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On the Graph Bisection Cut polytope

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Abstract

Given a graph $G = (V, E)$ with node weights $f_v \in \mathbb{N}_0$, $v \in V$, and some number $F \in \mathbb{N}_0$, the convex hull of the incidence vectors of all cuts $\delta(S)$, $S \subseteq V$ with $f(S) \leq F$ and $f(V \setminus S) \leq F$ is called the *bisection cut polytope*. We study the facial structure of this polytope which shows up in many graph partitioning problems with applications in VLSI-design or frequency assignment. We give necessary and in some cases sufficient conditions for the knapsack tree inequalities introduced in [9] to be facet-defining. We extend these inequalities to a richer class by exploiting that each cut intersects each cycle in an even number of edges. Finally, we present a new class of inequalities that are based on non-connected substructures yielding non-linear right-hand sides. We show that the supporting hyperplanes of the convex envelope of this non-linear function correspond to the faces of the so-called *cluster weight polytope*, for which we give a complete description under certain conditions.

Keywords: polyhedral combinatorics, minimum bisection problem, knapsack tree inequality, bisection knapsack walk inequality, cluster weight polytope

MSC 2000: 90C57

1 Introduction

Let $G = (V, E)$ be an undirected graph with $V = \{1, \ldots, n\}$ and $E \subseteq \{\{i, j\} : i, j \in V, i < j\}$. For given vertex weights $f_v \in \mathbb{N}_0$ for all $v \in V$ and edge costs $w_{\{i,j\}} \in \mathbb{R}$ for all $\{i, j\} \in E$, a partition of the vertex set V into two disjoint clusters S and $V \setminus S$ with sizes $f(S) \leq F$ and $f(V \setminus S) \leq F$, where $F \in \mathbb{N}_0 \cap \left[\frac{1}{2} f(V), f(V)\right]$, is called a *bisection*. Finding a bisection such that the total cost of edges in the cut $\delta(S) := \{\{i, j\} \in E : i \in S \wedge j \in V \setminus S\}$ is minimal is the *minimum bisection problem* (MB). The problem is known to be NP-hard [11].

In this paper we will investigate the *bisection cut polytope* P_{B} associated with MB. To define P_{B} note that a cut $\delta(S)$ can be described by its incidence vector $\chi^{\delta(S)}$ with respect to the edge set E. Then

$$P_{\mathrm{B}} := \operatorname{conv}\{\, y \in \mathbb{R}^{|E|} : \, y = \chi^{\delta(S)}, \, S \subseteq V, \, f(S) \leq F, \, f(V \setminus S) \leq F \,\}.$$

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MB as well as P_{B} are related to other problems and polytopes in the literature. Obviously, the bisection cut polytope is contained in the *cut polytope* [3, 7]

$$P_{\mathrm{C}} := \mathrm{conv}\left\{ y \in \mathbb{R}^{|E|} : y = \chi^{\delta(S)}, S \subseteq V \right\} \ . \tag{1}$$

If $F = f(V)$ then MB is equivalent to the maximum cut problem (using the negative cost function) and $P_{\mathrm{B}} = P_{\mathrm{C}}$. For $F = \lceil \frac{1}{2} f(V) \rceil$ MB is equivalent to the *equipartition problem* [6] and the bisection cut polytope equals the *equipartition polytope* [4, 5, 14]

$$P_{\mathrm{E}} := \mathrm{conv}\{\, y \in \mathbb{R}^{|E|} :\ y = \chi^{\delta(S)},\ S \subseteq V,\ f(S) = f(V \setminus S) \,\}$$

Furthermore, MB is a special case of the minimum node capacitated graph partitioning problem (MNCGP) [9] where $K \geq 2$ clusters are available for the partition of the node set and each cluster has a limited capacity. The objective in MNCGP is the same as in MB, i.e., to minimize the total cost of edges having endpoints in distinct clusters. Finally, we mention the *knapsack polytope* [18]

$$P_{\mathrm{K}} := \mathrm{conv}\left\{ x \in \{0,1\}^{|V|} : \sum_{v \in V} f_v x_v \leq F \right\} \ . \tag{2}$$

P_{K} plays a fundamental role in the inequalities which we are going to derive for the bisection cut polytope.

Graph partitioning problems in general have numerous applications, for instance in numerics [12], VLSI-design [16], compiler-design [15] and frequency assignment [8].

The main contributions of this paper are threefold. First, in [9] the so-called knapsack tree inequalities have been introduced. These inequalities relate the knapsack conditions on the nodes with the edge variables defining the cuts and turn out to be computationally very effective. However, no theoretical justification has been found so far for this behavior. In this paper, we give necessary conditions for the knapsack tree inequality to be facet-defining, which turn out to be also sufficient in certain cases. Second, we can generalize the knapsack tree inequalities in the case of bisections by exploiting the well-known fact that any cut intersects a cycle an even number of times. This new class of inequalities, called *bisection knapsack walk inequalities*, subsume the knapsack tree inequalities and yield computationally more flexibility in finding strong inequalities. The third class of inequalities, called *capacity reduced bisection knapsack walk inequalities*, extends both classes of inequalities to non-connected substructures. The idea is to exploit the weights of the nodes that are not end-nodes of walks to reduce the capacity of the corresponding knapsack inequality yielding this way stronger right-hand sides for the knapsack tree and bisection knapsack walk inequalities. These stronger conditions result in non-linear right-hand sides. We consider the convex envelope of this non-linear function and show that the supporting hyperplanes are in one-to-one correspondence to the faces of a certain polytope, called *cluster weight polytope*. For the case of a star without capacity restriction we are able to give a complete description of the cluster weight polytope yielding in this case the tightest strengthening possible for the capacity reduced bisection knapsack walk inequalities.

The outline of the paper is as follows. In Section 2 we introduce an integer programming formulation for MB building on the formulation of MNCGP given in [9]. Section 3 treats the known knapsack tree inequalities valid for both MB and MNCGP while Section 5 introduces the new bisection knapsack walk inequalities which are only valid for MB and which subsume the knapsack tree inequalities. Section 4 shows a strengthening only applicable to knapsack tree inequalities. Furthermore, we state necessary and sufficient conditions for knapsack tree inequalities to define facets of P_{B}. In Section 6 we are going to describe the relation between the bisection knapsack walk inequalities and the odd cycle inequalities for the cut polytope. Finally, Section 7 introduces a strengthening of the bisection knapsack walk inequalities. For this purpose we investigate the facial structure of the cluster weight polytope on stars.

We frequently denote an edge $\{i,j\}$ by ij. Let A and B be discrete sets such that $A \subseteq B$. The incidence vector of A with respect to B is a vector $\chi^A \in \{0,1\}^{|B|}$ with $\chi^A_a = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \in B \setminus A \end{cases}$. For a vector $x \in \mathbb{R}^{|B|}$ we define $x(A) = \sum_{a\in A} x_a$. $0^{|A|}$ is the zero vector of dimension $|A|$ and e_a is the unit vector of dimension $|A|$, which is indexed by the elements of A and has entry 1 in coordinate $a \in A$. For a graph $G = (V,E)$ the edge set of the subgraph induced by $\bar{V} \subseteq V$ will be denoted by $E\left(\bar{V}\right)$ and the node set of the subgraph induced by $\bar{E} \subseteq E$ by $V\left(\bar{E}\right)$. The convex hull of a set $A \subseteq \mathbb{R}^n$ will be denoted by $\operatorname{conv}\{A\}$.

2 An integer programming formulation of MB

The integer programming formulation for MB given below is based on the formulation for MNCGP presented in [9]. We introduce variables z_i^k for each node $i \in V$ and each cluster $k = 1,2$ and edge variables y_{ij} for each edge $ij \in E$. z_i^k is set to 1 if node i is in cluster k and 0 otherwise. Variable y_{ij} is set to 1 if edge ij is in the cut, i.e., i and j are not in the same cluster, and 0 otherwise. Then MB can be written as

$$\text{(MB)} \qquad \begin{array}{lll} \min & \displaystyle\sum_{e\in E} w_e y_e & \\ \text{s.t.} & z_i^1 + z_i^2 = 1 & \forall i \in V \\ & \displaystyle\sum_{i\in V} f_i z_i^k \le F & k = 1,2 \\ & y_{ij} \ge z_i^1 - z_j^1 & \forall ij \in E \\ & y_{ij} \ge z_j^1 - z_i^1 & \forall ij \in E \\ & y_{ij} \in \{0,1\} & \forall ij \in E \\ & z_i^k \in \{0,1\} & \forall i \in V, k = 1,2\ . \end{array}$$

The first constraints assure that each node i is packed into exactly one cluster k. The second constraints enforce the capacity restriction on each cluster k. The third and fourth constraints transmit for each edge $ij \in E$ the values of variables z_i^1 and z_j^1 to the edge variable y_{ij} in the sense that $y_{ij} = 1$ if and only if $z_i^1 \neq z_j^1$. The fifth and sixth constraints are the binary restrictions on the variables.

