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Optimality conditions for weak efficiency to vector optimization problems with composed convex functions

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Abstract. We consider a convex optimization problem with a vector valued function as objective function and finitely many inequalities involving convex functions as constraints. We suppose that each entry of the objective function is the composition of some convex functions. Our aim is to provide necessary and sufficient conditions for the weakly efficient solutions of this vector problem. Moreover, a multiobjective dual treatment is given and weak and strong duality assertions are proved.

Key Words. multiobjective optimization, composed convex functions, conjugate duality, weak efficiency

1 Introduction

Many optimization problems which arise from various fields of applications (like physics, economics, engineering) have not just one objective function, but a finite or even an infinite number of objectives, this being a reason why many mathematicians pay great attention to such kind of problems (see [5] and [6] and the references therein). As the complexity of the optimization

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problems is increasing, the study of problems which encompass as special cases the already treated ones is of large interest. Since many optimization problems involve composed convex functions, the attention of many researchers has turned to such kind of problems. From the large number of papers that have appeared during the last decades and treat composed convex optimization problems, we mention here [2], [8], [9], [11], [12], [13], [14], [15] and [17].

Given an optimization problem with a single-valued objective function, one can associate to it, by means of the very fruitful conjugate duality theory, various dual problems, like for example the classical Lagrange and Fenchel duals, but also the so-called Fenchel-Lagrange dual. The last one has been introduced by Boţ and Wanka and it is a "combination" of the classical ones (for more information see [1], [3], [4] and [7]). Regarding optimization problems which involve composed convex functions, the Fenchel-Lagrange duality has proved to be very useful in giving a compact formula for the dual and in deriving necessary and sufficient optimality conditions (see, for example, [1], [2]).

Let us consider a vector valued function whose entries are compositions of some convex functions. Having a problem with an objective function of this kind and with cone inequality constraints, our aim is to provide necessary and sufficient conditions for its weakly efficient solutions, expressed by using the conjugates of the functions involved. To this end we associate to our initial problem a family of scalar optimization problems and to each scalar problem we provide a Fenchel-Lagrange-type dual. Regarding the construction of the Fenchel-Lagrange-type dual of the scalar problem, we would like to mention that the approach we use is similar to the one used in [1] and [2]. Namely, we consider a problem which is equivalent to the scalar one in the sense that their optimal objective values are equal, but whose dual can be easier established. For the new problem we consider first the Lagrange dual problem. To the inner infimum of the Lagrange dual we attach the Fenchel dual problem and it can be easily seen that the final dual we obtain is actually a Fenchel-Lagrange-type dual of the primal problem. The construction of the dual is described here in detail and a constraint qualification ensuring strong duality is introduced. Further, using only the weak and the strong duality between the scalar problem and its dual, we derive the necessary and sufficient conditions which characterize the weakly efficient solutions of the primal vector problem. Moreover, a multiobjective dual to the initial problem is given, for which weak and strong duality theorems are proved.

The paper is organized as follows. In Section 2 we give some notions and results which are used later. The third section contains the main results of the paper. The multiobjective optimization problem we work with is presented together with a family of scalar problems associated to it. Moreover, to each of these scalar problems a dual problem is given and, using the weak and strong duality, some necessary and sufficient conditions for the weakly efficient solutions of the multiobjective problem are established. A multiobjective dual of the initial problem is given and weak and strong duality assertions are proved, too. In the last section of the paper some particular cases are considered.

2 Preliminary notions and results

In this section we present the notations we use throughout the paper. Some well-known notions and results which are used later are mentioned, too. All the vectors considered are column vectors. In order to transpose a column vector to a row vector we use an upper index ^{T}. Considering two arbitrary vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ from the real space $\mathbb{R}^n$, by $x^T y$ is denoted the usual inner product (i.e. we have $x^T y = \sum_{i=1}^n x_i y_i$). As usual, by "$\leqq_K$" is denoted the partial order introduced by the convex cone $K \subseteq \mathbb{R}^n$, defined by

$$x \leqq_K y \Leftrightarrow y - x \in K,\ x, y \in \mathbb{R}^n.$$

Let us mention that throughout this paper the cones are assumed to contain the element 0.

If $X \subseteq \mathbb{R}^n$ is given, its *relative interior* is denoted by $\mathrm{ri}(X)$. The *indicator function* of the set X is defined in the following way

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a given function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}(h) = \big\{x \in \mathbb{R}^n : h(x) < +\infty\big\}$ its *effective domain*. We say that the function is *proper* if its effective domain is a nonempty set and $h(x) > -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function h the *conjugate regarding to the set* X by

$$h_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad h_X^*(p) = \sup_{x \in X} \big\{p^T x - h(x)\big\}.$$

Regarding the conjugate, we would like to mention that the inequality (Young-Fenchel)

$$h(x) + h_X^*(x^*) - x^{*T}x \geq 0 \tag{1}$$

is fulfilled for all $x \in X$ and $x^* \in \mathbb{R}^n$. It is easy to see that for $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of h denoted by h^*. Even more, it can be easily proved that $h_X^* = (h + \delta_X)^*$.

The rules we adopt concerning the arithmetic calculation involving $+\infty$ and $-\infty$ are those in [16]. In this context, as

$$0(+\infty) = 0 \text{ and } 0(-\infty) = 0,$$

we can easily prove that

$$(0h)^*(x^*) = \begin{cases} 0, & x^* = 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2}$$

while

$$(\alpha h)^*(\alpha x^*) = \alpha h^*(x^*) \tag{3}$$

holds independently from this conventions for all $x^* \in \mathbb{R}^n$ and $\alpha > 0$.

Definition 2.1 Let $K \subseteq \mathbb{R}^k$ be a convex cone.

(i) The function $h : \mathbb{R}^k \to \overline{\mathbb{R}}$ is called K-*increasing* if for all $x, y \in \mathbb{R}^k$ such that $x \leqq_K y$ it holds $h(x) \leq h(y)$.

(ii) The function $H : \mathbb{R}^n \to \mathbb{R}^k$ is called K-*convex* if for all $x, y \in \mathbb{R}^n$ and for all $\alpha \in [0, 1]$ we have

$$H\big(\alpha x + (1 - \alpha)y\big) \leqq_K \alpha H(x) + (1 - \alpha)H(y).$$

Definition 2.3 Let $K \subseteq \mathbb{R}^n$ be a convex cone. By the *dual cone* of K we denote the set

$$K^* = \{x^* \in \mathbb{R}^n : x^{*T}x \geq 0, \forall x \in K\}.$$

Lemma 2.1 Let $K \subseteq \mathbb{R}^n$ be a convex cone and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ a proper and K-increasing function. Then $h^*(x^*) = +\infty$ for all $x^* \notin K^*$.

