# Local topological toughness and local factors

Frank Göring*[1], Gyula Y. Katona†[2]

[1] Department of Mathematics
Chemnitz University of Technology
09107 Chemnitz
Germany

[2] Dept. of Comp. Sci. and Inf. Th.
Budapest University of
Technology and Economics
Hungary

**Abstract.** We localize and strengthen Katona's idea of an edge-toughness to a local topological toughness. We disprove a conjecture of Katona concerning the conection between edge-toughness and factors. For the topological toughness we prove a theorem similar to Katona's $2k$-factor-conjecture, which turned out to be false for his edge-toughness. We prove, that besides this the topological toughness has nearly all known nice properties of Katona's edge-toughness and therefore is worth to be considered.

## 1. Preliminaries and Results

For notations not defined here we refer to [2]. Unless otherwise stated, $t$ is an arbitrary non negative real number, $k$ is an arbitrary integer, $G$ is an arbitrary finite graph (loops and multiple edges allowed), $U$ is an arbitrary subgraph of $G$, $X$ and $H$ are arbitrary disjoint subsets of $V(G)$, $Y$ is an arbitrary subset of $E(G-X-H)$, and $f$ is an arbitrary function that maps $H$ into the positive integers. An *$H$-path* is a path connecting two different vertices of $H$. A cycle covering $H$ is called an *$H$-cycle*. The union of internally disjoint $H$-paths is called an *$H$-local $k$-factor*, if all vertices of $H$ have degree $k$ in it, a *partial $H$-local $k$-factor*, if all vertices of $H$ have at most degree $k$ in it, an *$H$-local $f$-factor* if each vertex $h$ of $H$ has degree $f(h)$ in it, and a *partial $H$-local $f$-factor* if each vertex $h$ of $H$ has at most degree $f(h)$ in it. The *size* of $H$-local factors is the number of its $H$-paths. The maximum number of internally disjoint $H$-paths is denoted by $p_G(H)$. With $G[X]$ we denote the subgraph of $G$ induced by $X$, $[Y]$ denotes the graph with edge set $Y$ whose vertex set is the set of all vertices incident with edges of $Y$. Instead of $G[V([Y])]$ we shortly write $G[Y]$. $E'(G)$ denotes the set of all edges in $G$ except the loops. Let $\mathcal{C}(G)$ denote the set of components of $G$ and $\partial_G(U)$ denote the set of vertices of $U$ incident with

* Research supported by the "Mathematics in Information Society" project carried out by Alfréd Rényi Institute of Mathematics - Hungarian Academy of Sciences, in the framework of the European Community's "Confirming the International Role of Community Research" programme. e-mail: frank.goering@mathematik.tu-chemnitz.de

† Research supported by the Ministry of Education OTKA grant OTKA T 043520. e-mail: kiskat@cs.bme.hu

edges of $G - E(U)$. For $V(U) - \partial_G(U)$ we will write shortly $in_G(U)$. According to [10] we define the *permeability* of a pair $(X, Y)$ by:

$$\mathrm{perm}_G(X, Y) = |X| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{|\partial_{G-X}(C)|}{2} \right\rfloor$$

The following definitions generalize this concept:

Let $G$ be a graph, and $f$ be a function mapping $H^* \subseteq V(G)$ into the set of positive integers. An *$f$-separator* of $G$ is a pair $(X, Y)$ with $X \subseteq V(G)$, $Y \subseteq E(G-X)$ and $\partial_{G-X}Y$ disjoint to $H^*$ such that $G - X - Y$ has no $H^*$-paths.

The *permeability* of an $f$-separator is

$$\mathrm{perm}_{G,f}(X, Y) = |X \setminus H^*| + \sum_{v \in X \cap H^*} f(v) + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} \left( |\partial_{G-X}C| + \sum_{v \in V(C) \cap H^*} f(v) \right) \right\rfloor$$

In 1997 the second coauthor introduced the concept of edge-toughness (see [8]). It is strengthening the concept of toughness introduced by Chvátal in 1971. We will define these concepts.

$G$ is *$t$-tough* (in the sense of Chvátal, cf. [1]) if deleting of $k$ vertices of $G$ results in at most $\max\{1, \frac{k}{t}\}$ components. The *toughness of* $G$ (denoted by $t(G)$) is the supremum over all reals $t$ such that $G$ is $t$-tough. It turns out that $t(G) = \infty$ if any two vertices of $G$ are adjacent, and $t(G) = \min \left\{ \frac{|X|}{|\mathcal{C}(G-X)|} \;\middle|\; 2 \leq |\mathcal{C}(G - X)| \right\}$ otherwise.

$G$ is *$t$-edge-tough* if for all $(X, Y)$ with $[Y] = G[Y]$ we have:

$$|\mathcal{C}(G - X - Y - in_{G-X}([Y]))| \leq \max\{1, \frac{\mathrm{perm}_G(X, Y)}{t}\}$$

The *edge-toughness of* $G$ (denoted by $te(G)$) is the supremum over all reals $t$ such that $G$ is $t$ edge-tough.