Noting that the variables z_i^k do not appear in the objective function we can consider model

$$\begin{array}{ll} \min & \displaystyle\sum_{e\in E} w_e y_e \\ \text{s.t.} & y \in Y_{\text{MB}}, \end{array}$$

where $Y_{\text{MB}} \subseteq \mathbb{R}^{|E|}$ is the projection onto the y-space of the feasible region of model (MB). It can be worked out that $P_{\text{B}} = \operatorname{conv}(Y_{\text{MB}})$.

3 Known valid inequalities for MNCGP and MB

A large variety of valid inequalities for the polytope associated to MNCGP is known and, since MB is a special case of MNCGP, all those inequalities are also valid for P_{B}: odd cycle inequalities [3], tree inequalities [4], star inequalities [4], cycle inequalities [5], cycle with tails inequalities [9], suspended tree inequalities [14], path block cycle inequalities [14], cycle with ear inequalities [9], strengthened cycle with ear inequalities [9], knapsack tree inequalities [9] and strengthened knapsack tree inequalities [9].

In the remainder of the paper we specialize and improve the knapsack tree inequality for MB. First we recall its definition for MNCGP from [9].

Definition 1 (Knapsack tree inequality [9]). *Let* $\sum_{v\in V} a_v x_v \leq a_0$ *be a valid inequality for the knapsack polytope* P_{K} *with* $a_v \geq 0$ *for all* $v \in V$*. For a fixed node* $r \in V$ *and a subtree* (T, E_T) *of* G *rooted at* r *we define the* knapsack tree inequality

$$\sum_{v\in T} a_v \left(1 - \sum_{e\in P_{rv}} y_e\right) \leq a_0 \tag{3}$$

where for each $v \in T$ *the edge set of the path joining node* v *to root* r *in* (T, E_T) *is denoted by* P_{rv}*.*

If (T, E_T) *is a star rooted at* r*, i.e.,* $E_T = \{\{r,t\} : t \in T, t \neq r\}$*, then we call the inequality (3)* knapsack star inequality*.*

In general, there is an exponential number of these knapsack tree inequalities, since for each combination of a valid knapsack inequality with a choice of a rooted tree there is one knapsack tree inequality.

Proposition 2. [9] *The knapsack tree inequality (3) is valid for the polytope* P_{B}*.*

Proof. Let $S_1 \subseteq V$ and $S_2 = V \setminus S_1$ be a feasible bisection, then $\sum_{v\in V} a_v z_v^k \leq a_0$, $k = 1, 2$, holds for the given valid inequality for the knapsack polytope (it is valid for both clusters since on both of them the capacity constraint is F). W.l.o.g. let $r \in S_1$, i.e. $z_r^1 = 1$. So for all $v \in S_1$ we have $z_v^1 = 1$. Otherwise $v \in S_2$ and $z_v^1 = 0$. Now let $y = \chi^{\delta(S_1)} \in P_{\mathrm{B}}$. Note that, $1 - \sum_{e\in P_{rv}} y_e$ is equal to one if and only if all vertices of P_{rv} are contained in S_1 and less or equal to zero otherwise. Therefore,

$$\sum_{v\in T} a_v \left(1 - \sum_{e\in P_{rv}} y_e\right) \leq \sum_{v\in T\cap S_1} a_v \cdot 1 \leq \sum_{v\in V} a_v z_v^1 \leq a_0$$

where the second inequality uses $a_v \geq 0$ and $z^1 = \chi^{S_1}$. □

It will be useful to write the inequality (3) in the form

$$\sum_{e\in E_T} \left(\sum_{v:e\in P_{rv}} a_v\right) y_e \geq \sum_{v\in T} a_v - a_0 \ . \tag{4}$$

The term on the right-hand side may be interpreted as the excess if all vertices $v \in T$ are packed into the cluster containing the root node r while we are only allowed to pack a total weight of a_0. The left-hand side has to compensate for this, i.e., it has to force some edges into the cut so that not all vertices are placed into the same cluster as the current root. We will use this reformulation to apply a folklore approach to strengthen coefficients in general binary programs.

Lemma 3. *Let* $S \in \{0,1\}^{|E|}$, $P = \mathrm{conv}(S)$ *and* $\sum_{e\in E} \alpha_e y_e \geq \alpha_0$ *an inequality valid for* P. *Define*

$$\tilde{\alpha}_e := \min\left\{\alpha_e, \max\left\{0, \alpha_0 - \sum_{e\in E:\alpha_e<0} \alpha_e\right\}\right\} .$$

Then the strengthened inequality $\sum_{e\in E} \tilde{\alpha}_e y_e \geq \alpha_0$ *is valid for* P, *too.*

Proof. Let $\bar{y} \in P \cap \{0,1\}^{|E|}$. If $y_e = 0$ for all $e \in \{e : \tilde{\alpha}_e \neq \alpha_e\}$ then $\bar{y}$ also satisfies $\sum_{e\in E} \tilde{\alpha}_e y_e \geq \alpha_0$. Otherwise $y_{\bar{e}} = 1$ for at least one $\tilde{\alpha}_{\bar{e}} \neq \alpha_{\bar{e}}$. Then

$$\begin{aligned}\sum_{e\in E} \tilde{\alpha}_e y_e - \alpha_0 &= \sum_{e\in E:\tilde{\alpha}_e\geq 0} \tilde{\alpha}_e y_e + \sum_{e\in E:\tilde{\alpha}_e<0} \tilde{\alpha}_e y_e - \alpha_0 \geq \tilde{\alpha}_{\bar{e}} - \left(\alpha_0 - \sum_{e\in E:\alpha_e<0} \alpha_e\right) \\ &= \max\left\{0, \alpha_0 - \sum_{e\in E:\alpha_e<0} \alpha_e\right\} - \left(\alpha_0 - \sum_{e\in E:\alpha_e<0} \alpha_e\right) \geq 0 . \qquad \square\end{aligned}$$

Remark 4. *Lemma 3 applied to the reformulated knapsack tree inequality (4) for MB yields*

$$\sum_{e\in E_T} \min\left\{\sum_{v:e\in P_{rv}} a_v, \sum_{v\in T} a_v - a_0\right\} y_e \geq \sum_{v\in T} a_v - a_0. . \tag{5}$$

We call this inequality truncated knapsack tree inequality. *Note that it is the same as the first case proposed in Proposition 3.12 of [9] applied to the knapsack tree inequality for MNCGP. For MNCGP those authors also proposed a second case of strengthening, namely (in our notation) to reduce* α_e *to* a_0. *But for MB the second case never applies, since we always have* $\alpha_0 = \sum_{v\in T} a_v - a_0 \leq a_0$ *due to* $a_0 \geq \frac{1}{2}\sum_{v\in V} a_v$.

4 Minimum root strengthening of knapsack tree inequalities

Given a knapsack inequality $\sum_{v\in V} a_v x_v \leq a_0$ with $a_v \geq 0$, $v \in V$, let a corresponding knapsack tree inequality be defined on a tree (T, E_T) rooted at r. If we replace r by another node from T the paths change. The corresponding change of the coefficients of the inequality will be exploited in the strengthening presented in this section. We are going to show that the strongest or in some cases even facet-defining inequality is achieved if r corresponds to a sort of equilibrium with respect to the cumulated node weights on the paths to r. Since further improvements pay off only if a stronger inequality than the truncated knapsack tree inequality (5) is achieved, we act on inequalities in this form. To emphasize that the coefficients in (5) depend on the root node r we introduce the notation

$$\alpha_0 := \sum_{v\in T} a_v - a_0, \qquad \alpha_e^r := \min\{\sum_{v:e\in P_{rv}} a_v, \alpha_0\}, \, e \in E_T, \tag{6}$$

and consider (5) in the form

$$\sum_{e\in E_T} \alpha_e^r y_e \geq \alpha_0. \tag{7}$$

Note that if we change the root of (T, E_T) the right-hand side of (7) remains the same, since by this operation we do not eliminate nodes of (T, E_T).

At first we derive some relations based on the definition of the coefficients α_e^r, $r \in T$, $e \in E_T$, which we will exploit in the proofs of the results presented in this section. In the next lemma we investigate the change of coefficients if the root is moved from a node r to an adjacent node s.

