \, dW_t^q + P_t \, dZ_t$$

where $\sigma(t)$ is a continuous function and Z_t is a continuous random process of zero square variation, possibly dependent on P, fulfilling weak regularity conditions. For details see [5].

For our Model 1.1 we consider the function $C \in \mathbb{C}\left([0,T] \times (0,\infty)\right) \cap \mathbb{C}^{1,2}\left([0,T) \times (0,\infty)\right)$ defined by

$$C(t,x) := \begin{cases} x\,\Phi\left(\dfrac{\ln \frac{x}{K} + r(T-t) + \frac{1}{2}\int_t^T \sigma^2(s)\,ds}{\sqrt{\int_t^T \sigma^2(s)\,ds}}\right) - Ke^{-r(T-t)}\Phi\left(\dfrac{\ln \frac{x}{K} + r(T-t) - \frac{1}{2}\int_t^T \sigma^2(s)\,ds}{\sqrt{\int_t^T \sigma^2(s)\,ds}}\right) & t < T \\ \max(x-K, 0) & t = T \end{cases}$$

$$= U_{BS}\left(x, K, r, T-t, S(T)-S(t)\right) .$$

We consider the trading strategy consisting at time t of

$$a_t := C_x(t, P_t)$$

shares of the asset and

$$b_t := e^{-rt}\left(C(t,P_t) - P_t C_x(t,P_t)\right)$$

units of the bond. Then the processes a_t and b_t possess continuous realisations, which implies

$$\int_0^T |b_t|\, dt < \infty \quad \text{a.s.} \qquad \text{and} \qquad \int_0^T |a_t P_t|^2\, dt < \infty \quad \text{a.s.}$$

Concerning the wealth process $V(t, P_t)$, defined by

$$V(t, P_t) = a_t P_t + b_t \exp(rt)$$

it yields

$$V(t,P_t) = P_t\,C_x(t,P_t) + C(t,P_t) - P_t\,C_x(t,P_t)$$
$$= C(t,P_t)\,.$$

It can be shown by elementary considerations that $V(t,P_t) \geq 0$ holds. Obviously, the trading strategy has a. s. the same final value $V(T,P_T)$ as the call option which is to valuate. Applying Itô's Lemma we obtain

$$\begin{aligned} V(t,P_t) =& V(0,P_0) + \int_0^t C_t(s,P_s)\,ds + \int_0^t C_x(s,P_s)\,dP_s \\ &+ \frac{1}{2}\int_0^t C_{xx}(s,P_s)P_s^2\sigma^2(s)\,ds\,. \end{aligned}$$

On the other hand it can be derived easily that $C(t,x)$ fulfils the Black-Scholes differential equation

$$C_t + \frac{1}{2}x^2\sigma^2(t)C_{xx} + rxC_x - rC = 0$$

and consequently

$$\begin{aligned} V(t,P_t) =& V(0,P_0) + \int_0^t C_x(s,P_s)\,dP_s \\ &+ \int_0^t r\left(C(s,P_s) - P_s\,C_x(s,P_s)\right)\,ds \\ =& V(0,P_0) + \int_0^t a_s\,dP_s + \int_0^t r\,b_s\exp(rs)\,ds\,, \end{aligned}$$

which proves that the considered trading strategy is self-financing.

It is important to remark that the hedging strategy $(a_t,b_t)_{t\in[0,T]}$ is even adapted to the filtration $\left(\mathcal{F}_t^P\right)_{t\in[0,T]}$ (the augmented filtration generated by the process P). From this fact it follows that in order to perform the hedging strategy at the time point t it is not necessary to know the value of X_t.

By finding the hedging strategy, the main work for calculating the fair option price is done. From easy non-arbitrage arguments one usually concludes that this fair price is the value of the hedging strategy, i. e., the fair option price at time t is equal to $C(t,P_t)$, which would prove our assertion. However, a more careful consideration has to take into account the fact that even in the classical Black-Scholes model there are still some pathological strategies (namely the so-called suicide strategies) which make things complicated. To be precise, in many situations there exists self-financing admissible strategies with arbitrary starting value $V_0 > 0$ and $V_T \equiv 0$ a. s.