Proof. Take an arbitrary $x^* \notin K^*$. By definition there exists $\overline{x} \in K$ such that $x^{*T}\overline{x} < 0$. Since for some arbitrary $\widetilde{x} \in \text{dom}(h)$ and for all $\alpha > 0$ we have $h(\widetilde{x} - \alpha\overline{x}) \leq h(\widetilde{x})$, it is not hard to see that

$$\begin{aligned} & h^*(x^*) = \sup_{x \in \mathbb{R}^n} \{x^{*T}x - h(x)\} \geq \sup_{\alpha > 0}\{x^{*T}(\widetilde{x} - \alpha\overline{x}) - h(\widetilde{x} - \alpha\overline{x})\} \\ \geq \ & \sup_{\alpha>0}\{x^{*T}(\widetilde{x} - \alpha\overline{x}) - h(\widetilde{x})\} = x^{*T}\widetilde{x} - h(\widetilde{x}) + \sup_{\alpha>0}\{-\alpha x^{*T}\overline{x}\} = +\infty, \end{aligned}$$

and the proof of the lemma is complete. □

Definition 2.4 We call *infimal convolution* of the proper functions $h_1, ..., h_k : \mathbb{R}^n \to \overline{\mathbb{R}}$, the function

$$h_1 \square ... \square h_k : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad (h_1 \square ... \square h_k)(x) = \inf \left\{ \sum_{i=1}^k h_i(x_i) : x = \sum_{i=1}^k x_i \right\}.$$

The following statement closes this preliminary section.

Theorem 2.1 (cf. [16]) Let $h_1, ..., h_k : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^k \text{ri}(\text{dom}(h_i))$ is nonempty, then

$$\left(\sum_{i=1}^k h_i \right)^* (p) = (h_1^* \square ... \square h_k^*)(p) = \inf \left\{ \sum_{i=1}^k h_i^*(p_i) : p = \sum_{i=1}^k p_i \right\},$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

3 The composite multiobjective problem

In the first subsection of this section we present the multiobjective problem we treat within the paper. A family of scalar optimization problems is then attached to it and a characterization of the weakly efficient solutions is given. In the second subsection we provide a dual problem to the scalar problem derived in the first subsection and a weak and a strong duality theorem are proved. Moreover, necessary and sufficient optimality conditions for weak efficiency are presented. In the last subsection a multiobjective dual of the primal one is also introduced and weak and strong duality assertions for the vector primal and dual problems are proved.

3.1 The general framework

In the following let $X \subseteq \mathbb{R}^n$ be a nonempty convex set, $K \subseteq \mathbb{R}^m$ a convex cone containing 0 and $g : \mathbb{R}^n \to \mathbb{R}^m$, $g = (g_1, ..., g_m)^T$, be a K-convex function. For $i = 1, ..., k$, let $K_i \subseteq \mathbb{R}^{n_i}$ be a convex cone ($0 \in K_i$) and consider the functions $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$ and $F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ such that f_i is a proper, convex and K_i-increasing function, while F_i is a K_i-convex one.

The primal vector optimization problem we treat within the present paper is

$$(P) \qquad \operatorname*{v-min}_{\substack{x \in X, \\ g(x) \leqq_K 0}} \big(f_1 \circ F_1(x), ..., f_k \circ F_k(x)\big)^T.$$

Moreover, we suppose that

$$\mathcal{A} \subseteq \bigcap_{i=1}^{k} F_i^{-1}(\mathrm{dom}(f_i)),$$

where $\mathcal{A} = \{x \in X : g(x) \leqq_K 0\} \neq \emptyset$ is the feasible set of the problem (P) and $F_i^{-1}(\mathrm{dom}(f_i)) = \{x \in \mathbb{R}^n : F_i(x) \in \mathrm{dom}(f_i)\}$.

Definition 3.1 A feasible element $\overline{x} \in \mathcal{A}$ is called *weakly efficient solution* of the problem (P) if there exists no $x \in \mathcal{A}$ such that $f_i \circ F_i(x) < f_i \circ F_i(\overline{x})$ for all $i = 1, ..., k$.

The proof of the following proposition is omitted as it is trivial.

Proposition 3.1 Under the previous assumptions each function $f_i \circ F_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., k$, is a proper convex function.

To an arbitrary $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k_+$ we associate the set $I_\lambda = \{i \in \{1, ..., k\} : \lambda_i > 0\}$. One has $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$ if and only if $I_\lambda \neq \emptyset$.

By Proposition 3.1, (P) is a multiobjective convex optimization problem and, in order to characterize its weakly efficient solutions, to (P) we associate a family of scalar optimization problems. Namely, for each $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k_+ \setminus \{0\}$ we consider the optimization problem

$$(P_\lambda) \qquad \inf_{\substack{x\in X,\\ g(x)\leqq_K 0}} \sum_{i=1}^{k} \lambda_i (f_i \circ F_i)(x).$$

or, equivalently,

$$(P_\lambda) \qquad \inf_{\substack{x\in X,\\ g(x)\leqq_K 0}} \sum_{i\in I_\lambda} \lambda_i (f_i \circ F_i)(x).$$

The following well-known result gives a characterization of the weakly efficient solutions of a convex vector optimization problem via linear scalarization (see, for instance, [10]).

Theorem 3.1 A feasible point $\overline{x}$ of the problem (P) is weakly efficient if and only if there exists $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$ such that $\overline{x}$ is an optimal solution of the problem (P_λ).

3.2 Optimality conditions for weak efficiency

Let us consider an arbitrary $\lambda \in \mathbb{R}^k_+$ such that $I_\lambda \neq \emptyset$. We construct a dual problem to (P_λ) and from the strong duality assertion we derive the optimality conditions which characterize a weakly efficient solution for (P). To this end we associate to the problem (P_λ) the following convex optimization problem

$$(P'_\lambda) \qquad \inf_{\substack{x\in X,\, g(x)\leqq_K 0,\\ y_i\in\mathbb{R}^{n_i}, F_i(x)-y_i\leqq_{K_i} 0,\\ i\in I_\lambda}} \sum_{i\in I_\lambda} \lambda_i f_i(y_i).$$

In what follows by $v(P)$ we understand the optimal objective value of an optimization problem (P). Regarding the optimal values of the problems (P_λ) and (P'_λ), the following result can be established.

Theorem 3.2 It holds $v(P_\lambda) = v(P'_\lambda)$.

Proof. For an arbitrary x feasible to (P_λ) take $y_i = F_i(x)$ for all $i \in I_\lambda$, and so the tuple formed by x and y_i, $i \in I_\lambda$, is feasible to (P'_λ). Thus $\sum_{i\in I_\lambda} \lambda_i f_i\big(F_i(x)\big) = \sum_{i\in I_\lambda} \lambda_i f_i(y_i) \geq v(P'_\lambda)$, and this implies $v(P_\lambda) \geq v(P'_\lambda)$.