Lemma 5. *Let (T, E_T) be a tree in G and r, s two adjacent nodes with $\bar{e} = rs$. We have*

- *(a) $\alpha_e^r = \alpha_e^s$ for all $e \neq rs$,*
- *(b) $\alpha_{\bar{e}}^r = \min\{a_s + \sum_{e\in\delta(\{s\})\setminus\{\bar{e}\}} \alpha_e^s, \alpha_0\}$,*
- *(c) $\sum_{e\in E_T} \alpha_e^r \leq \sum_{e\in E_T} \alpha_e^s$ if and only if $\alpha_{\bar{e}}^r \leq \alpha_{\bar{e}}^s$, and the equality holds if and only if $\alpha_{\bar{e}}^r = \alpha_{\bar{e}}^s$.*

Proof. (*a*) For $e \neq rs$ we have $\{v : e \in P_{rv}\} = \{v : e \in P_{sv}\}$ and thus $\sum_{v:e\in P_{rv}} a_v = \sum_{v:e\in P_{sv}} a_v$.

(*b*) We have:

$$\sum_{v:\bar{e}\in P_{rv}} a_v = a_s + \sum_{e\in\delta(\{s\})\setminus\{\bar{e}\}} \left(\sum_{v:e\in P_{sv}} a_v \right).$$

(*c*) The claim follows directly from (*a*). □

Lemma 6. *Let (T, E_T) be a tree in G rooted at node r and let e and f be two edges on a path to r such that e is closer to r than f with respect to the number of edges. Then*

$$\alpha_f^r \leq \alpha_e^r. \tag{8}$$

Proof. W.l.o.g. we assume that e and f are adjacent. Setting $e := ij$ and $f := jk$ we obtain

$$\sum_{v:e\in P_{rv}} a_v = \sum_{v:f\in P_{rv}} a_v + a_j + \sum_{\bar{e}\in\bar{E}} \left(\sum_{v:\,\bar{e}\in P_{rv}} a_v \right) \geq \sum_{v:f\in P_{rv}} a_v,$$

where $\bar{E}$ contains edges incident to j except e and f. Hence if $\alpha_f^r = \alpha_0$ then also $\alpha_e^r = \alpha_0$, otherwise $\alpha_f^r \leq \min\{\sum_{v:e\in P_{rv}} a_v,\, \alpha_0\} = \alpha_e^r$. □

In the following theorem we claim that the strength of truncated knapsack tree inequalities depends on the position of the root r in the underlying tree. We show how to select r so that the best possible reduction of the coefficients and thus the strongest truncated knapsack tree inequality for a given tree can be achieved.

Theorem 7. *Let (T, E_T) be a tree in G. The strongest truncated knapsack tree inequality, with respect to the knapsack inequality $\sum_{v\in V} a_v x_v \leq a_0$, $a_v \geq 0$, $v \in V$, defined on (T, E_T) is obtained for a root $r \in \mathcal{R} := \operatorname{Argmin}_{v\in T} \sum_{e\in E_T} \alpha_e^v$, i.e., if $r \in \mathcal{R}$ then*[1]

$$\sum_{e\in E_T} \alpha_e^s y_e \geq \sum_{e\in E_T} \alpha_e^r y_e \geq \alpha_0 \tag{9}$$

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[1] $\mathcal{R}$ is the set of all nodes minimizing the given sum.

holds for all $s \in T$ *and all* $y \in P_{\mathrm{B}}$*. In particular,*

$$\sum_{e \in E_T} \alpha_e^r y_e = \sum_{e \in E_T} \alpha_e^s y_e \tag{10}$$

holds for all $r, s \in \mathcal{R}$ *and all* $y \in P_{\mathrm{B}}$*.*

Proof. Let $\Pi = (V_\Pi, E_\Pi)$ be the path joining nodes $r \in \mathcal{R}$ and $s \in T$, $r \neq s$, with $V_\Pi = \{v_1, \dots, v_p\}$, $p \geq 2$, where $v_1 = r$, $v_p = s$ and v_k, v_{k+1}, $1 \leq k \leq p-1$ are adjacent. Applying recursively Lemma 5 (a) to nodes v_i, v_{i+1}, $i = 1, \dots, p-1$ we obtain

$$\alpha_e^r = \alpha_e^s \qquad \forall\, e \in E_T \setminus E_\Pi. \tag{11}$$

By Lemma 6 we have

$$\begin{array}{ccccccc} \alpha_{rv_2}^r & \geq & \alpha_{v_2v_3}^r & \geq \dots \geq & \alpha_{v_{p-2}v_{p-1}}^r & \geq & \alpha_{v_{p-1}s}^r \\ \alpha_{rv_2}^s & \leq & \alpha_{v_2v_3}^s & \leq \dots \leq & \alpha_{v_{p-2}v_{p-1}}^s & \leq & \alpha_{v_{p-1}s}^s. \end{array} \tag{12}$$

Since r is a minimal root $\sum_{e \in E_T} \alpha_e^r \leq \sum_{e \in E_T} \alpha_e^{v_2}$ holds. Applying Lemma 5 (c) to nodes r and v_2 and Lemma 5 (a) to nodes v_2 and s we obtain

$$\alpha_{rv_2}^r \leq \alpha_{rv_2}^{v_2} = \alpha_{rv_2}^s.$$

This together with (12) yields

$$\forall\, e \in E_\Pi \quad \alpha_e^r \leq \alpha_e^s. \tag{13}$$

Hence for each $e \in E_\Pi$ the inequality $(\alpha_e^r - \alpha_e^s) y_e \leq 0$ is trivially valid for P_{B} and by (11) we obtain

$$\sum_{e \in E_T} \alpha_e^r y_e - \sum_{e \in E_T} \alpha_e^s y_e = \sum_{e \in E_\Pi} (\alpha_e^r - \alpha_e^s) y_e \leq 0.$$

Thus (9) holds. The equation (10) follows directly from (9). ☐

The relation (13) in the above proof implies the following statement.

Remark 8. *Let* (T, E_T) *be a tree in* G *and* $r, s \in T$ *then the inequality*

$$\sum_{e \in E_T} \min\{\alpha_e^r, \alpha_e^s\}\, y_e \geq \alpha_0$$

is valid for P_{B}*.*

In the sequel the elements of the set $\mathcal{R}$ will be called *minimal roots* of a given tree (T, E_T). In Theorem 7 we showed that all minimal roots of (T, E_T) deliver the same truncated knapsack tree inequality and thus to obtain the strongest one it is sufficient to identify any minimal root. Assume we are given a tree (T, E_T) rooted at some node r. In order to find a minimal root one can proceed iteratively as follows. Select a node $s \in T$ adjacent to r such that $\alpha_{rs}^r = \max\{\alpha_{rv}^r : rv \in E_T\}$. If $\alpha_{rs}^r > \alpha_{rs}^s$ then also $\sum_{e \in E_T} \alpha_e^r > \sum_{e \in E_T} \alpha_e^s$, by Lemma 5 (c). Hence r can be discarded and s is marked as root of (T, E_T). Otherwise $\sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0$ is the strongest truncated knapsack tree inequality with respect to all possible choices of roots in (T, E_T). The following propositions show that our strengthening procedure delivers correct results. Due to following Proposition 9 it is sufficient to search in the direct neighborhood of the current root for a possible improvement. Proposition 10 assures that it is enough to examine the node adjacent to r maximizing α_{rv}^r, $rv \in E_T$.