The ideas are as follows: Every path passing a vertex of $X$ needs another one of these vertices, every path not counted by this and passing a component $C$ of [Y] needs two more vertices of the boundary of $C$ in $G - X$. If $G$ is hamiltonian, then deleting $X$, $Y$ and $in_{G-X}(Y)$ therefore results in at most $\mathrm{perm}_G(X, Y)$ components incident with edges not contained in $Y$. This idea gives Chvatal's toughness if we restrict $(X, Y)$ to that pairs with $Y = \emptyset$. Otherwise – as defined before – we get edge-toughness.

We are interested in a local toughness concept, since the topic of the existence of cycles through prescribed vertices of a graph seems to be of interest (cf. [6, 7, 5, 3, 10, 11]).

Local versions of Katona's edge toughness and Chvatal's toughness are naturally defined as follows:
$H$ is called *$k$-edge-tough* (or *$k$-tough*) *in* $G$ if for all $(X, Y)$ with $[Y] = G[Y]$ (or $(X, Y)$ with $Y = \emptyset$) the graph $G - X - Y$ has at most $\max\{1, \frac{\mathrm{perm}_G(X,Y)}{k}\}$ components containing a vertex of $H$. The local version of Chvatal's toughness occurs for instance in [5].

What would be useful properties we expect from a local version of toughness? First of all, we should be sure that in a graph $G$ for a subset $H$ of its vertices not being 1-tough in $G$ the graph $G$ contains no $H$-cycle. Second, if a set $H$ is $k$-tough in $G$, then every subset of $H$ with at least two elements should be $k$-tough, too. Third, the toughness of $H$ in $G$ should not depend on the length of paths in $G$, the inner vertices of which have degree 2 in $G$.

Obviously the local versions of the mentioned toughnesses fulfill the first and the second condition, but break the third. The latter is easy to see (e.g. intersect each edge of a complete graph).

The toughness concepts we have discussed by now deal with disconnecting graphs. Our idea is to complementary - it deals with connecting vertices.

Every cycle for every $k$ element subset $H$ of vertices has exactly $k$ internally disjoint $H$-paths. This simple observation leads to the following definition: $H$ is *topological $t$-tough in $G$* iff for all $H' \subseteq H$ with $|H'| \geq 2$ the graph $G$ contains (at least) $t|H'|$ internally disjoint $H'$-paths. We chose this name because subdividing edges has obviously no effect on this value. Moreover, this definition ensures that the topological toughness fulfills all our three conditions.

If $V(G)$ itself is topological $t$-tough in $G$, we will say shorter that $G$ is topological $t$-tough. The *topological toughness* of $H$ in $G$ is the maximal $t$ such that $H$ is $t$-tough in $G$, the topological toughness of $G$ is the topological toughness of $V(G)$ in $G$.

We want to compare the ideas of edge toughness and topological toughness. For this we need Mader's theorem about the number of internally disjoint $H$-paths (cf. [12]) in $G$. We use it in the version of [2].

**Theorem 1 (Mader, 1978)** $p_G(H) = |E'(G[H])| + \min\{perm_G(X,Y) \mid \forall C \in \mathcal{C}(G - X - Y - E'(G[H])) : |V(C) \cap H| \leq 1\}$

Mader's theorem is often understood as a generalization of Menger's theorem (cf. [14]).

**Theorem 2 (Menger, 1927)** *Let $a$ and $b$ be nonadjacent vertices of $G$. The maximum number $p_G(\{a,b\})$ of internally disjoint $ab$-paths in $G$ equals the minimum number of vertices of $G - \{a,b\}$ separating $a$ from $b$ in $G$.*

In [10] the concept of $A$-separators is introduced. In our notation we will replace $A$ by $H$ and call it small $H$-separator. For an independent set $H$ a pair $(X,Y)$ is called an *$H$-separator* if $G$ has no $H$-path avoiding $X$ and $Y$, and *small $H$-separator*, if additionally $\mathrm{perm}_G(X,Y) < |H|$ holds. Obviously, $G$ can't have an $H$-cycle if $G$ has a small $H$-separator. Our topological toughness by Theorem 1 generalizes the idea of small $H$-separators: An independent set $H$ is topological 1-tough in $G$ if and only if $G$ has no $H$-separator.

For a graph $G$ being $t$-edge-tough means having a system of at least $t|H|$ $H$-paths for certain (but not all!) subsets $H$ of the vertex set of $G$. Especially using Mader's theorem one can prove easily the following lemma which classifies the edge-toughness in a connector-language:

**Lemma 3** *If for each independent set $H$ there are at least $t|H|$ internally disjoint $H$-paths in $G$, then $G$ is $t$-edge-tough. If $G$ is $t$- edge-tough, then for each induced subgraph $U$ there are at least $|\mathcal{C}(U)|t$ subgraphs of $G$ being $U$-paths or cycles not disjoint to $U$ which are disjoint out of $U$.*

Obviously Lemma 3 combined with the definition of topological toughness leads to

**Corollary 4** *Every topological $t$-tough graph $G$ is $t$-edge-tough and therefore $t$-tough in Chvátal's sense.*

All the toughnesses are constructed to detect non-hamiltonicity by a toughness value less than one (which one can prove by presenting a single separator). Let $NC$ be the set of graphs not being 1-tough, $NE$ be the set of graphs not being 1-edge-tough and