In order to prove the opposite inequality, let us consider some x and y_i, $i \in I_\lambda$, feasible to (P'_λ). Since $g(x) \leqq_K 0$, it follows immediately that x is feasible to (P_λ). By the hypothesis that f_i is a K_i-increasing function the inequality $F_i(x) - y_i \leqq_{K_i} 0$ implies $f_i(F_i(x)) \leq f_i(y_i)$, $\forall i \in I_\lambda$. We have $v(P_\lambda) \leq \sum_{i\in I_\lambda} \lambda_i f_i\big(F_i(x)\big) \leq \sum_{i \in I_\lambda} \lambda_i f_i(y_i)$. Taking the infimum on the right-side regarding x and y_i, $i \in I_\lambda$, feasible to (P'_λ) we obtain $v(P_\lambda) \leq v(P'_\lambda)$. □

Our next step is to construct a dual problem to (P'_λ) (see also [1], [2]) and to give sufficient conditions in order to achieve strong duality, i.e. the situation when the optimal objective value of the primal coincides with the optimal objective value of the dual and the dual has an optimal solution.

First of all, we consider the Lagrange dual problem to (P'_λ)

$$(D_\lambda) \qquad \sup_{\substack{q\in K^*,\\ u_i \in K_i^*, i\in I_\lambda}} \inf_{\substack{x\in X,\\ y_i\in\mathbb{R}^{n_i}, i\in I_\lambda}} \left\{ \sum_{i\in I_\lambda} \lambda_i f_i(y_i) + q^T g(x) + \sum_{i\in I_\lambda} u_i^T (F_i(x) - y_i) \right\},$$

where $q \in K^*$ and $u_i \in K_i^*$, $i \in I_\lambda$, are the dual variables. Regarding the inner infimum, by the definition of the conjugate relative to X one obtains

$$\begin{aligned}
&\inf_{\substack{x\in X,\\ y_i\in\mathbb{R}^{n_i}, i\in I_\lambda}} \left\{ \sum_{i\in I_\lambda} \lambda_i f_i(y_i) + q^T g(x) + \sum_{i\in I_\lambda} u_i^T (F_i(x) - y_i) \right\}\\
= \; & -\sup_{x\in X} \left\{ - q^T g(x) - \sum_{i\in I_\lambda} u_i^T F_i(x) \right\} - \sum_{i\in I_\lambda} \sup_{y_i\in\mathbb{R}^{n_i}} \left\{ u_i^T y_i - \lambda_i f_i(y_i) \right\}\\
= \; & -\left(\sum_{i\in I_\lambda} u_i^T F_i + q^T g \right)_X^* (0) - \sum_{i\in I_\lambda} (\lambda_i f_i)^*(u_i).
\end{aligned}$$

Moreover, by Theorem 2.1 we get further

$$\begin{aligned}
&\left(\sum_{i\in I_\lambda} u_i^T F_i + q^T g \right)_X^* (0) = \left(\sum_{i\in I_\lambda} u_i^T F_i + (q^T g + \delta_X) \right)^* (0)\\
= \; & \inf_{v_i\in\mathbb{R}^n, i\in I_\lambda} \left\{ \sum_{i\in I_\lambda} (u_i^T F_i)^*(v_i) + (q^T g + \delta_X)^* \left(- \sum_{i\in I_\lambda} v_i \right) \right\} \qquad (4)\\
= \; & \inf_{v_i\in\mathbb{R}^n, i\in I_\lambda} \left\{ \sum_{i\in I_\lambda} (u_i^T F_i)^*(v_i) + (q^T g)_X^* \left(- \sum_{i\in I_\lambda} v_i \right) \right\}.
\end{aligned}$$

Taking into consideration the previous relations, the dual (D_λ) can be equivalently rewritten as

$$(D_\lambda) \quad \sup_{\substack{q\in K^*,\\ u_i\in K_i^*, i\in I_\lambda}} \sup_{\substack{v_i\in\mathbb{R}^n,\\ i\in I_\lambda}} \left\{-\sum_{i\in I_\lambda}(\lambda_i f_i)^*(u_i) - \sum_{i\in I_\lambda}(u_i^T F_i)^*(v_i) - (q^T g)_X^*\left(-\sum_{i\in I_\lambda} v_i\right)\right\}.$$

Introducing the new variables $\beta_i := \left(\frac{1}{\lambda_i}\right)u_i$ and $p_i := \left(\frac{1}{\lambda_i}\right)v_i$, $i \in I_\lambda$, the dual problem can be written as (we use relation (3))

$$(D_\lambda) \quad \sup_{\substack{q\in K^*,\\ \beta_i\in K_i^*, p_i\in\mathbb{R}^n,\\ i\in I_\lambda}} \left\{-\sum_{i\in I_\lambda}\lambda_i f_i^*(\beta_i) - \sum_{i\in I_\lambda}\lambda_i(\beta_i^T F_i)^*(p_i) - (q^T g)_X^*\left(-\sum_{i\in I_\lambda}\lambda_i p_i\right)\right\}.$$

It is well-known that the optimal objective value of the problem (P'_λ) is always greater than or equal to the optimal objective value of its Lagrange dual, i.e. $v(P'_\lambda) \geq v(D_\lambda)$. Because of Theorem 3.2 the problem (D_λ) is a dual problem to (P_λ), too, and thus the following assertion arises easily.

Theorem 3.3 Between the primal problem (P_λ) and the dual problem (D_λ) weak duality always holds, i.e. $v(P_\lambda) \geq v(D_\lambda)$.

In order to ensure the equality of the optimal objective values of the two problems we have to impose a constraint qualification. The idea we follow is similar to the one presented in [1] and to this aim some preliminary work is necessary. Let us consider that $I_\lambda = \{i_1, ..., i_l\}$ $(l \leq k)$ and take $Y = \mathrm{dom}(f_{i_1}) \times ... \times \mathrm{dom}(f_{i_l}) \subseteq \mathbb{R}^N$, where $N = n_{i_1} + ... + n_{i_l}$. It is not hard to see that the optimization problem (P'_λ) can be equivalently written as

$$(P''_\lambda) \qquad \inf_{\substack{(x,y)\in X\times Y,\\ B(x,y)\leqq_Q 0}} A(x,y),$$

where $Q = K \times K_{i_1} \times ... \times K_{i_l}$, $y = (y_{i_1}, ..., y_{i_l}) \in \mathbb{R}^{n_{i_1}} \times ... \times \mathbb{R}^{n_{i_l}} = \mathbb{R}^N$,

$$A : \mathbb{R}^n \times \mathbb{R}^N \to \overline{\mathbb{R}},\ A(x,y) = \lambda_{i_1} f_{i_1}(y_{i_1}) + ... + \lambda_{i_l} f_{i_l}(y_{i_l})$$

and

$$B : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^m \times R^N,\ B(x,y) = \big(g(x), F_{i_1}(x) - y_{i_1}, ..., F_{i_l}(x) - y_{i_l}\big)^T.$$

Let us notice that Q is a convex cone containing 0 and that (P''_λ) is a convex optimization problem. Using the results and considerations in [1] (cf. the proof of Proposition 1 the closedness assumption for Q is there superfluous) it follows that between (P''_λ) and its Fenchel-Lagrange dual problem

$$(D''_\lambda) \qquad \sup_{\substack{(x^*,y^*)\in\mathbb{R}^n\times\mathbb{R}^N,\\ \gamma\in Q^*}} \{-A^*(x^*,y^*) - (\gamma^T B)^*_{X\times Y}(-x^*,-y^*)\}$$

strong duality holds if the following condition is fulfilled

$$0 \in B\big(\operatorname{ri}(X\times Y)\big) + \operatorname{ri}(Q). \tag{5}$$