Proposition 9. *r is a minimal root if and only if $\alpha^r_{rv} \leq \alpha^v_{rv}$ holds for all v adjacent to r.*

Proof. Let v be a node adjacent to r. We assume first that $r \in \mathcal{R}$. Then $\sum_{e\in E_T} \alpha^r_e \leq \sum_{e\in E_T} \alpha^v_e$ holds and thus $\alpha^r_{rv} \leq \alpha^v_{rv}$ due to Lemma 5 (c). Now assume that

$$\forall\, rv \in E_T \quad \alpha^r_{rv} \leq \alpha^v_{rv}. \tag{14}$$

We show that this implies $\sum_{e\in E_T} \alpha^r_e \leq \sum_{e\in E_T} \alpha^s_e$ for any $s \in T$ and thus r is a minimal root. Using (14) and Lemma 5 (c) we obtain that $\sum_{e\in E_T} \alpha^r_e \leq \sum_{e\in E_T} \alpha^v_e$ holds for all v adjacent to r. Now let s be a node not adjacent to r. Similarly to the proof of Theorem 7 we consider a path Π joining r and s and derive relations (11) and (12). We apply next (14) to (12) to obtain (13). From (11) and (13) follows that $\sum_{e\in E_T} \alpha^r_e \leq \sum_{e\in E_T} \alpha^s_e$. □

Proposition 10. *Suppose $\alpha^s_{rs} < \alpha^r_{rs}$, then $\sum_{e\in E_T} \alpha^s_e < \sum_{e\in E_T} \alpha^v_e$ holds for all nodes $v \neq s$ adjacent to r. Furthermore, $\alpha^r_{rs} = \max\{\alpha^r_{rv} : rv \in E_T\}$.*

Proof. Let $v \neq s$ be a node adjacent to r. By Lemma 5 (a) $\alpha^s_e = \alpha^v_e$ holds for all $e \in E_T \setminus \{sr, rv\}$. Furthermore, we obtain the chain of inequalities

$$\alpha^s_{rv} \leq \alpha^s_{rs} < \alpha^r_{rs} = \alpha^v_{rs} \leq \alpha^v_{rv},$$

where the first inequality follows from (8), the second from the assumption of this lemma, the equality from Lemma 5 (a) applied to nodes r, v and the last inequality again from (8). Thus $\sum_{e\in E_T} \alpha^s_e < \sum_{e\in E_T} \alpha^v_e$. From the relations

$$\alpha^r_{rv} = \alpha^s_{rv} \leq \alpha^s_{rs} < \alpha^r_{rs}, \quad v \neq s,$$

we obtain the second claim of the lemma, where the first equality follows from Lemma 5 (a) applied to r and s. □

In the remainder of this section we show that the assumption on r to be a minimal root is not only a necessary condition as it follows from Theorem 7 but in some cases also sufficient for a truncated knapsack tree inequality to be facet-defining for the polytope P_{B}.

For this purpose we assume that $G = (T, E_T)$ is a tree and $f_v = 1$ for all $v \in T$. Then the knapsack polytope P_{K} is defined by the inequality $\sum_{v\in T} x_v \leq F$ and the corresponding knapsack tree inequality (3) defined on (T, E_T) takes the form

$$\sum_{v\in T} (1 - \sum_{e\in P_{rv}} y_e) \leq F.$$

Applying the strengthening (5) and notation (6) we obtain $\alpha_0 = |V| - F$ and $\alpha^r_e = \min\{|V^r_e|, |V| - F\}$ for all $e \in E$, where V^r_e is the set of nodes, whose path to $r \in T$ contains the edge e, see e.g. Figure 3. To emphasize the special case that we treat in the sequel we set $\kappa^r_e := \alpha^r_e$, $\bar{F} := \alpha_0$ and consider the inequality $\sum_{e\in E_T} \kappa^r_e y_e \geq \bar{F}$ or $(\kappa^r)^T y \geq \bar{F}$ for short. For ease of exposition we call κ^r_e the *knapsack weight* of $e \in E$ with respect to the root r of (T, E_T). If $\kappa^r_e = \bar{F}$ and $\bar{F} < |V^r_e|$ we say that e has the *reduced knapsack weight.* Furthermore, we introduce the term *branch-less path*, which is a path in (T, E_T), whose inner nodes are all of degree 2.

Theorem 11. *Assume that* $G = (T, E_T)$ *is a tree,* $f_v = 1$ *for all* $v \in T$ *and that* $\frac{|T|}{2} + 1 \leq F < |T|$. *Furthermore, let* (T, E_T) *be rooted at* $r \in T$ *and assume that all branch-less paths to* r *in* (T, E_T) *contain only edges with reduced knapsack weights.* $(\kappa^r)^T y \geq \bar{F}$ *is a facet-defining inequality for* P_{B} *if and only if* r *is a minimal root.*

Remark 12. *Given a graph* $G = (V, E)$, P_{B} *is full-dimensional under assumptions that* $f_v = 1$ *for all* $v \in V$ *and* $F \geq \frac{|V|}{2} + 1$, *see [9]. For sake of simplicity we will restrict ourselves to this case and refer to [10] for the remaining case* $F = \frac{|V|}{2}$.

Furthermore, note that in case $F = |V|$ *the knapsack inequality* $\sum_{v \in V} x_v \leq F$ *is redundant for* P_{K} *and thus the corresponding truncated knapsack tree inequalities are redundant for* P_{B}. *Therefore we assume that* $F < |V|$, *in particular,* $\bar{F} > 0$.

Due to the complexity of the proof of Theorem 11 we complete it in several steps. First we outline the general idea of the sufficiency part. Let $\mathcal{F}$ be a face of P_{B} induced by $(\kappa^r)^T y \geq \bar{F}$ and $\mathcal{F}_b$ be the facet of P_{B} defined by the inequality $b^T y \geq b_0$ such that $\mathcal{F} \subseteq \mathcal{F}_b$. To show that $(\kappa^r)^T y \geq \bar{F}$ is a facet-defining inequality for P_{B}, we prove that $\mathcal{F} = \mathcal{F}_b$, i.e., there exists $\gamma \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{array}{rcl} b_e & = & \gamma \kappa_e, \quad \forall\, e \in E \\ b_0 & = & \gamma \bar{F}. \end{array} \tag{15}$$

hold.

We introduce now further definitions and lemmas required to prove the above relations. Given a partition of the node set T we denote by V_r the cluster containing r, see e.g. Figure 3. We say that two edges $e, f \in E$ are *related*, if there exists a path to the root containing both e and f. An edge e is related to itself. For an edge e, the set $B_e = \{f \in E \,:\, f \text{ is related to } e\}$ is called *branch* (induced by e). If any two edges e and f are adjacent and related and such that f is closer to the root than e (with respect to the number of edges), then f is the *father* of e and e is a *son* of f. We call an edge a *leaf* if one of its endpoints is of degree 1. A branch-less path $\Pi = (V_\Pi, E_\Pi)$ of a subgraph $G' \subseteq (T, E_T)$ is called *maximal* in G' if

$$|V_\Pi| = \max\{|V_p| \,:\, (V_p, E_p) \subseteq G' \text{ and } (V_p, E_p) \text{ is a branch-less path}\},$$

i.e., it is a branch-less path in G' with the maximal number of nodes. We say that a bisection cut $\delta(V_r)$ is *tight* for $(\kappa^r)^T y \geq \bar{F}$ if $(\kappa^r)^T \chi^{\delta(V_r)} = \bar{F}$ is satisfied. As we will show soon, $|V_r| = F$ holds if $\delta(V_r)$ is tight for $(\kappa^r)^T y \geq \bar{F}$ and all $e \in \delta(V_r)$ satisfy $\kappa^r = |V_e^r| \leq \bar{F}$, i.e., all edges in the cut do not have reduced knapsack weights. In this case we will call the cut $\delta(V_r)$ *double-tight* for $(\kappa^r)^T y \geq \bar{F}$.

We derive next some properties of bisection cuts tight for $(\kappa^r)^T y \geq \bar{F}$.

Lemma 13. *No two edges in a bisection cut tight for* $(\kappa^r)^T y \geq \bar{F}$ *are related.*

Proof. Let $\delta(V_r)$ be a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$ such that $|\delta(V_r)| > 1$. Let δ_0 be the subset of edges in $\delta(V_r)$ with an even distance to r , i.e.,

$$\delta_0 := \{e \in \delta(V_r) \,:\, |\delta(V_r) \cap P_e^r| \text{ is even }\},$$

where P_e^r is the set of edges of the path, which contains e and joins e with r. Let $\delta_1 = \delta(V_r) \backslash \delta_0$. Since $\delta(V_r)$ is tight for $(\kappa^r)^T y \geq \bar{F}$ we have

$$\sum_{e \in \delta_1} \kappa_e^r + \sum_{e \in \delta_0} \kappa_e^r = \bar{F}.$$

On the other side, using the fact that if an edge belongs to a cut then its endpoints belong to different clusters and that the total weight of $T \setminus V_r$ cannot fall below $\bar{F}$ we obtain

$$\sum_{e \in \delta_1} \kappa_e^r - \sum_{e \in \delta_0} \kappa_e^r \geq \bar{F}.$$

Both relations can be satisfied only if $\sum_{e \in \delta_0} \kappa_e^r$ vanishes. Hence $\delta_0 = \emptyset$ and its construction indicates that each path to r can be cut only once. $\square$

Lemma 14. *Let (T, E_T) be rooted at $r \in T$. A bisection cut $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$ if and only if $|V_r| = F$ and $(V_r, E(V_r))$ is connected.*