$NT$ be the set of graphs not being topological 1-tough. The following observation tells us, that beyond the mentioned versions of toughness, topological toughness detects non-hamiltonicity best:

**Observation 1** *The following holds: $NC \subset NE \subset NT$*

Replacing edge-toughness by the topological toughness unfortunately doesn't preserve the strong (linear) connection to Chvátal's toughness:

**Observation 2** *Let $G$ be a complete graph on $2k^2+1$ vertices after deleting one of it's edges. Then the toughness of $G$ is $k^2 - \frac{1}{2}$ and the topological toughness is $2k - \frac{3}{2}$.*

The toughness of $G$ is $k^2 - \frac{1}{2}$ because by deleting vertices one can only separate the endvertices of the missing edge and has to delete all other vertices for this purpose. For an $h$-element subset $H$ of the vertex set of $G$ we find at least $(2k^2+1-h) + \binom{h}{2} - 1$ internally disjoint $H$-paths and this bound is tight (if $H$ contains the endvertices of the missing edge this becomes obvious). For the topological toughness of $G$ we get therefore

$$t = \min\left\{ \frac{\binom{h}{2} + (2k^2+1-h) - 1}{h} \;\middle|\; h = 2, \ldots, k^2+1 \right\} \tag{1}$$

$$= -\frac{3}{2} + \min\left\{ \frac{h}{2} + \frac{2k^2}{h} \;\middle|\; h = 2, \ldots, k^2+1 \right\} \tag{2}$$

$$= -\frac{3}{2} + \left[ \frac{h}{2} + \frac{2k^2}{h} \right]_{h=2k} \tag{3}$$

$$= 2k - \frac{3}{2} \tag{4}$$

However, we prove the following:

**Theorem 5** *If $H$ is $(4t^2+2t)$-tough in $G$ and $|H| \geq \frac{(t+\frac{3}{2})^2}{2}$, then $H$ is topological $t$-tough in $G$.*

Here $4t^2 + 2t$ is not best possible, as it is seen in the next theorem for small values of $t$.

**Theorem 6** *If $t \leq 1$, $|H| \geq 3$ and $H$ is $2t$-tough in $G$, then $H$ is topological $t$-tough in $G$.*

The connection between $k$-factors and toughness first was proved in [4]:

**Theorem 7 (Enomoto, Jackson, Katerinis, Saito, 1985)** *Let $k$ be a positive integer and $G$ be a $k$-tough graph such that $k|V(G)|$ is even. Then $G$ has a $k$-factor.*

We know that a graph being 1-tough may have a hamiltonian cycle – which is a 2-factor (more precisely, we know that a graph not being 1-tough cannot be hamiltonian). Therefore the idea of toughness creates the conjecture that every $k$-tough graph has a $2k$-factor. This was conjectured by the second coauthor for the edge-toughness (see [9]) and proved for $k = 1$.

Unfortunately, this is not true in general.

Let $C_q^p$ denote the $p^{\text{th}}$ power of a cycle on $q$ vertices. (In the $p^{\text{th}}$ power of a cycle on vertices $v_1, \ldots, v_q$ the vertices $v_i$ and $v_j$ are connected iff $|i-j| \leq p$ or $|i-j| \geq q-p$.) Then take $m$ disjoint copies of $C_{k^2+k-1}^{k-1}$ and denote these by $H_1, \ldots, H_m$. Moreover, take a complete graph $K_x$, where $x$ is the largest integer satisfying $x < m\frac{k^2+k-1}{k}$ and connect each vertex of $K_x$ to each vertex of each $H_i$. The resulting graph is denoted by $G_{k,m}$. It is worth of mentioning that the smallest such construction for $k = 2$ is obtained from a $K_7$ by deleting the edges of a cycle of length 5.

**Theorem 8** *Then $G_{k,m}$ is $k$-edge-tough but has no $2k$-factor for all $m \geq 1$ and $k \geq 2$ integers.*

Even a local version of Theorem 7 is not true.

**Observation 3** *Let $G$ be a copy of $K_{24}$ after deleting an edge. Let $H$ be a set of 6 vertices of $G$ not inducing $K_6$. Then $H$ is 11-tough in $G$ but $G$ has no $X$-local 11-factor because $G$ has no 33 internally disjoint $H$-paths.*

However, the situation changes if we consider the topological toughness:

**Theorem 9** *Every topological $k$-tough graph has a $2k$-factor.*

This is a consequence of our main result:

**Theorem 10** *A set $H^*$ of vertices of a graph $G$ is topological $k$-tough in $G$ if and only if for every $H \subseteq H^*$ with $|H| \geq 2$ the graph $G$ has an $H$-local $2k$-factor.*

This theorem is a little surprising because it says that it is sufficient to have enough $H$-paths for each $H \subseteq H^*$, $|H| \geq 2$, to be able to arrange them in a regular way for each such $H$.