Since

$$\operatorname{ri}(Q) = \operatorname{ri}(K) \times \operatorname{ri}\left(K_{i_1}\right) \times ... \times \operatorname{ri}\left(K_{i_l}\right),$$

relation (5) requires the existence of some $x' \in \operatorname{ri}(X)$ and $y' = (y'_{i_1}, ..., y'_{i_l}) \in \operatorname{ri}(Y)$ such that

$$0 \in \left(g(x'), F_{i_1}(x') - y'_{i_1}, ..., F_{i_l}(x') - y'_{i_l}\right) + \operatorname{ri}(K) \times \operatorname{ri}\left(K_{i_1}\right) \times ... \times \left(K_{i_l}\right).$$

The last relation is equivalent with

$$g(x') \in -\operatorname{ri}(K) \text{ and } F_{i_j}(x') \in y'_{i_j} - \operatorname{ri}\left(K_{i_j}\right), j = 1, ..., l,$$

and from here the condition

$$(CQ_\lambda) \quad \exists x' \in \operatorname{ri}(X) \text{ such that } \begin{cases} F_i(x') \in \operatorname{ri}(\operatorname{dom}(f_i)) - \operatorname{ri}(K_i), & i \in I_\lambda, \\ g(x') \in -\operatorname{ri}(K). \end{cases}$$

can be easily derived.

In the following we prove that the dual problems (D_λ) and (D''_λ) are identical. To this aim let us take some arbitrary $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^N$ and $\gamma \in Q^*$. This is equivalent with the existence of some vectors $y^*_{i_1} \in \mathbb{R}^{n_{i_1}}$, ..., $y^*_{i_l} \in \mathbb{R}^{n_{i_l}}$ and $q \in K^*$, $\beta_{i_1} \in K^*_{i_1}$, ..., $\beta_{i_l} \in K^*_{i_l}$ such that $y^* = (y^*_{i_1}, ..., y^*_{i_l})$ and $\gamma = (q, \beta_{i_1}, ..., \beta_{i_l})$, respectively.

Using the definition of the conjugate function we obtain

$$\begin{aligned}
A^*(x^*, y^*) &= \sup_{\substack{x\in\mathbb{R}^n,\\ y\in\mathbb{R}^N}} \left\{ x^{*T}x + y^{*T}y - A(x,y) \right\} \\
&= \sup_{\substack{x\in\mathbb{R}^n,\\ y_{i_j}\in\mathbb{R}^{n_{i_j}},\\ j=1,\dots,l}} \left\{ x^{*T}x + \sum_{j=1}^{l} {y_{i_j}^*}^T y_{i_j} - \sum_{j=1}^{l} \lambda_{i_j} f_{i_j}(y_{i_j}) \right\} \\
&= \sup_{x\in\mathbb{R}^n} x^{*T}x + \sum_{j=1}^{l} \sup_{y_{i_j}\in\mathbb{R}^{n_{i_j}}} \left\{ {y_{i_j}^*}^T y_{i_j} - \lambda_{i_j} f_{i_j}(y_{i_j}) \right\} \\
&= \sup_{x\in\mathbb{R}^n} \{x^{*T}x\} + \sum_{j=1}^{l} (\lambda_{i_j} f_{i_j})^*(y_{i_j}^*),
\end{aligned}$$

while

$$(\gamma^T B)^*_{X\times Y}(-x^*, -y^*) = \sup_{\substack{x\in X,\\ y\in Y}} \left\{ -x^{*T}x - y^{*T}y - \gamma^T B(x,y) \right\}$$

$$= \sup_{\substack{x\in X,\\ y_{i_j}\in \mathrm{dom}(f_{i_j}),\\ j=1,\dots,l}} \left\{ -x^{*T}x - \sum_{j=1}^{l} {y_{i_j}^*}^T y_{i_j} - q^T g(x) - \sum_{j=1}^{l} \beta_{i_j}^T (F_{i_j}(x) - y_{i_j}) \right\}$$

$$= \sup_{x\in X} \left\{ -x^{*T}x - q^T g(x) - \sum_{j=1}^{l} \beta_{i_j}^T F_{i_j}(x) \right\} + \sum_{j=1}^{l} \sup_{y_{i_j}\in \mathrm{dom}(f_{i_j})} \left\{ -{y_{i_j}^*}^T y_{i_j} + \beta_{i_j}^T y_{i_j} \right\}$$

$$= \left(q^T g + \sum_{j=1}^{l} \beta_{i_j}^T F_{i_j} \right)^*_X (-x^*) + \sum_{j=1}^{l} \delta^*_{\mathrm{dom}(f_{i_j})}(\beta_{i_j} - y_{i_j}^*).$$

Since it is binding to have $x^* = 0$ (otherwise $\sup_{x\in\mathbb{R}^n}\{x^{*T}x\} = +\infty$) we get

$$v(D''_\lambda) = \sup_{\substack{(x^*,y^*)\in\mathbb{R}^n\times\mathbb{R}^N,\\ \gamma\in Q^*}} \left\{ -A^*(x^*, y^*) - (\gamma^T B)^*_{X\times Y}(x^*, -y^*) \right\}$$

$$= \sup_{\substack{q\in K^*,\\ \beta_{i_j}\in K_{i_j}^*,\\ y_{i_j}^*\in\mathbb{R}^{n_{i_j}},\\ j=1,\dots,k}} \left\{ \sum_{j=1}^{l} \left\{ -(\lambda_{i_j} f_{i_j})^*(y_{i_j}^*) - \delta^*_{\mathrm{dom}(f_{i_j})}(\beta_{i_j} - y_{i_j}^*) \right\} - \left(q^T g + \sum_{j=1}^{l} \beta_{i_j}^T F_{i_j} \right)^*_X (0) \right\}.$$

As by Theorem 2.1

$$\sup_{y^*_{i_j}\in\mathbb{R}^{n_{i_j}}}\left\{-(\lambda_{i_j}f_{i_j})^*(y^*_{i_j})-\delta^*_{\mathrm{dom}(f_{i_j})}(\beta_{i_j}-y^*_{i_j})\right\}=-(\lambda_{i_j}f_{i_j})^*(\beta_{i_j}),$$

$j=1,...,l$, and

$$\left(q^Tg+\sum_{j=1}^{l}\beta^T_{i_j}F_{i_j}\right)^*_X(0)=\inf_{\substack{x^*_{i_j}\in\mathbb{R}^n,\\ j=1,...,l}}\left\{\sum_{j=1}^{l}(\beta^T_{i_j}F_{i_j})^*(x^*_{i_j})+(q^Tg)^*_X\left(-\sum_{j=1}^{l}x^*_{i_j}\right)\right\},$$

we obtain

$$v(D''_\lambda)=\sup_{\substack{q\in K^*,\\ x^*_{i_j}\in\mathbb{R}^n,\\ \beta_{i_j}\in K^*_{i_j},\\ j=1,...,k}}\left\{-\sum_{j=1}^{l}(\lambda_{i_j}f_{i_j})^*(\beta_{i_j})-\sum_{j=1}^{l}(\beta^T_{i_j}F_{i_j})^*(x^*_{i_j})-(q^Tg)^*_X\left(-\sum_{j=1}^{l}x^*_{i_j}\right)\right\}.$$