Proof. Assume first that $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$. By Lemma 13 any two edges in $\delta(V_r)$ are not related. This implies that V_r induces a connected subgraph of (T, E_T). Hence $T \setminus V_r = \bigcup_{e \in \delta(V_r)} V_e^r$ and $V_e^r \cap V_f^r = \emptyset$ for any $e, f \in \delta(V_r)$. Furthermore, $\kappa_e^r = |V_e^r|$ holds for each $e \in \delta(V_r)$ and we obtain $|T \setminus V_r| = \sum_{e \in \delta(V_r)} \kappa_e^r = \bar{F}$, i.e., $|V_r| = F$.
Now consider a bisection $(V_r, T \setminus V_r)$ such that $(V_r, E(V_r))$ is connected and $|V_r| = F$ (i.e., $|T \setminus V_r| = \bar{F}$). We show first that $\delta(V_r)$ contains only edges, whose knapsack weights are not reduced. Assume for contradiction that $\delta(V_r)$ contains an edge f with reduced knapsack weight. Since $\kappa_f^r = \bar{F}$, this is the only edge in $\delta(V_r)$, otherwise $\delta(V_r)$ is not tight for $(\kappa^r)^T y \geq \bar{F}$. Hence $\delta(V_r) = \{f\}$ and $|T \setminus V_r| = |V_f^r| > \bar{F}$ holds contradicting the assumption that $|V_r| = F$. To show that $\delta(V_r)$ is tight for $(\kappa^r)^T y \geq \bar{F}$, we use the assumption that $(V_r, E(V_r))$ is connected. We have

$$\sum_{e \in \delta(V_r)} \kappa_e^r = \sum_{e \in \delta(V_r)} |V_e^r| = |\bigcup_{e \in \delta(V_r)} V_e^r| = \bar{F}.$$

Hence $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$. $\square$

Next we provide some results following from the assumption that (T, E_T) is rooted at a minimal root. As we will show in the following lemmas, this assures the existence of bisection cuts tight for $(\kappa^r)^T y \geq \bar{F}$, which we will consider in the proof of further lemmas preceding the proof of Theorem 11.

Lemma 15. *Let $B = (V_\Pi, E_\Pi) \subseteq (T, E_T)$ be a branch incident on a node $r \in T$. If r is a minimal root of (T, E_T) then $|V_B \setminus \{r\}| \leq F$.*

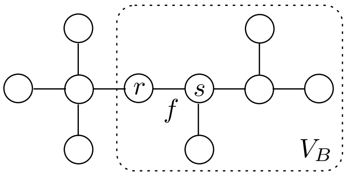

Figure 1: Node set V_B.

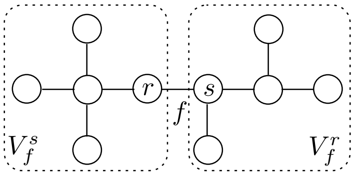

Figure 2: Node sets V_f^r and V_f^s.

Proof. Let $B = (V_B, E_B)$ be a branch incident on r and assume that $|V_B \setminus \{r\}| > F$. We are going to show that in this case there exists a node $s \in T \setminus \{r\}$ such that $\sum_{e \in E_T} \kappa_e^s < \sum_{e \in E_T} \kappa_e^r$, i.e., r cannot be a minimal root. Let s be the node in V_B adjacent to r, and let $f = rs \in E_B$, see Figure 1 and 2. Note that $V_f^r \dot{\cup} V_f^s = V$. Since $|V_f^r| = |V_B \setminus \{r\}| > F > \bar{F}$ we have $\kappa_f^r = \bar{F}$. On the other side $\kappa_f^s = \min\{|V \setminus V_f^r|, \bar{F}\} < \bar{F}$. Hence $\kappa_f^s < \kappa_f^r$ and Lemma 5 (c) yields $\sum_{e \in E_T} \kappa_e^s < \sum_{e \in E_T} \kappa_e^r$. Thus r is not a minimal root. $\square$

Lemma 16. *Assume (T, E_T) is rooted at a minimal root and has an edge $e \in E_T$ such that $\kappa^r_e = \bar{F}$. Then the cut $\delta(V^r_e) = \{e\}$ is a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$.*

Proof. Note that in the considered case we have $V_r = T \setminus V^r_e$. We show that the cut $\delta(V_r)$, which is obviously tight for $(\kappa^r)^T y \geq \bar{F}$, is also a bisection cut. Assume that $\delta(V_r)$ is not a bisection cut. Then either $|V_r| < \bar{F}$ or $|T \setminus V_r| < \bar{F}$. In the first case, let s be a node incident to e such that the path $\Pi_{rs} = (V_{rs}, E_{rs})$ joining r and s contains e, see Figure 3. For all $f \in E_T \setminus E_{rs}$ holds $\kappa^r_f = \kappa^s_f$ due to Lemma 5 (a). For $f \in E_{rs}$ we obtain by Lemma 6 that $\kappa^r_f \geq \kappa^r_e = \bar{F} > |V_r| \geq |V^s_f| = \kappa^s_f$. Hence there exists a node $s \in T \setminus \{r\}$ such that $\sum_{e \in E_T} \kappa^s_e < \sum_{e \in E_T} \kappa^r_e$ contradicting the assumption that r is a minimal root. The case $|T \setminus V_r| < \bar{F}$ is also not possible due to the assumption $\kappa^r_e = \bar{F}$. $\square$

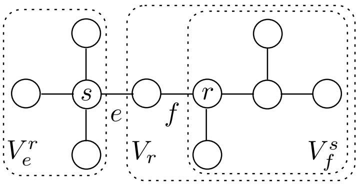

Figure 3: Node sets V_r, V^r_e and V^r_f.

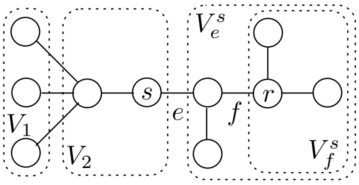

Figure 4: Node sets V_1, V_2, V^s_f and V^s_e.

Lemma 17. *Let (V_B, E_B) be a branch in (T, E_T) incident to the root $r \in T$. Let $E_1 \subseteq E_B$ be such that all edges in E_1 are not related and $\sum_{e \in E_1} \kappa^r_e \leq \bar{F}$. If r is a minimal root, then there exists a bisection $(V_r, T \setminus V_r)$ such that $E_1 \subseteq \delta(V_r)$ and $\delta(V_r)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$.*

Proof. For ease of exposition we set $V_1 := \bigcup_{e \in E_1} V^r_e$. By the assumption $\bar{F} \geq \sum_{e \in E_1} \kappa^r_e = |V_1|$. Therefore $|T \setminus V_1| \geq F$ and there exists a set $V_2 \subseteq T \setminus V_1$ such that $(V_2, E(V_2))$ is connected and $|V_2| = F$. Thus the partition $(V_2, T \setminus V_2)$ is a bisection. For ease of exposition we set $\bar{V}_2 := T \setminus V_2$. We select V_2 so that the corresponding bisection cut $\delta(V_2)$ includes E_1. It suffices to show that $r \in V_2$ to obtain by Lemma 14 that $\delta(V_2)$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$. As it will turn out this holds if r is minimal. Assume that $r \notin V_2$. Note that $r \notin V_1$ as well. Hence $r \in \bar{V}_2 \setminus V_1$. Furthermore, r and V_1 are in one cluster and thus $\delta(V_2) \setminus E_1 \neq \emptyset$. Let $e \in \delta(V_2) \setminus E_1$ and let $s \in V_2$ be a node incident to e, such that the path $\Pi_{rs} = (V_{rs}, E_{rs})$ joining r and s contains e, see Figure 4. Since $V^s_e \subseteq \bar{V}_2 \setminus V_1$, for all $f \in E_{rs}$ we have $\bar{F} < |V^s_f| = \kappa^s_f$. On the other side $V_1 \cup V_2 \subseteq V^r_f$ and $|V^r_f| > F$ hold for all $f \in E_{rs}$ as well as $\kappa^r_f = \min\{|V^r_f|, \bar{F}\} = \bar{F} > \kappa^s_f$. For all $f \in E_T \setminus E_{rs}$ we have $\kappa^r_f = \kappa^s_f$ due to Lemma 5 (a). Therefore $\sum_{f \in E_T} \kappa^r_e > \sum_{f \in E_T} \kappa^s_e$ and r cannot be a minimal root. $\square$

Remark 18. *From Lemma 16 and 17 follows that for each $e \in E_T$ there exists a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$ and containing e if r is a minimal root of (T, E_T). It can be shown (see [10]) that the condition on the root of (T, E_T) to be minimal is also necessary for the existence of a bisection cut for each $e \in E_T$.*

The next lemma provides a tool, which we will use to obtain new bisection cuts tight for $(\kappa^r)^T y \geq \bar{F}$ from a given one.