To prove Theorem 10 we use the following theorem, which is equivalent to Theorem 2 in [13]:

**Theorem 11** *Let $G$ be a graph, $H^* \subseteq V(G)$ be independent in $G$, and $f$ be a function that maps $H^*$ to the positive integers. Then the maximal size of a partial $H^*$-local $f$-Factor equals the minimum of $perm_{G,f}(X,Y)$ taken over all $f$-separators $(X,Y)$ of $G$.*

Theorem 11 also has the following corollary, which provides a necessary and sufficient condition for the existence of $H^*$-local $f$-Factors:

**Corollary 12** *For $G$, $H^*$ and $f$ defined as in Theorem 11 $G$ has an $H^*$-local $f$-factor if and only if for each $f$-separator $(X,Y)$ we get:*

$$perm_{G,f}(X,Y) \geq \frac{1}{2} \sum_{h \in H^*} f(h)$$

Corollary 12 has the following special case ($f(h) = 2k$ for each $h \in H^*$):

**Corollary 13** *Let $G$ and $H^*$ be defined as above. Then $G$ has an $H^*$-local $2k$-factor if and only if if for each $(X,Y)$ such that $X \subseteq V(G)$, $Y \subseteq V(G-X)$, $\partial_{G-X}Y \subseteq V(G-H^*)$, and $G - X - Y$ has no $H^*$-path, we get:*

$$|X \setminus H^*| + 2k|H^* \cap X| + k|H^* \cap V([Y])| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor \geq k|H^*|$$

## 2. Proofs

We only need to prove the Theorems 5, 6, 8 and 10. Because the equivalence of Theorem 11 and Theorem 2 in [13] is not easy to deduce, we will give a proof of Theorem 11, too.

**Proof of Theorem 5.** Suppose $H$ is not topological $t$-tough in $G$. Then there is a set $H' \subseteq H$ with $|H'| \geq 2$ such the maximum number of internally disjoint $H'$-paths in $G$ is smaller than $t|H'|$. By Mader's Theorem (Theorem 1) there is a total separator $(X', Y')$ of $H'$ in $G - E(G[H'])$ satisfying $\mathrm{perm}_G(X', Y') + |E'(H')| < t|H'|$. Let $\alpha$ be the independence number of $G[H']$ and set $x = \frac{|H'|}{\alpha}$. Let $\overline{G}$ be the simple graph on $H'$ in which two vertices are adjacent only if they are not adjacent in $G$. Clearly $\overline{G}$ is $K_{\alpha+1}$-free. By Turán's Theorem (cf. [2]) it has at most as many edges as a complete $r$-partite graph with nearly equal partition classes, such that the sizes of classes differ by at most one. Thus $G[H']$ has at least $\alpha\frac{x(x-1)}{2}$ edges. Therefore we have $\alpha\frac{x(x-1)}{2} \leq |H'|t = \alpha x t$. This leads to $x \leq 2t + 1$.

Consider for the first case $\alpha \geq 2$. Let $H''$ be an independent subset of $H'$ in $G$ with $|H''| = \alpha$. Then in $(X', Y')$, we can replace each component $C$ of $\mathcal{C}(Y')$ by all but one vertices of $\partial_{G-X}(C)$, and replace the edges induced by $H'$ by the vertices of $H' \setminus H''$. This operation leads to a total separator $(X'', \emptyset)$ of $H''$ with $|X''| \leq 2\,(\mathrm{perm}_G(X', Y') + |E(G[H'])|)$. For the Chvatal-toughness $t_c$ of $G$ we get:

$$t_c \leq \frac{|X''|}{|H''|} = \frac{|X''|}{\alpha} = \frac{x|X''|}{|H'|} \leq \frac{2(\mathrm{perm}_G(X', Y') + |E(G[H'])|)x}{|H'|} < 2tx \leq 4t^2 + 2t$$

Consider now the other case $\alpha = 1$. If every vertex of $H \setminus H'$ is contained in $X'$, we get $t|H'| > |H| - |H'| + \binom{|H'|}{2}$. This leads to:

$$|H| < \frac{1}{2}(2t + 3 - |H'|)|H'| \leq \frac{(t + \frac{3}{2})^2}{2}.$$

Therefore, such $H$ is not considered in the theorem we have to prove. Hence we may assume that $H$ contains a vertex $v$ not contained in $H' \cup X'$. Furthermore, we may assume that for every component $C$ of $[Y']$ we have $\left\lfloor \frac{|\partial_{G-X'}(C)|}{2} \right\rfloor \geq 1$, that is, every component has a nonzero contribution to the permeability of $(X', Y')$ in $G$. Let $X^*$ contain all vertices of $X'$ and for each component $C$ of $[Y']$ all but one element of $\partial_{G-X'}(C)$, chosen such that $X^*$ does not contain $v$. Such $X^*$ exists because $X'$ does not contain $v$. Clearly, $G - X^* - E(G[H'])$ has no component containing two vertices of $H'$, but it may have a component containing $v$ and a vertex $w$ of $H'$, but no other vertex of $H'$. If this is the case, set $X'' = X^* \cup \{w\}$, otherwise set $X'' = X^*$. In both cases let $H''$ consist of $v$ and one element of $H' \setminus X''$. Then $X''$ separates $H''$. We get $|X''| \leq 2\mathrm{perm}_G(X', Y') + 1 < 2(t|H'| - |E'(G[H'])|) + 1 \leq 2(t|H'| - \binom{|H'|}{2}) + 1 = (2t + 1 - |H'|)|H'| + 1 < (t + \frac{1}{2})^2 + 1$. Finally, the toughness of $H'$ in $G$ is at most $\frac{|X''|}{2} < 4t^2 + 2t$. □

**Proof of Theorem 6.** It suffices to prove the following for all $H$: If $|H| \geq 3$ and there is a set $H'$ with $|H'| \geq 2$ such that $G$ has no $t|H'|$ internally disjoint $H'$-paths, then $H$ has an independent subset $H''$ such that there is a set $X'' \subseteq V(G - H'')$ being a total separator of $H''$ with $2t|H''| > X''$.