Introducing the new variables $\overline{q}=q$, $\overline{\beta}_{i_j}:=\left(\frac{1}{\lambda_{i_j}}\right)\beta_{i_j}$ and $\overline{x}^*_{i_j}:=\left(\frac{1}{\lambda_{i_j}}\right)x^*_{i_j}$, $j=1,...,l$, the previous relation can be rewritten as (we use relation (3))

$$v(D''_\lambda)=\sup_{\substack{\overline{q}\in K^*,\\ \overline{x}^*_{i_j}\in\mathbb{R}^n,\\ \overline{\beta}_{i_j}\in K^*_{i_j},\\ j=1,...,k}}\left\{-\sum_{j=1}^{l}\lambda_{i_j}f^*_{i_j}(\overline{\beta}_{i_j})-\sum_{j=1}^{l}\lambda_{i_j}(\overline{\beta}^T_{i_j}F_{i_j})^*(\overline{x}^*_{i_j})-(\overline{q}^Tg)^*_X\left(-\sum_{j=1}^{l}\lambda_{i_j}\overline{x}^*_{i_j}\right)\right\},$$

and it can be easily seen that the dual problems (D''_λ) and (D_λ) coincide.

We consider now the following constraint qualification for (P)

$$(CQ)\quad \exists x'\in\mathrm{ri}(X)\text{ such that}\begin{cases}F_i(x')\in\mathrm{ri}(\mathrm{dom}(f_i))-\mathrm{ri}(K_i), & i=1,...,k,\\ g(x')\in-\mathrm{ri}(K).\end{cases}$$

The following assertion displays the strong duality between the optimization problems (P_λ) and (D_λ).

Theorem 3.4 Suppose that the constraint qualification (CQ) is fulfilled. Then strong duality holds between (P_λ) and (D_λ), i.e. $v(P_\lambda)=v(D_\lambda)$ and the dual problem (D_λ) has an optimal solution.

Proof. Since (CQ) is fulfilled strong duality holds between the problems (P''_λ) and (D''_λ), i.e. $v(P''_\lambda) = v(D''_\lambda)$ and the dual has an optimal solution. Since this implies $v(P_\lambda) = v(P'_\lambda) = v(P''_\lambda) = v(D''_\lambda) = v(D_\lambda)$ and the existence of a solution for the problem (D_λ), the proof is complete. $\square$

Remark. Although for the proof of the previous theorem we need just the weaker assumption (CQ_λ), we decided to consider (CQ) since this constraint qualification is independent from the set I_λ.

Based on the just proved strong duality property we are able to point out necessary and sufficient optimality conditions for the solutions of problem (P). Theorem 3.5 is devoted to that matter.

Theorem 3.5

(a) Suppose that the condition (CQ) is fulfilled and let $\overline{x}$ be a weakly efficient solution of the problem (P). Then there exist $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in K^*$, $p_i \in \mathbb{R}^n$ and $\beta_i \in K_i^*$, $i \in I_\lambda = \{i \in \{1, ..., k\} : \lambda_i > 0\}$, such that

   (i) $f_i \circ F_i(\overline{x}) + f_i^*(\beta_i) - \beta_i^T F_i(\overline{x}) = 0$, $i \in I_\lambda$;

   (ii) $\beta_i^T F_i(\overline{x}) + (\beta_i^T F_i)^*(p_i) - p_i^T \overline{x} = 0$, $i \in I_\lambda$;

   (iii) $q^T g(\overline{x}) + (q^T g)^*_X \bigg(- \sum\limits_{i \in I_\lambda} \lambda_i p_i \bigg) + \sum\limits_{i \in I_\lambda} \lambda_i p_i^T \overline{x} = 0$;

   (iv) $q^T g(\overline{x}) = 0$.

(b) If there exists $\overline{x}$ feasible to (P) such that for some $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in K^*$, $p_i \in \mathbb{R}^n$ and $\beta_i \in K_i^*$, $i \in I_\lambda$, the conditions $(i) - (iv)$ are satisfied, then $\overline{x}$ is a weakly efficient solution of (P).

Proof. (a) Since $\overline{x}$ is a weakly efficient solution of (P), by Theorem 3.1 there exists $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k_+ \setminus \{0\}$ such that $\overline{x}$ is an optimal solution of the problem (P_λ). As (CQ) is fulfilled, Theorem 3.4 ensures the strong duality between (P_λ) and (D_λ). Thus there exist $q \in K^*$, $p_i \in \mathbb{R}^n$ and $\beta_i \in K_i^*$, $i \in I_\lambda$, such that

$$\sum_{i \in I_\lambda} \lambda_i (f_i \circ F_i)(\overline{x}) = - \sum_{i \in I_\lambda} \lambda_i f_i^*(\beta_i) - \sum_{i \in I_\lambda} \lambda_i (\beta_i^T F_i)^*(p_i) - (q^T g)^*_X \bigg(- \sum_{i \in I_\lambda} \lambda_i p_i \bigg).$$

The last equality is nothing else than

$$\begin{aligned}
0 &= \sum_{i\in I_\lambda}\lambda_i(f_i\circ F_i)(\overline{x})+\sum_{i\in I_\lambda}\lambda_i f_i^*(\beta_i)+\sum_{i\in I_\lambda}\lambda_i(\beta_i^T F_i)^*(p_i)+(q^Tg)_X^*\Big(-\sum_{i\in I_\lambda}\lambda_i p_i\Big)\\
&= \sum_{\in I_\lambda}\lambda_i\Big[(f_i\circ F_i)(\overline{x})+f_i^*(\beta_i)-\beta_i^T F_i(\overline{x})\Big]\\
&\quad+\sum_{i\in I_\lambda}\lambda_i\Big[\beta_i^T F_i(\overline{x})+(\beta_i^T F_i)^*(p_i)-p_i^T\overline{x}\Big]\\
&\quad+\Big[q^Tg(\overline{x})+(q^Tg)_X^*\Big(-\sum_{i\in I_\lambda}\lambda_i p_i\Big)-\Big(-\sum_{i\in I_\lambda}\lambda_i p_i^T\overline{x}\Big)\Big]\\
&\quad-q^Tg(\overline{x}).
\end{aligned}$$

As $g(\overline{x}) \leqq_K 0$ ($\overline{x}$ is a feasible solution to (P)) and $q\in K^*$ we have $-q^Tg(\overline{x})\geq 0$. Moreover, all the other terms within the brackets of the previous sum are non-negative (see relation (1)). Thus each term must be equal to 0 and the relations $(i)-(iv)$ follows.