Lemma 19. *Let r be a root of (T, E_T). Let $(V_r, T \setminus V_r)$ be a bisection of T such that $|V_r| = F$.*

- (a) *For each $s \in \bar{V}_r$ adjacent to a node $u \in V_r$ there exists a bisection $(V', T \setminus V')$ such that $V' = \{s\} \cup V_r \setminus \{v\}$ for some $v \in V_r$, see Figure 5.*

(*b*) *If $v \neq r$, then $\delta(V')$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$.*

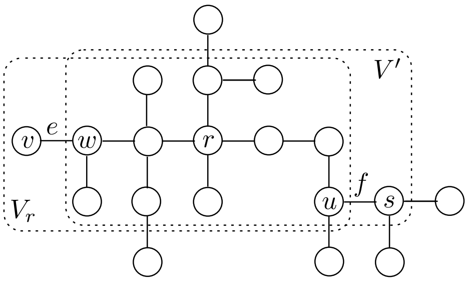

Figure 5: Swapping v against s ($\bar{F} = 6$).

Proof. (*a*) Due to the definition of V' we have $|V'| = |V_r|$ and $|T \setminus V'| = |T \setminus V_r|$. Hence $\bar{F} \leq |V'|, |T \setminus V'| \leq F$ holds and thus $(V', T \setminus V')$ is also a bisection.
(*b*) If $v \neq r$ then $r \in V'$. By definition $(V', E(V'))$ is connected and $|V'| = F$. Hence $\delta(V')$ is double-tight by Lemma 14. □

Note that if $v \neq r$ then V' is obtained from V_r by *swapping* the nodes v and s and thus preserving the weight F of the new cluster containing r, which is V'.

Corollary 20. *Let $\mathcal{F}, \mathcal{F}_b$ be faces of P_{B} defined by $(\kappa^r)^T y \geq \bar{F}$ and $b^T y \geq b_0$, respectively, such that $\mathcal{F} \subseteq \mathcal{F}_b$. In the setting of Lemma 19 (b) let w be the node in V' adjacent to v, $e = vw$ and $f = us$. Then*

$$b_f - \sum_{\bar{e} \in S_f} b_{\bar{e}} = b_e - \sum_{\bar{e} \in S_e} b_{\bar{e}} \tag{16}$$

where S_e and S_f are the sets of sons of e and f, respectively.

Proof. $r \in V'$, since we assume that $v \neq r$. We have $\delta(V_r) = \{f\} \cup S_e \cup D$ and $\delta(V') = \{e\} \cup S_f \cup D$, where D is the set of the remaining edges (equal) in both cuts, see Figure 5. $\delta(V_r)$ and $\delta(V')$ are tight for $(\kappa^r)^T y \geq \bar{F}$ by Lemma 19. Hence their incidence vectors also lie in $\mathcal{F}_b$. Therefore $b_0 = \sum_{e \in \delta(V_r)} b_e = \sum_{e \in \delta(V')} b_e$, i.e.,

$$b_f + \sum_{\bar{e} \in S_e} b_{\bar{e}} + \sum_{\bar{e} \in D} b_{\bar{e}} = b_e + \sum_{\bar{e} \in S_f} b_{\bar{e}} + \sum_{\bar{e} \in D} b_{\bar{e}}.$$

This yields directly the relation (16). □

Lemma 21. *Let $\mathcal{F}, \mathcal{F}_b$ be faces of P_{B} defined by $(\kappa^r)^T y \geq \bar{F}$ and $b^T y \geq b_0$, respectively, such that $\mathcal{F} \subseteq \mathcal{F}_b$. Let $B = (V_B, E_B) \subset (T, E_T)$ be a branch incident to $r \in T$ and $\Pi = (V_\Pi, E_\Pi) \subseteq B$ be the maximal branch-less path to r in B. Assume that r is a minimal root of (T, E_T), then there is a $\gamma_b \neq 0$ such that*

(*a*) *$b_e = b_f$ holds for any two edges $e, f \in E_T$ with $\kappa^r_e = \kappa^r_f = \bar{F}$,*

(*b*) *$b_e = b_f := \gamma_B$ holds for any two leaves $e, f \in B$,*

(*c*) *$b_e = \gamma_B \kappa_e$ holds for $e \in E_B \setminus E_\Pi$,*

(*d*) *$b_e = \gamma_B \kappa_e$ holds for $e \in E_\Pi$, if E_Π contains only edges with reduced knapsack weight.*

Proof. (a) Let e and f be any two edges in E_T with $\kappa^r_e = \kappa^r_f = \bar{F}$. By Lemma 16 both $\delta(V^r_e) = \{e\}$ and $\delta(V^r_f) = \{f\}$ are bisection cuts tight for $(\kappa^r)^T y \geq \bar{F}$. Since $\chi^{\{e\}}$ and $\chi^{\{f\}}$ are in $\mathcal{F}_b$ we obtain $b_e = b_f$.

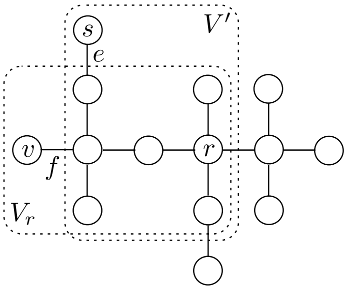

Figure 6: Swapping s against v considered in case (b) $(\bar{F} = 6)$.

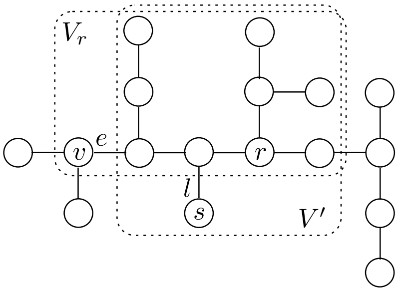

Figure 7: Swapping s against v considered in case $(c.1)$ $(\bar{F} = 7)$.

(b) Let e and f be leaves in E_B and s, v be their respective end-nodes of degree 1. By Lemma 15 there exists a bisection $(V_r, T \setminus V_r)$ such that $V_B \setminus \{s\} \subseteq V_r$ and $|V_r| = F$. Using Lemma 19 we swap s against v and get a new bisection $(V', T \setminus V')$, see Figure 6. Since $v \neq r$ we apply Corollary 20 and by (16) obtain $b_e = b_f = \gamma_B$.

(c) If $B \subseteq \Pi$ there is nothing to prove, so assume that $B \neq \Pi$. Let $e \in E_B \setminus E_\Pi$. If e is a leaf, $b_e = \gamma_B = \gamma_B \kappa^r_e$ follows from (b). Hence we assume that e is not a leaf and consider further two cases:

$(c.1)$ κ^r_e is not reduced.

Since $e \notin E_\Pi$ there exists a leaf $l \in E_B \setminus E_\Pi$ not related to e. By Lemma 6 all sons of e, say S_e, do not have reduced knapsack-weights. We have

$$\sum_{f \in S_e} \kappa^r_f + \kappa^r_l = \sum_{f \in S_e} |V^r_f| + |V^r_l| = |V^r_e| = \kappa^r_e \leq \bar{F}$$

and by Lemma 17 there exists a bisection $(V_r, T \setminus V_r)$ such that the cut $\delta(V_r) = S_e \cup \{l\} \cup D$ is double-tight for $(\kappa^r)^T y \geq \bar{F}$. D is the (possibly empty) set of edges neither related to e nor l. Thus $|V_r| = F$ by Lemma 14. Let s be the end-node of l of degree 1 and v be the node adjacent to e and its sons, see Figure 7. We swap v against s and obtain a new bisection by Lemma 19. Since $v \neq r$ we use Corollary 20. By (16) and (b) it holds

$$b_e = b_l + \sum_{f \in S_e} b_f = \gamma_B + \sum_{f \in S_e} b_f. \tag{17}$$

We assume first that all edges in S_e are leaves. By (17) and (b) we obtain

$$b_e = b_l + \sum_{f \in S_e} b_f = \gamma_B + \gamma_B |S_e| = \gamma_B \kappa^r_e. \tag{18}$$

If e has higher level we apply (17) and (18) recursively and also get $b_e = \gamma_B \kappa^r_e$.

$(c.2)$ κ^r_e is reduced.