If $G[H']$ is connected, then it has at least $|H'| - 1$ edges. Its edges are internally disjoint $H'$-paths. Therefore it has at most $|H'| - 1$ edges. Thus it is a tree and $t > \frac{1}{2}$ holds. Furthermore, $G - E(G[H'])$ has no $H'$-path.

If, furthermore, $|H'| = 2$, then there is a vertex $h \in H \setminus H'$. In this case let $H' = \{h_1, h_2\}$. Clearly, either $G-h_1$ has no $\{h, h_2\}$-path or $G-h_2$ has no $\{h, h_1\}$-path. Suppose w.l.o.g. the latter is the case. Then with $H'' = \{h, h_1\}$ and $X'' = \{h_2\}$ we are done.

If, otherwise, $|H'| > 2$, then $G[H']$ has a cutvertex $x$. Let $H''$ consist of two vertices of different components of $G[H'] - x$. We are done with $X'' = \{x\}$.

So we may suppose that $G[H']$ contains at least 2 components. Choose for $H''$ one vertex of each of these components.

By Theorem 1 there is a set $X' \subseteq V(G - H')$ and a set $Y' \subseteq V(G - H' - X')$ such that $\mathrm{perm}_G(X', Y') + |E'(G[H'])| < t|H'|$ and $G - X' - Y' - E'(G[H'])$ has no $H'$-paths. Therefore, by the construction of $H''$, the graph $G-X'-Y'$ has no $H''$-paths. Furthermore, we get $\mathrm{perm}_G(X', Y') < t|H'|-|E'(G[H'])| < t|H''|$. Let $X''$ consist of all vertices of $X'$ and all but one vertices of $\partial_{G-X'}(C)$ for all $C \in \mathcal{C}([Y])$. Then $|X''| \le \mathrm{perm}_G(X', Y') < t|H''|$ but $G - X''$ has no $H''$-path. □

**Proof of Theorem 8.** First we prove that $G_{k,m}$ has no $2k$-factor. Since each $H_i$ is $2k-2$ regular, and they are disjoint from each other, each vertex of each $H_i$ in a $2k$-factor must send at least 2 edges to $K_x$, so there must be at least $2m(k^2+k-1)$ edges ending in $K_x$. However, $x$ is given such that this is not possible, because in a $2k$-factor each vertex of $K_x$ is incident to at most $2k$ of these edges.

Now we prove that $G_{k,m}$ is $k$-edge-tough. By Lemma 3 it is enough to prove that for any independent vertex set $H \subset V(G_{k,m})$ there are at least $k|H|$ internally disjoint $H$-paths. It is clear that $H$ cannot contain any vertex of $K_x$. Let $h_i = |H \cap V(B_i)|$. Since all $B_i$ are copies of $C^{k-1}_{k^2+k-1}$, there are at most $k$ independent vertices in it, so $h_i \le k$. One can easily see that if $i \le j - k$, then there are $k - 1$ disjoint paths from $v_i$ to $v_j$ using only vertices $v_{i'}$ with $i < i' < j$. This implies that in each $B_i$ there are at least $(k-1)h_i$ internally disjoint $H$-paths. Thus there we have $d := x + \sum_{i=1}^{m}(k-1)h_i$ internally disjoint $H$-paths in $G_{k,m}$. Since $x > m\frac{k^2+k-1}{k} - 1$ and $\sum_{i=1}^{m} h_i \le km$ straightforward calculation gives that $d \ge k\sum_{i=1}^{m} h_i$ holds if $m \ge 2$, which proves our claim in this case. If $m = 1$, then $x = k$ and similar argument shows that our claim holds, so the proof is complete. □

Before we start to prove Theorem 11, we add some notation and provide some lemmas.

The *neighborhood* of a vertex $v$ in a graph $G$ is the set of $v$ and all vertices adjacent to $v$ in $G$ and we denote it by $N_G(v)$. Let $(X, Y)$ be an *optimal $H^*$-separator*, if $(X, Y)$ is an $H^*$-separator and $\mathrm{perm}_G(X, Y) = p_G(H^*)$. Since $H^*$ is independent in $G$, an optimal $H^*$-separator exists by Theorem 1. In a first lemma we refine this Observation.