(b) Following the same steps as in (a), but in the reverse order, the desired conclusion can be easily reached. □

Remark. For the assertion (b) of Theorem 3.5, i.e. the sufficiency of the conditions (i), ..., (iv) for the weak efficiency of $\overline{x}$ the fulfillment of (CQ) is not necessary.

3.3 The vector dual of (P)

For an arbitrary $\lambda=(\lambda_1,...,\lambda_k)^T\in\mathbb{R}^k_+\setminus\{0\}$ let be $|\lambda|=\sum_{i=1}^k\lambda_i$. We introduce the following multiobjective dual problem to (P)

$$(D) \qquad \underset{(\lambda,q,p,\beta,t)\in\mathcal{B}}{\text{v-max}}\big(h_1(\lambda,q,p,\beta,t),...,h_k(\lambda,q,p,\beta,t)\big)^T,$$

where

$$h_i(\lambda,q,p,\beta,t)=-f_i^*(\beta_i)-(\beta_i^TF_i)^*(p_i)-\frac{1}{|\lambda|}(q^Tg)_X^*\Big(\sum_{i\in I_\lambda}\lambda_i p_i\Big)+t_i$$

for all $i = 1, ..., k$, and the dual variables are $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k$, $q = (q_1, ..., q_m)^T \in \mathbb{R}^m$, $p = (p_1, ..., p_k) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$, $\beta = (\beta_1, ..., \beta_k) \in \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$ and $t = (t_1, ..., t_k)^T \in \mathbb{R}^k$. The feasible set of the problem (D) is described by

$$\mathcal{B} = \left\{ (\lambda, q, p, \beta, t) : \lambda \in \mathbb{R}^k_+ \setminus \{0\}, q \in K^*, \beta_i \in K_i^*, i = 1, ..., k, \sum_{i=1}^{k} \lambda_i t_i = 0 \right\}.$$

As for the primal problem (P) we consider for the dual problem weakly efficient solutions, too.

Definition 3.2 A feasible element $(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t}) \in \mathcal{B}$ is called weakly efficient solution of the problem (D) if there exists no $(\lambda, q, p, \beta, t) \in \mathcal{B}$ such that $h_i(\lambda, q, p, \beta, t) > h_i(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t})$ for all $i = 1, ..., k$.

Theorem 3.6 (*weak vector duality*) There is no $x \in \mathcal{A}$ and no $(\lambda, q, p, \beta, t) \in \mathcal{B}$ such that $f_i \circ F_i(x) < h_i(\lambda, q, p, \beta, t)$ for all $i = 1, ..., k$.

Proof. In order to prove the theorem we suppose that there exist $x \in \mathcal{A}$ and $(\lambda, q, p, \beta, t) \in \mathcal{B}$ such that $f_i \circ F_i(x) < h_i(\lambda, q, p, \beta, t)$. Since $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$, the inequality

$$\sum_{i=1}^{k} \lambda_i f_i \circ F_i(x) < \sum_{i=1}^{k} \lambda_i h_i(\lambda, q, p, \beta, t) \tag{6}$$

follows immediately. But

$$\sum_{i=1}^{k} \lambda_i h_i(\lambda, q, p, \beta, t) = \sum_{i \in I_\lambda} \lambda_i h_i(\lambda, q, p, \beta, t)$$
$$= \sum_{i \in I_\lambda} \lambda_i \left[- f_i^*(\beta_i) - (\beta_i^T F_i)^*(p_i) - \frac{1}{|\lambda|} (q^T g)^*_X \left(\sum_{i \in I_\lambda} \lambda_i p_i \right) + t_i \right],$$

and, since $|\lambda| = \sum_{i=1}^{k} \lambda_i = \sum_{i \in I_\lambda} \lambda_i$ and $\sum_{i \in I_\lambda} \lambda_i t_i = \sum_{i=1}^{k} \lambda_i t_i = 0$, we get

$$\sum_{i \in I_\lambda} \lambda_i h_i(\lambda, q, p, \beta, t) = - \sum_{i \in I_\lambda} \lambda_i f_i^*(\beta_i) - \sum_{i \in I_\lambda} \lambda_i (\beta_i^T F_i)^*(p_i) - (q^T g)^*_X \left(- \sum_{i \in I_\lambda} \lambda_i p_i \right). \tag{7}$$

The inequalities

$$- \sum_{i \in I_\lambda} \lambda_i f_i^*(\beta_i) \leq \sum_{i \in I_\lambda} \lambda_i f_i \circ F_i(x) - \sum_{i \in I_\lambda} \lambda_i \beta_i^T F_i(x) \tag{8}$$

and

$$-\sum_{i\in I_\lambda}\lambda_i(\beta_i^T F_i)^*(p_i) \leq \sum_{i\in I_\lambda}\lambda_i\beta_i^T F_i(x) - \sum_{i\in I_\lambda}\lambda_i p_i^T x \tag{9}$$

are easy consequences of the Young-Fenchel inequality as well as

$$-(q^T g)_X^*\bigg(-\sum_{i\in I_\lambda}\lambda_i p_i\bigg) \leq \sum_{i\in I_\lambda}\lambda_i p_i^T x + (q^T g)(x).$$

Since $q^T g(x) \leq 0$ ($q \in K^*$ and $g(x) \in -K$) there follows the inequality

$$-(q^T g)_X^*\bigg(-\sum_{i\in I_\lambda}\lambda_i p_i\bigg) \leq \sum_{i\in I_\lambda}\lambda_i p_i^T x. \tag{10}$$

Adding up relations (8), (9) and (10) we get

$$\begin{aligned}
&\sum_{i=1}^k \lambda_i h_i(\lambda, q, p, \beta, t)\\
= \;& -\sum_{i\in I_\lambda}\lambda_i f_i^*(\beta_i) - \sum_{i\in I_\lambda}\lambda_i(\beta_i^T F_i)^*(p_i) - (q^T g)_X^*\bigg(-\sum_{i\in I_\lambda}\lambda_i p_i\bigg)\\
\leq \;& \sum_{i\in I_\lambda}\lambda_i f_i \circ F_i(x).
\end{aligned}$$

This leads us to a contradiction to (6). Thus the initial assumption is false and the proof of the theorem is complete. □

Theorem 3.7 (*strong vector duality*) Assume that (CQ) is fulfilled. If $\overline{x}$ is a weakly efficient solution of the primal problem (P), then there exists $(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t}) \in \mathcal{B}$ that is a weakly efficient solution to the dual problem (D) and for all $i = 1, ..., k$ applies

$$f_i \circ F_i(\overline{x}) = h_i\big(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t}\big).$$