By Lemma 16 the cut $\delta(V^r_e) = \{e\}$ is a bisection cut tight for $(\kappa^r)^T y \geq \bar{F}$. Since κ^r_e is reduced there exists a set, say S_e, of edges that are not reduced and that are related to e but not to each other with the total knapsack weight equal to $\bar{F}$, see Figure 8. Note that S_e may

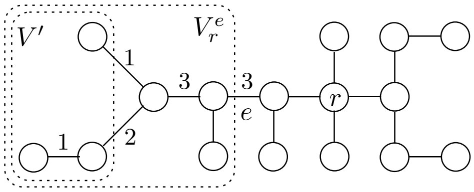

Figure 8: Sets V_r and V' considered in case $(c.2)$ $(\bar{F} = 3)$.

contain not only sons but also further descendants of e. Both cuts $\delta(V_e^r)$ and $\delta(V') = S_e$ are tight for $(\kappa^r)^T y \geq \bar{F}$. Thus by case $(c.1)$ we have

$$b_e = \sum_{f \in S_e} b_e = \sum_{\bar{f} \in S_e} \gamma_B \kappa_f^r = \gamma_B \bar{F} = \gamma_B \kappa_e^r.$$

(d) Since all edges in E_Π have reduced knapsack weight, we apply the same method as in case $(c.2)$. □

Summing up the results presented so far we can turn to the proof of Theorem 11.

Proof of Theorem 11. Theorem 7 yields that the condition on r to be a minimal root in (T, E_T) is necessary for $(\kappa^r)^T y \geq \bar{F}$ to be a facet-defining inequality for P_{B}. It remains to show that it is also a sufficient condition. Let $B_1 = (V_1, E_1), \ldots, B_k = (V_k, E_k)$, $k = deg(r)$, be all branches in (T, E_T) incident to root r. We have $E = \bigcup_{1 \leq i \leq k} E_i$. By Lemma 21 (c) and (d) we have that for all $1 \leq i \leq k$ there exists γ_i such that

$$\forall\, e \in E_i \qquad b_e = \gamma_i \kappa_e^r.$$

By the assumption each branch in (T, E_T) contains at least one edge with reduced knapsack weight. For any two i, j $(1 \leq i, j \leq k)$ we consider an edge $e_i \in E_i$ and an edge $e_j \in E_j$ with reduced knapsack weight and apply Lemma 21 (a). We have $\gamma_{B_i} \bar{F} = b_{e_i} = b_{e_j} = \gamma_{B_j} \bar{F}$. Hence $\gamma_i = \gamma_j := \gamma$ holds for all i, j, $1 \leq i, j \leq k$ and we obtain $b_e = \gamma \kappa_e^r$ for all $e \in E_T$. Now, let $\delta(V_r)$ be a bisection cut, whose incidence vector $\chi^{\delta(V_r)}$ lies on the face $\mathcal{F}$ $(\subseteq \mathcal{F}_b)$. We have

$$b_0 = \sum_{e \in E_T} b_e \chi_e^{\delta(V_r)} = \sum_{e \in \delta(V_r)} b_e = \gamma \sum_{e \in \delta(V_r)} \kappa_e^r = \gamma \bar{F}.$$

and thus (15) holds. □

The results in Theorem 11 can be generalized as we describe in the next theorem.

Theorem 22. *Assume that $G = (T, E_T)$ is a tree rooted at a node $r \in T$, $f_v = 1$ for all $v \in T$ and $\frac{|T|}{2} + 1 \leq F < |T|$. The truncated knapsack tree inequality $(\kappa^r)^T y \geq \bar{F}$ is facet-defining for P_{B} if and only if one of the following conditions is satisfied*

- (a) *r is a minimal root and each branch-less path in (T, E_T) contains less than F nodes*
- (b) *r is a minimal root and (T, E_T) contains a branch-less path with exactly F nodes and one end-edge of this path is a leaf,*
- (c) *$F = |T| - 1$.*

The proof of Theorem 22 requires a further distinction of cases depending on the existence and the allocation of edges with reduced knapsack weight and the degree of the root r. They are handled in a similar manner as in the proof of Theorem 11. Due to their length and complexity we abandon their presentation here and refer to [10] for all details.

5 The bisection knapsack walk inequalities for MB

In this section we exploit the special structure of MB in order to derive an improved version of the knapsack tree inequality. Note that in the MNCGP case with $K > 2$ a walk $\{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}\}$ with $y_{e_1} = y_{e_2} = 1$ does not imply any relation between nodes v_1 and v_3 while in the MB case where $K = 2$ it follows from $y_{e_1} = y_{e_2} = 1$ that v_1 and v_3 belong to the same cluster.

More generally, whenever there is a walk between two nodes of the graph with an even number of edges in the cut we know in the case of MB that the two end nodes of the walk have to be in the same cluster. We may therefore replace the indicator term $1 - \sum_{e \in P_{rv}} y_e$ of (3) by

$$1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \tag{19}$$

where $H_v \subseteq P_{rv}$ with even cardinality. So if $y \in \{0,1\}^{|E|}$ is a valid solution of MB and P_{rv} is a walk from r to v in G with $H_v = \{e \in P_{rv} : y_e = 1\}$ and $|H_v|$ even, then expression (19) is equal to one, indicating that r and v belong to the same cluster. If, however, $H_v \neq \{e \in P_{rv} : y_e = 1\}$ the value of (19) is less than or equal to zero.

Lemma 23. *Let a specified root node $r \in V$, walks $P_{rv} \subseteq E$ and even subsets $H_v \subseteq P_{rv}$ for all $v \in V$ be given. Let S_1, S_2 be a partition of V with $r \in S_1$. Then for $y = \chi^{\delta(S_1)}$ (and therefore for all $y \in P_{\mathrm{B}}$) and for all $v \in V$*

$$1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \leq z_v^1 \ . \tag{20}$$

Proof. (20) is true if $v \in S_1$, because $y_e \geq 0$ and $1 - y_e \geq 0$ for all $e \in E$ and $z_v^1 = 1$. If $v \notin S_1$ the set $C = \{e \in P_{rv} : y_e = 1\}$ must be of odd cardinality (otherwise r and v would be together in S_1). Since H_v is of even cardinality and both C and H_v are subsets of P_{rv} there exists an $e \in P_{rv}$ with $e \in C \setminus H_v$ or $e \in H_v \setminus C$. If $e \in C \setminus H_v$ then $y_e = 1$ and the left-hand side of (20) is smaller or equal to $1 - y_e = 0 = z_v^1$. If $e \in H_v \setminus C$ then $y_e = 0$ and the left-hand side of (20) is smaller or equal to $1 - (1 - y_e) = 0 = z_v^1$. □

Now we are ready to sum up all the evaluation terms.

Definition 24 (Bisection knapsack walk inequality)**.** *Let $\sum_{v \in V} a_v x_v \leq a_0$ be a valid inequality for the knapsack polytope P_{K} with $a_v \geq 0$ for all $v \in V$. For a subset $V' \subseteq V$, a fixed root node $r \in V'$, walks $P_{rv} \subseteq E$, and sets $H_v \subseteq P_{rv}$ with $|H_v|$ even, the* bisection knapsack walk inequality *reads*

$$\sum_{v \in V'} a_v \left(1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \right) \leq a_0 \ . \tag{21}$$

Then Lemma 23 directly implies

Proposition 25. *The bisection knapsack walk inequality (21) is valid for the polytope* P_{B}.

Note that knapsack tree inequalities are a special case of the bisection knapsack walk inequalities where the walks P_{rv} form a tree, all nodes on these walks are contained in V' and all $H_v = \emptyset$. Again, we may rewrite the bisection knapsack walk inequality so as to pronounce its strength in forcing cut variables to increase:

$$\sum_{e\in E}\left(\sum_{v\in V':e\in P_{rv}} a_v - \sum_{v\in V':e\in H_v} 2a_v\right) y_e \geq \sum_{v\in V'} a_v - a_0 - \sum_{v\in V'} a_v|H_v| \ .$$

Lemma 3 can be applied to strengthen bisection knapsack walk inequalities in this form to yield the so-called *truncated bisection knapsack walk inequalities*.

Remark 26. *Note that one can also show* $\sum_{e\in P_{rv}\setminus U_v} y_e + \sum_{e\in U_v}(1-y_e) \geq z_v^1$ *for all* $v \in V\setminus\{r\}$ *if* $|U_v|$ *odd and* $r \in S_1$. *Furthermore, a valid knapsack inequality* $\sum_{v\in V} a_v x_v \leq a_0$ *implies in case of (MB) validity of* $\sum_{v\in V'} a_v z_v^1 \geq a(V') - a_0$ *for all* $V' \subseteq V$. *Thus the so-called* odd bisection knapsack walk inequality

$$a_r + \sum_{v\in V'\setminus\{r\}} a_v \left(\sum_{e\in P_{rv}\setminus U_v} y_e + \sum_{e\in U_v}(1-y_e)\right) \geq a(V') - a_0$$

is valid for P_{B}, *too. Due to their close relation to the (even) bisection knapsack walk inequalities (21) we will not treat these inequalities further in this paper but refer the interested reader to [1].*

6 Relation between odd cycle inequalities and bisection knapsack walk inequalities

Another class of valid inequalities for P_{B}, which are closely connected to bisection knapsack walk inequalities, are the odd cycle inequalities [3] which completely describe the cut polytope on graphs not contractible to the complete graph on five nodes [2].