**Lemma 14** *Let $G'$ be a graph and $H^*$ be an independent subset of $V(G')$. Then $G'$ has an optimal $H^*$-separator $(X, Y)$ satisfying the following conditions:*

1. *If $u, v, w \in V(G - H - X)$, $w \in N_G(v) = N_G(w)$, and $\{u, v\} \in Y$, then $\{u, w\} \in Y$.*
2. *If $u, v \in V(G - H)$, $N_G(u) \subseteq N_G(v)$, and $u \in X$, then $v \in X$.*
3. *For each component $C$ of $[Y]$ the graph $G'$ obtained from $G$ by deleting all elements of $X \cup Y \setminus E(C)$ and contracting $C$ to a vertex $c$ has no pair $G'_1, G'_2$ of subgraphs such that $H^* \cap V(G'_2) = \emptyset$, $|V(G'_1 \cap G'_2)| = 1$, and $G'_1 \cup G'_2 = G'$ ( $G'$ has no endblock disjoint from $H^*$).*

**Proof of Lemma 14.** We give a constructive proof of the conditions, starting with an arbitrary optimal separator $(X, Y)$. In each step of the construction we apply a transformation, which changes $X$ and $Y$ while $\mathrm{perm}_G(X, Y)$ is nonincreasing and $(X, Y)$ remains an $H^*$-separator of $G$. Hence, $\mathrm{perm}_G(X, Y)$ is constant, and $(X, Y)$ remains an optimal $H^*$-separator of $G$.

This is the general step with its three cases:

**Case 1)**

If $(X,Y)$ infringes condition 1 we proceed as follows: Since having the same neighborhood in $G-H^*-X$ is an equivalence relation, it induces a partition of $V(G-H^*-X)$ into a finite set $\mathcal{A}$ of classes. All $A \in \mathcal{A}$ for which there are vertices $v,w \in A$ and a vertex $u \in N_G(v) = N_G(w)$, such that $\{u,v\} \in Y$, and $\{u,w\} \notin Y$ we call *asymmetric classes.* The vertices $u$ we call the *asymmetric neighbors* of $A$, the elements of $N_G(v) \setminus v$ we call the *neighbors* of $A$.

Infringement of condition 1 yields at least one asymmetric class $A$. If there is a neighbor of $A$, which is not contained in $V([Y])$, we delete all edges from $Y$, which connect an asymmetric neighbor of $A$ with an element of $A$. Otherwise we add all edges incident with vertices in $A$ to $Y$.

In the first situation, some components of $[Y]$ may split, but $\partial Y$ may only loose vertices. Hence $\mathrm{perm}_G(X,Y)$ will not increase. Furthermore, the ends of the deleted edges where before this step connected in $G-X-Y$. Hence, an $H^*$-path of $G-X-Y$ after this step yields an $H^*$-path of $G-X-Y$ befor this step. Thus, if before this step $(X,Y)$ was an optimal $H^*$-separator of $G$, in this step it remains an optimal $H^*$-separator

In the second situation, all at most $|A|$ components of $[Y]$ containing vertices of $A$ glue together; but if there are more than one such components, then $\partial_{G-X}Y$ looses at least $|A|$ vertices.

Hence, $\mathrm{perm}_G(X,Y)$ will not increase, too. Thus, clearly, $(X,Y)$ remains an optimal $H^*$-separator of $G$, too.

Both variants of our transformation cannot produce asymmetric neighbors of another class $A' \in \mathcal{A}$.

Hence, this step decreases the number of asymmetric classes.

**Case 2)**

If, otherwise, $(X,Y)$ infringes condition 2, then there are vertices $u,v \in V(G-H)$ such that $N_G(u) \subseteq N_G(v)$, $u \in X$, and $v \notin X$. In this case, we delete $u$ from $X$ and add all edges to $Y$ that connect $u$ to a neighbor of $v$ in $Y$.

In $G-X-Y$ this transformation adds $u$ to the component which contains $v$, but all components remain separated. Hence $(X,Y)$ stays an $H^*$-separator of $G$.

In $\mathrm{perm}_G(X,Y)$ the term $|X|$ decreases by one, while $u$ is added $\partial_{G-X}C$ with $C$ being the component of $[Y]$ containing $v$ (which must exist, because otherwise we have a contradiction to the optimality of $(X,Y)$ before this step). Since nothing else changes in $\mathrm{perm}_G(X,Y)$, the $H^*$-separator $(X,Y)$ stays optimal.

In this case, condition 1 obviously remains satisfied, whereas $|X|$ increases.

**Case 3)**

If, finally, $(X,Y)$ only infringes condition 3, the graph $G'$ obtained from $G$ by deleting all elements of $X \cup Y \setminus E(C)$ and contracting $C$ to a vertex $c$ has a pair $G_1', G_2'$ of subgraphs such that $H^* \cap V(G_2') = \emptyset$, $|V(G_1' \cap G_2')| = 1$, and $G_1' \cup G_2' = G'$.

This may glue some components $C_1, \dots C_k$ of $[Y]$ together resulting in a component $C'$. Hence, in $\mathrm{perm}_G(X,Y)$ the only change is, that the part $\sum_{i=1}^{k} \left\lfloor \frac{1}{2} |\partial_{G-X} C_i| \right\rfloor$ of the sum-term will be replaced by $\left\lfloor \frac{1}{2} |\partial_{G-X} C'| \right\rfloor$. Furthermore, $\partial_{G-X} C' \subseteq \bigcup_{i=1}^{k} \partial_{G-X} C_i$. If $C_i \neq C$ we get additionally $|\partial_{G-X}C_i \setminus \partial_{G-X}C'| \geq 1$. Hence,

$$\left\lfloor \frac{1}{2}|\partial_{G-X}C'| \right\rfloor \leq \left\lfloor \frac{1}{2}1 - k + \sum_{i=1}^{k} |\partial_{G-X}C_i| \right\rfloor$$
$$\leq \sum_{i=1}^{k} \left\lfloor \frac{1}{2}|\partial_{G-X}C_i| \right\rfloor$$

Consequently, $(X, Y)$ stays an optimal $H^*$-separator of $G$. In this last case, condition 1 remains satisfied, while $|X|$ stays constant and $|Y|$ increases.