Proof. Since $\overline{x}$ is a weakly efficient solution of (P) and the condition (CQ) is fulfilled, by Theorem 3.5 there exist $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in K^*$, $p_i \in \mathbb{R}^n$ and $\beta_i \in K_i^*$, $i \in I_\lambda$, such that the conditions $(i) - (iv)$ of the above mentioned theorem are fulfilled. Take an arbitrary $i \in \{1, ..., k\} \setminus I_\lambda$. Since the function f_i is proper and convex, the function f_i^* is proper and convex, too (for more details see [16]). Therefore there exists $\widetilde{\beta}_i \in K_i^*$ (see Lemma 2.1) such that $f_i^*(\widetilde{\beta}_i) \in \mathbb{R}$. Moreover, since $\widetilde{\beta}_i^T F_i$ is proper and convex, we can find at least one $\widetilde{p}_i \in \mathbb{R}^n$ such that $(\widetilde{\beta}_i^T F_i)^*(\widetilde{p}_i) \in \mathbb{R}$. Choose

$$\overline{\lambda} := \lambda, \quad \overline{q} := q, \quad \overline{p}_i := \begin{cases} p_i, & i \in I_\lambda, \\ \widetilde{p}_i, & i \notin I_\lambda, \end{cases} \quad \overline{\beta}_i := \begin{cases} \beta_i, & i \in I_\lambda, \\ \widetilde{\beta}_i, & i \notin I_\lambda, \end{cases} \quad \text{and}$$

$$\overline{t}_i := \begin{cases} p_i^T \overline{x} + \frac{1}{|\lambda|}\big(q^T g\big)_X^*\Big(-\sum\limits_{i\in I_\lambda} \lambda_i p_i\Big), & i \in I_\lambda, \\ f_i \circ F_i(\overline{x}) + f_i^*(\widetilde{\beta}_i) + (\widetilde{\beta}_i^T F_i)^*(\widetilde{p}_i) + \frac{1}{|\lambda|}(q^T g)_X^*\Big(-\sum\limits_{i\in I_\lambda} \lambda_i p_i\Big), & i \notin I_\lambda. \end{cases}$$

It is clear that $\overline{t}_i \in \mathbb{R}$ since all terms occuring in the definition of $\overline{t}_i$ are finite, $\forall\, i = 1, ..., k$, and that (see Theorem 3.5 (iii) and (iv))

$$\sum_{i=1}^{k} \overline{\lambda}_i \overline{t}_i = \sum_{i\in I_\lambda} \overline{\lambda}_i \overline{t}_i = \sum_{i\in I_\lambda} \lambda_i (p_i^T \overline{x}) + (q^T g)_X^* \bigg(- \sum_{i\in I_\lambda} \lambda_i p_i \bigg) = 0$$

It remains to prove that $f_i \circ F_i(\overline{x}) = h_i\big(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t}\big)$ for all $i \in I_{\overline{\lambda}}$ (for $i \notin I_{\overline{\lambda}}$ this is trivial as a consequence of the definition of $\overline{t}$). We have

$$\begin{aligned} &h_i(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t}) = -f_i^*(\overline{\beta}_i) - (\overline{\beta}_i^T F_i)^*(\overline{p}_i) - \tfrac{1}{|\overline{\lambda}|}(\overline{q}^T g)_X^* \bigg(\sum_{i\in I_{\overline{\lambda}}} \overline{\lambda}_i \overline{p}_i \bigg) + \overline{t}_i \\ &= -f_i^*(\beta_i) - (\beta_i^T F_i)^*(p_i) - \tfrac{1}{|\lambda|}(q^T g)_X^* \bigg(\sum_{i\in I_\lambda} \lambda_i p_i \bigg) + p_i^T \overline{x} + \tfrac{1}{|\lambda|}\big(q^T g\big)_X^*\bigg(- \sum_{i\in I_\lambda} \lambda_i p_i \bigg) \\ &= -f_i^*(\beta_i) - (\beta_i^T F_i)^*(p_i) + p_i^T \overline{x} = -f_i^*(\beta_i) + \beta_i^T F_i(\overline{x}) = f_i \circ F_i(\overline{x}). \end{aligned}$$

For the last equalities we used Theorem 3.5 (i) and (ii). The fact that $\big(\overline{\lambda}, \overline{q}, \overline{p}, \overline{\beta}, \overline{t}\big)$ is a weakly efficient solution of the dual problem (D) is a straightforward consequence of Theorem 3.6. $\square$

4 Special cases

Within this section two special cases are treated. In the first case we consider the functions F_i being linear, while in the second case we show how the ordinary convex optimization problem can be derived as a special case of our general result.

4.1 Composition with a linear operator

In the following let $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$ be proper convex functions and F_i be linear functions, $i = 1, ..., k$. More precisely, we consider the functions

$$F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}, \quad F_i(x) = A_i x,$$

where A_i is an $n_i \times n$ real matrix for each $i = 1, ..., k$. Our initial problem becomes in this special case

$$(P^A) \qquad \operatorname*{v-min}_{\substack{x\in X,\\ g(x)\leqq_K 0}} \big(f_1(A_1x), ..., f_k(A_kx)\big)^T.$$

Let us consider $K_i = \{0\} \subset \mathbb{R}^{n_i}$ for all $i = 1, ..., k$. It is not hard to prove that the functions f_i are K_i-increasing, while F_i are K_i-convex. Moreover, since $\mathrm{ri}(K_i) = \{0\}$, $i = 1, ..., k$, the condition (CQ) becomes in this special case

$$(CQ^A) \qquad \exists x' \in \mathrm{ri}(X) \text{ such that } \begin{cases} A_i x' \in \mathrm{ri}(\mathrm{dom}(f_i)), & i = 1, ..., k, \\ g(x') \in -\,\mathrm{ri}(K). \end{cases}$$

Since for all $i = 1, ..., k$ and for all $\beta_i \in \mathbb{R}^{n_i}_+$ we have

$$(\beta_i^T F_i)^*(p_i) = \begin{cases} 0, & A_i^T \beta_i = p_i, \\ +\infty, & \text{otherwise}, \end{cases}$$

the next results arise as easy consequences of the ones presented within the previous section.

Theorem 4.1

(a) Suppose that the condition (CQ^A) is fulfilled and let $\overline{x}$ be a weakly efficient solution of the problem (P^A). Then there exists $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in K^*$ and $\beta_i \in \mathbb{R}^{n_i}$, $i \in I_\lambda$, such that

(i^A) $f_i(A_i\overline{x}) + f_i^*(\beta_i) - \beta_i^T(A_i\overline{x}) = 0$, $i \in I_\lambda$,

(ii^A) $q^T g(\overline{x}) + (q^T g)^*_X\Big(- \sum\limits_{i\in I_\lambda} \lambda_i A_i^T \beta_i\Big) + \sum\limits_{i\in I_\lambda} \lambda_i \beta_i^T (A_i\overline{x}) = 0$,

(iii^A) $q^T g(\overline{x}) = 0$.

(b) If there exists $\overline{x}$ feasible to (P) such that for some $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in K^*$ and $\beta_i \in \mathbb{R}^{n_i}$, $i \in I_\lambda$, the conditions $(i^A) - (iii^A)$ are satisfied, then $\overline{x}$ is a weakly efficient solution of (P^A).