For this reason, the fair option price of an attainable claim at time $t = 0$ has to be defined as

$$\begin{aligned} c_0 := \inf\Big\{& x \geq 0 : \text{there exists an admissible self-financing duplication} \\ & \text{strategy } (\tilde{a}_t,\tilde{b}_t) \text{ for the option with } \tilde{a}_0P_0 + \tilde{b}_0 = x\Big\}\,. \end{aligned} \tag{3.3}$$

In order to check that our strategy (a_t,b_t) considered above leads to the correct option price, i. e.,

$$c_0 = C(0,P_0) \tag{3.4}$$

it is sufficient to show that its discounted wealth process $e^{-rt}V(t, P_t)$ follows a martingale with respect to a martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ (i. e., a measure $\mathbb{Q}$ under which the discounted asset price process $e^{-rt}P_t$ follows a martingale). To prove the latter statement let us assume for the moment that we have found such a measure. Clearly, Definition (3.3) implies $c_0 \leq a_0 P_0 + b_0$. On the other hand it is easy to show that the discounted wealth process of any admissible self-financing duplication strategy $(\tilde{a}_t, \tilde{b}_t)$ is a non-negative local martingale and consequently a supermartingale. This leads to

$$\begin{aligned}\tilde{a}_0 P_0 + \tilde{b}_0 &\geq \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}\left(\tilde{a}_T P_T + \tilde{b}_T e^{rT}\right)\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}C\right)\\ &= \mathbb{E}_{\mathbb{Q}}\left(e^{-rT}\left(a_T P_T + b_T e^{rT}\right)\right)\\ &= a_0 P_0 + b_0\end{aligned}$$

for all admissible self-financing strategies $\left(\tilde{a}_t, \tilde{b}_t\right)$ and therefore $c_0 \geq a_0 P_0 + b_0$, which leads to $c_0 = a_0 P_0 + b_0$.

It remains to show that there exists an equivalent martingale measure and that the discounted trading strategy introduced at the beginning of this section is a martingale with respect to this measure.

For this we define the stochastic process

$$Z_t := \exp\left(-\int_0^t \alpha_s \, dW_s - \frac{1}{2}\int_0^t \alpha_s^2 \, ds\right)$$

with

$$\alpha_t := \frac{-\gamma p_t + \gamma\mu t + \gamma d + \lambda X_t + \mu + \frac{\sigma^2(t)}{2} - r}{\sigma(t)}. \tag{3.5}$$

If it is possible to show that Z_t is a martingale, then by Girsanovs Theorem

$$\tilde{W}_t^q := W_t^q + \int_0^t \alpha_s \, ds$$

and

$$\tilde{W}_t^X := W_t^X$$

form a two-dimensional Wiener process with respect to the measure $\mathbb{Q}$ defined by

$$\mathbb{Q}(A) = \mathbb{E}(\mathbf{1}_A Z_T) \quad (A \in \mathcal{F}_T).$$

To check that Z_t is a martingale it is sufficient to show that the Novikov condition

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T \alpha_s^2 \, ds\right) < \infty$$

is fulfilled. In [2][Chapter 6.2] it is shown that in case of Gaussian processes α_t with

$$\sup_{t\leq T} \mathbb{E}|\alpha_t| < \infty \quad \text{and} \quad \sup_{t\leq T} \mathbb{D}^2\alpha_t < \infty$$

Novikovs condition holds true. From the considerations of Section 2 it is clear that the process γ_t defined in (3.5) is such a Gaussian process.

Under the measure $\mathbb{Q}$ the asset price has the dynamics

$$dP_t = rP_t\,dt + \sigma(t)P_t\,d\tilde{W}_t^q$$

which implies

$$d\left(e^{-rt}P_t\right) = \sigma(t)e^{-rt}P_t\,d\tilde{W}_t^q$$

and clearly $\mathbb{Q}$ is a (not the unique one) martingale measure.

Finally, it remains to show that the discounted value process $e^{-rt}V_t$ is a honest $\mathbb{Q}$-martingale. From the self-financing property it follows easily

$$\begin{aligned} d\left(e^{-rt}V_t\right) &= -re^{-rt}V_t\,dt + e^{-rt}\,dV_t \\ &= -re^{-rt}a_tP_t\,dt - re^{-rt}b_te^{rt}\,dt + e^{-rt}a_t\,dP_t + re^{-rt}b_te^{rt}\,dt \\ &= a_t\,d\left(e^{-rt}P_t\right) = a_t\sigma(t)e^{-rt}P_t\,d\tilde{W}_t^q\,, \end{aligned}$$

which shows that $e^{-rt}V_t$ is a local $\mathbb{Q}$-martingale. A continuous non-negative local martingale is always a supermartingale. Performing an elementary calculation (which is the same as done during the calculation of the classical Black-Scholes formula using the expectation in the risk neutral world) one gets

$$\mathbb{E}_{\mathbb{Q}}e^{-rT}V_T = V_0\,,$$

which proves that $e^{-rt}V_t$ is a honest martingale. Thus, the fair option price at time $t = 0$ is given by (3.4). The proof of this property was the aim of this section.

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