This algorithm will stop after a finite number of steps, because $G$ is finite. The resulting optimal $H^*$-separator $(X, Y)$ of $G$ obviously proves the lemma. □

The next lemma is a consequence of Lemma 14.

**Lemma 15** *Let $G$ be a graph and $H$ be an independent subset of $V(G)$. Then, $G$ has an optimal $H$-separator $(X, Y)$ such that for each $h$ with the property, that the neighborhoods of the neighbors of $h$ are identical, one of the following conditions holds*

1. *The neighborhood of $h$ is disjoint to $X$ and no edge of $Y$ is incident to a vertex of it.*
2. *Each vertex adjacent to $h$ is contained in $X$.*
3. *Each edge incident with a neighbor of $h$ but not incident with $h$ is in $Y$.*

**Proof of Lemma 15.** Let $(X, Y)$ be an optimal $H$-separator of $G$ satisfying the conditions of Lemma 15 and let $h$ be an arbitrary vertex with the property, that the neighborhoods of the neighbors of $h$ are identical.

Suppose there is a vertex of $X$ in the neighborhood of an $h$. By condition 2 of Lemma 15 condition 2 of Theorem 15 holds.

If, otherwise, conditions 1 of Lemma 15 are violated, $Y$ contains an edge connecting a neighbor $v$ of $h$ with a vertex $w \neq h$.

By condition 1 of Lemma 14, each neighbor of $h$ is connected with $w$ in $[Y]$. If, furthermore, condition 3 of Lemma 15 is violated, too, $G - X - Y$ contains an edge $e$ connecting $v$ with a vertex $u \neq h$.

Let $C$ be the component of $[Y]$ containing $w$ and $D$ be the component of $G - X - Y$ containing $e$. We have $V(D) \cap H = \{h\}$. After contracting $C$ in $D$ to a vertex $c$, this vertex becomes a cutvertex of $D$. This contradicts condition 3 of Lemma 14, and the proof is done. □

The following lemma is a big step toward the proof of Theorem 11.

**Lemma 16** *Let $G$ be a graph, $H^*$ be an independent subset of $V(G)$, and $f$ be a function mapping $H^*$ into the positive integers. Let $G^*$ be obtained from $G$ by deleting each edge incident (in $G$) with a vertex of $H^*$ and, for each $h \in H^*$, adding $f(h)$ new vertices connected to $h$ and all neighbors (in $G$) of $h$.*

*Then the maximal size of a partial $f$-Factor of $G$ is $p_{G^*}(H^*)$.*

**Proof of Lemma 16.** If we have a partial $f$-factor of $G$ of size $s$, it is obvious, that $G'$ has $s$ internally disjoint $H$-paths. For the other direction, consider a set $S$ of $p_{G^*}(H^*)$ internally disjoint paths. By construction of $G^*$, $S$ is a partial $f$-factor of $G'$.

Beyond all possibilities choose $S$ with a minimal number of edges in the union of its paths. This additional condition yields, that a path $P$ containing a neighbor of a vertex $h \in H^*$ also contains the edge connecting it to $h$. Hence, by if we contract all edges incident with an element of $H^*$, $G^*$ becomes $G$ and $S$ becomes a partial $f$-factor of $G$ of the same size $|S|$. This completes the proof. $\square$

**Proof of Lemma 11.** Let $G^*$ be the graph obtained from $G$ as described in Lemma 16.

First, let $(X^*, Y^*)$ be an optimal $H^*$-separator of $G^*$ satisfying the conditions of Lemma 14 and set $X = (V(G) \cap X^*) \cup \{h \in H^* \mid N_{G^*}(h) \subseteq X\}$ and $Y = (E(G) \cap Y^*) \cup \{\{h, x\} \in E(G) \mid N_{[Y^*]}(x) = N_{G^*}(h)\}$. Then, by Lemma 15, $(X, Y)$ is an $f$-separator of $G$ with $\mathrm{perm}_{G,f}(X, Y) = \mathrm{perm}_{G^*}(X^*, Y^*)$.

Second, let $(X, Y)$ be an $f$-separator of $G$ and set $X^* = (X \setminus H) \cup \bigcup\limits_{x \in X \cap H} N_{G^*}(x)$ and $Y^* = E([Y] - H) \cup \{\{v, w\} \in E(G^*) \mid \exists h \in H^* : v \in N_{G^*}(h) \wedge w \neq h\}$. Then $(X^*, Y^*)$ is an $H^*$-separator of $G^*$ with $\mathrm{perm}_{G^*}(X^*, Y^*) = \mathrm{perm}_{G,f}(X, Y)$. Thus, the maximum number of internally disjoint $H^*$-paths equals the minimum of $\mathrm{perm}_{G,f}(X, Y)$ taken over all $f$-separators of $G$. With Lemma 16, the proof is complete. $\square$

**Proof of Theorem 10.** If all the local $2k$-factors exist, the assertion is trivial. Hence we only have to prove the other direction. Therefore $H^*$ is assumed to be topological $k$-tough in $G$. Consequently, every subset of $H^*$ is topological $k$-tough in $G$ and thus it suffices to prove that there is an $H^*$-local $2k$-factor.