To the problem (P^A) we attach as a special case of (D) (cf. 3.3) the vector dual problem

$$(D^A) \qquad \underset{(\lambda,q,\beta,t)\in\mathcal{B}^A}{\text{v-max}} \big(h_1^A(\lambda,q,\beta,t),...,h_k^A(\lambda,q,\beta,t)\big)^T,$$

where for each $i = 1,...,k$ we have

$$h_i^A(\lambda,q,\beta,t) = -f_i^*(\beta_i) - \frac{1}{|\lambda|}(q^T g)_X^* \left(\sum_{i\in I_\lambda} \lambda_i A_i^T \beta_i\right) + t_i$$

and the dual variables are $\lambda = (\lambda_1,...,\lambda_k)^T \in \mathbb{R}^k$, $q = (q_1,...,q_m)^T \in \mathbb{R}^m$, $\beta = (\beta_1,...,\beta_k) \in \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$ and $t = (t_1,...,t_k)^T \in \mathbb{R}^k$. The feasible set turns out to be

$$\mathcal{B}^A = \left\{(\lambda,q,\beta,t) : \lambda \in \mathbb{R}^k_+ \setminus \{0\}, q \in K^*, \sum_{i=1}^k \lambda_i t_i = 0\right\}.$$

Now we get from Theorem 3.6 and Theorem 3.7 the corresponding weak and strong vector duality results.

Theorem 4.2 There is no $x \in \mathcal{A}$ and no $(\lambda,q,\beta,t) \in \mathcal{B}^A$ such that $f_i(A_i x) < h_i^A(\lambda,q,\beta,t)$ for all $i = 1,...,k$.

Theorem 4.3 Assume that (CQ^A) is fulfilled. If $\overline{x}$ is a weakly efficient solution of the problem (P^A), then there exists $(\overline{\lambda},\overline{q},\overline{\beta},\overline{t}) \in \mathcal{B}^A$ that is a weakly efficient solution to (D^A) and for all $i = 1,...,k$ one has

$$f_i(A_i\overline{x}) = h_i^A\big(\overline{\lambda},\overline{q},\overline{\beta},\overline{t}\big).$$

4.2 The ordinary multiobjective optimization problem

Let us consider now $n_1 = ... = n_k = n$ and let

$$F_i : \mathbb{R}^n \to \mathbb{R}^n, \quad F_i(x) = x,$$

for all $i = 1, ..., k$. For $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$ proper and convex functions, $i = 1, ..., k$, our initial problem becomes

$$(P^B) \qquad \operatorname*{v-min}_{\substack{x \in X, \\ g(x) \leqq_K 0}} \big(f_1(x), ..., f_k(x)\big)^T,$$

Obviously, the previous problem is a particular case of (P^A) with $A_i = I$ (the identical operator), $i = 1, ..., k$. The constraint qualification (CQ^A) becomes

$$(CQ^B) \qquad \exists x' \in \mathrm{ri}(X) \bigcap_{i=1}^{k} \mathrm{ri}\big(\mathrm{dom}(f_i)\big) \text{ such that } g(x') \in -\,\mathrm{ri}(K).$$

Theorem 4.4

(a) Suppose that the condition (CQ^B) is fulfilled and let $\overline{x}$ be a weakly efficient solution of the problem (P^B). Then there exists $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in \mathbb{R}^m_+$, and $p_i \in \mathbb{R}^n$, $i \in I_\lambda$, such that

(i^B) $f_i(\overline{x}) + f_i^*(p_i) - p_i^T \overline{x} = 0$, $i \in I_\lambda$,

(ii^B) $q^T g(\overline{x}) + (q^T g)^*_X\Big(- \sum\limits_{i \in I_\lambda} \lambda_i p_i\Big) + \sum\limits_{i \in I_\lambda} \lambda_i p_i^T \overline{x} = 0$,

(iii^B) $q^T g(\overline{x}) = 0$.

(b) If there exists $\overline{x}$ feasible to (P^B) such that for some $\lambda \in \mathbb{R}^k_+ \setminus \{0\}$, $q \in \mathbb{R}^m_+$ and $p_i \in \mathbb{R}^n$, $i \in I_\lambda$, the conditions $(i^B) - (iii^B)$ are satisfied, then $\overline{x}$ is a weakly efficient solution of (P^B).

As before to (P^B) we associate a vector dual problem, namely

$$(D^B) \qquad \operatorname*{v-max}_{(\lambda,q,p,t) \in \mathcal{B}^B} \big(h_1^B(\lambda,q,p,t), ..., h_k^B(\lambda,q,p,t)\big)^T,$$

where

$$h_i^B(\lambda,q,p,t) = -f_i^*(p_i) - \frac{1}{|\lambda|}(q^T g)^*_X\left(\sum_{i \in I_\lambda} \lambda_i p_i\right) + t_i$$

for all $i = 1, ..., k$, and the dual variables are $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k$, $q = (q_1, ..., q_m)^T \in \mathbb{R}^m$, $p = (p_1, ..., p_k) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $t = (t_1, ..., t_k)^T \in \mathbb{R}^k$. Let

$$\mathcal{B}^B = \left\{(\lambda,q,p,t) : \lambda \in \mathbb{R}^k_+ \setminus \{0\}, q \in K^*, \sum_{i=1}^{k} \lambda_i t_i = 0\right\}.$$

Theorem 4.5 There is no $x \in \mathcal{A}$ and no $(\lambda, q, p, t) \in \mathcal{B}^B$ such that $f_i(x) < h_i^B(\lambda, q, p, t)$ for all $i = 1, ..., k$.

Theorem 4.6 Assume that (CQ^B) is fulfilled. If $\overline{x}$ is a weakly efficient solution of the problem (P^B), then there exists $(\overline{\lambda}, \overline{q}, \overline{p}, \overline{t}) \in \mathcal{B}^B$ that is a weakly efficient solution to (D^B) and for all $i = 1, ..., k$ applies

$$f_i(\overline{x}) = h_i^B\big(\overline{\lambda}, \overline{q}, \overline{p}, \overline{t}\big).$$

Remark. We would like to mention that for $K = \mathbb{R}_+^m$ the results presented in this paper are true if instead of $g(x') \in -\operatorname{ri}(\mathbb{R}_+^m) = -\operatorname{int}(\mathbb{R}_+^m)$ we impose the weaker assumption (see [16])

$$\begin{cases} g_j(x') \leq 0, & j \in L, \\ g_j(x') < 0. & j \in N, \end{cases}$$

where $L := \big\{j \in \{1, ..., m\} : g_j \text{ is an affine function}\big\}$ and $N := \{1, ..., m\} \backslash L$.

5 Conclusions

In this paper we consider a multiobjective optimization problem the objective function of which has as entries compositions of some convex functions, while the constraints are given by cone inequality constraints. To the problem we treat we associate a family of scalar optimization problems and to each member of this family a Fenchel-Lagrange-type dual is formulated. Using the weak and strong duality statements for the scalar problems optimality conditions for weakly efficient solutions of the original problem are presented, where only the involved functions and their conjugates are used. A vectorial dual of the general problem we treat is given and weak and strong duality assertions are proved. Moreover, some special cases are considered.

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