We'll do this indirectly, i.e. in the sequel we assume, that there is no $H^*$-local $2k$-factor, and we have to show, that there is an $H \subseteq H^*$, such that $p_G(H) < k|H|$ (i.e. $G$ has no $k|H|$ internally disjoint $H$-paths).

Because intersecting edges by additional vertices will neither destroy $H^*$-local $2k$-factors, nor change the maximum number of internally disjoint $H^*$-paths, we may and will assume in the sequel, that $H^*$ is independent in $G$.

By Corollary 13 (with $f(h) = 2k$ for all $h \in H^*$) there is a pair $(X, Y)$ such that

1. $X \subseteq V(G)$,
2. $Y \subseteq E(G - X)$,
3. $\partial_{G-X}[Y] \subseteq V(G - H^*)$
4. $G - X - Y$ has no $H^*$-path, and
5. $|X \setminus H^*| + 2k|X \cap H^*| + k|V([Y]) \cap H^*| + \sum\limits_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor < k|H^*|$.

From property 5, we deduce $|H^* \setminus (X \cup V([Y]))| \geq 1$.

In a first case we study equality. In this case let $H$ be the set of the unique vertex $h \in H^* \setminus (X \cup V([Y]))$, and an arbitrary other vertex $h'$ from $H^*$. Note, that here $Y$ must (and will) not be contained in $G - X - H$.

Property 5 in this case yields $|X \cap H^*| = 0$. Hence $[Y]$ has a component $C_h$ containing $h'$. By property 3 we get that $X \cup \partial_{G-X} C_h$ in $G$ separates $h$ from $h'$. Finally, property 5 yields $|X| + \left\lfloor \frac{|\partial_{G-X} C|}{2} \right\rfloor \leq k - 1$, and hence $|X \cup \partial_{G-X} C_h| \leq 2k - 1$, which by Theorem 2 completes the proof in this case.

In the remaining case we set $H = H^* \setminus (X \cup V([Y]))$. Here $X \subseteq V(G - H)$ and $Y \subseteq E(G - H - X)$ hold, and $G - X - Y$ has no $H$-path. Finally, Theorem 1 together with property 5 yield

$$
\begin{aligned}
p_G(H) &\le |X| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor \\
&\le |X \setminus H^*| + 2k|X \cap H^*| + k|V([Y]) \cap H^*| - k|H^* \cap (X \cup V([Y]))| + \\
&\quad + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor \\
&< k(|H^*| - |H^* \cap (X \cup V([Y]))|) = k|H^* \setminus (X \cup V([Y]))| = k|H|
\end{aligned}
$$

□

## References


1. V. Chvátal, *Tough graphs and hamiltonian circuits*, Discrete Mathematics **5,** (1973) 215-228.
2. R. Diestel, *Graph Theory* (Springer, 2000) Graduate Texts in Mathematics 173(2000).
3. G. A. Dirac, *4-chromatische Graphen und vollständige 4-Graphen*, Mathematische Nachrichten **22,** (1960) 51-60.
4. H. Enomoto, B. Jackson, P. Katerinis, and A. Saito, *Toughness and the existence of k-factors*, Journal of Graph Theory **9,** (1985) 87-95.
5. T. Gerlach, F. Göring, J. Harant, M. Tkáč, *On cycles through specified vertices*, Discrete Mathematics **306,** (2006) 831-835.
6. F. Göring, J. Harant, E. Hexel, Zs. Tuza, *On short cycles through prescribed vertices of a graph*, Discrete Mathematics **286,** (2004) 67-74.
7. J. Harant, *On paths and cycles through specified vertices*, Discrete Mathematics **286,** (2004) 95-98.
8. G. Y. Katona, *Toughness and edge-toughness*, Discrete Mathematics **164(1-3),** (1997) 187-196.
9. G. Y. Katona, *Properties of edge-tough graphs*, Graphs and Combinatorics **15 (3),** (1999) 205-212.
10. A. K. Kelmans and M. V. Lomonosov, *When m vertices in a k-connected graph cannot be walked round along a simple cycle*, Discrete Mathematics **38,** (1982) 53-62.
11. M. V. Lomonosov, *Cycles through prescribed elements in a graph* in *Paths, flows, and VLSI-layout* (Bonn, 1988) 215-234; Algorithms and Combinatorica **9,** Springer (Berlin 1990).
12. W. Mader, *Über die Maximalzahl kreuzungsfreier H-Wege*, Arch. Math. **31,** (1978) 387-402.
13. W. Mader, *Über ein graphentheoretisches Problem von T.Gallai*, Arch. Math. **33,** (1979) 239-257.
14. K. Menger, *Zur allgemeinen Kurventheorie*, Fund. Math. **10,** (1927) 96-115.