Definition 27 (Odd cycle inequality [3])**.** *For a cycle* $C = (V_C, E_C)$ *in* G *and a subset* $U \subseteq E_C$ *with* $|U|$ *odd we define the* odd cycle inequality

$$\sum_{e\in E_C\setminus U} y_e - \sum_{e\in U} y_e \geq 1 - |U| \ .$$

If $|E_C| = 3$ *the odd cycle inequality is called* triangle inequality.

Proposition 28. [9] *The odd cycle inequalities are valid for the polytope* P_{B}.

Proof. Since P_{B} is contained in the cut polytope and the odd cycle inequalities are valid for the cut polytope they are also valid for P_{B}. □

In order to exhibit the tight relation of odd cycle inequalities to the bisection knapsack walk inequalities, consider the key relation (20) of Lemma 23 which we used in the proof of Proposition 25. If $\{r,v\} \in E$, $r \in S_1$, and $y = \chi^{\delta(S_1)}$ then $z_v^1 = z_r^1 - y_{rv} = 1 - y_{rv}$. In this case (20) reads

$$1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \leq 1 - y_{rv} = z_v^1 \ . \tag{22}$$

If P_{rv} is a path then this is the odd cycle inequality with edge set $E_C = P_{rv} \cup \{rv\}$ and odd set $U = H_v \cup \{rv\}$ which is valid for all $y \in P_{\mathrm{B}}$. Thus an alternative way to show that (20) holds for paths P_{rv} is to use the odd cycle inequality to bound z_v^1 as in (22) from below and to insert this relation into the valid knapsack inequality $\sum_{v \in V} a_v z_v^1 \leq a_0$. Note that this shows directly that (20) is valid for paths P_{rv} in case $rv \in E$. Since the variable y_{rv} is not contained in (20) it is also valid for paths P_{rv} if we project out the edge rv thus taking care of the case $rv \notin E$.

The observations above lead us to an assertion on the strength of bisection knapsack walk inequalities on a subgraph induced by the node set of a star if all odd cycle inequalities are fulfilled.

Proposition 29. *Suppose $(V', E_{V'})$ is a star contained in G with center $r \in V'$ and let $\sum_{v \in V} a_v x_v \leq a_0$ be a valid inequality for the knapsack polytope P_{K} with $a_v \geq 0$. Let $y \in \{0,1\}^{|E|}$ satisfy all odd cycle inequalities on the subgraph $(V', E(V'))$ of G induced by V'. Then the strongest bisection knapsack walk inequality on V' rooted at r is the knapsack star inequality*

$$a_r + \sum_{v \in V' \setminus \{r\}} a_v \, (1 - y_{rv}) \leq a_0 \ .$$

Proof. Let an arbitrary bisection knapsack walk inequality rooted at r first be given only via paths P_{rv} and sets H_v. Then use (22) to see that

$$a_r + \sum_{v \in V' \setminus \{r\}} a_v \left(1 - \sum_{e \in P_{rv} \setminus H_v} y_e - \sum_{e \in H_v} (1 - y_e) \right) \leq a_r + \sum_{v \in V' \setminus \{r\}} a_v \, (1 - y_{rv}) \leq a_0 \tag{23}$$

holds. Using walks instead of paths P_{rv} does not increase the left-hand side of the above relation: Let a cycle contained in P_{rv} be denoted by C and its set of complemented edges by H_C. If $|H_C|$ is odd the fulfilled odd cycle inequality $\sum_{e \in C \setminus H_C} y_e + \sum_{e \in H_C} (1 - y_e) \geq 1$ shows that the walk P_{rv} contributes at most zero to the left-hand side of (23), i.e., removing v from V' may increase the left-hand side of (23). If $|H_C|$ is even, the cycle can be left out of the walk while increasing the left-hand side of (23) by $\sum_{e \in C \setminus H_C} y_e + \sum_{e \in H_C} (1 - y_e)$. Thus the latter walks can be reduced to paths with no smaller left-hand side of (23). □

Influenced by Proposition 29 one might now be tempted to expect that in the presence of all odd cycle inequalities the strongest bisection knapsack walk inequalities are obtained by taking P_{rv} as the shortest path (with respect to number of edges) in G connecting r to v. This is not true, as the following example shows.

Example 30. *Let G be the cycle on five nodes of Figure 9. The solution $y = (y_{12}, y_{23}, y_{34}, y_{45}, y_{15})^T = (0.5, 0.5, 0, 0, 0)^T$ fulfills all odd cycle inequalities because it is a convex combination of the two cuts $(0,0,0,0,0)^T$ and $(1,1,0,0,0)^T$. Now look at the bisection knapsack walk inequalities with $V' = \{1,3\}$ and $r = 1$. The shorter path P_{13}^s from root* *node 1 to node 3 uses the edge set* $\{\{1,2\},\{2,3\}\}$ *with* $H_3^s = \emptyset$ *or* $H_3^s = \{\{1,2\},\{2,3\}\}$*, the longer path* P_{13}^l *uses the edge set* $\{\{3,4\},\{4,5\},\{1,5\}\}$ *with* $H_3^l = \emptyset$*,* $H_3^l = \{\{3,4\},\{4,5\}\}$*,* $H_3^l = \{\{3,4\},\{1,5\}\}$ *or* $H_3^l = \{\{4,5\},\{1,5\}\}$*. For the shorter path of the two possible bisection knapsack walk inequalities the left-hand side value is* $a_3 \cdot 0$ *whereas the best possible bisection knapsack walk inequality on the longer path uses* $H_3^l = \emptyset$ *and yields left-hand side value* $a_3 \cdot 1$*.*

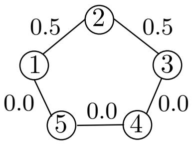

Figure 9: Graph for the counter example of Ex. 30

7 Capacity improved bisection knapsack walk inequalities and the lower envelope for stars

To motivate another strengthening for bisection knapsack walk inequalities consider the case of a disconnected graph with two components, one of them being a single edge $\{u,v\}$, the other connected one being $V' = V \setminus \{u,v\}$. Even though one cannot include the edge $\{u,v\}$ directly in a bisection knapsack walk inequality rooted at some $r \in V'$, one can at least improve the inequality if $y_{uv} = 1$. In this case u and v belong to different clusters and therefore the capacity F of both clusters can be reduced by $\min\{f_u, f_v\}$. Since F is the right-hand side of the inequality $\sum_{v\in V} f_v x_v \le F$ used to define the knapsack polytope P_{K}, this reduction may help to derive stronger bisection knapsack walk inequalities. For instance, one can look at a given valid inequality $\sum_{v\in V} a_v x_v \le a_0$ for the original knapsack polytope with capacity F and in case $y_{uv} = 1$ we are allowed to reduce the right-hand side a_0 by $\min\{a_u, a_v\}$, thus also improving the bisection knapsack walk inequality.

To generalize this idea we define for $\bar{G} \subseteq G$ with $\bar{V} \subseteq V$, $\bar{E} \subseteq E(\bar{V})$ and $a \in \mathbb{R}_+^{|\bar{V}|}$ a function $\beta_{\bar{G}} : \{0,1\}^{|\bar{E}|} \to \mathbb{R}$ with

$$\beta_{\bar{G}}(y) = \inf\left\{a(S), a(\bar{V}\setminus S) : S \subseteq \bar{V}, \max\left\{a(S), a(\bar{V}\setminus S)\right\} \le a_0, y = \chi^{\delta_{\bar{G}}(S)}\right\} .$$

Now we look at the convex envelope $\breve{\beta}_{\bar{G}} : \mathbb{R}^{|\bar{E}|} \to \mathbb{R}$ of $\beta_{\bar{G}}(y)$, i.e.,

$$\breve{\beta}_{\bar{G}}(y) = \sup\left\{\breve{\beta}(y) : \quad \breve{\beta} : \mathbb{R}^{|\bar{E}|} \to \mathbb{R},\ \breve{\beta}(y) \le \beta_{\bar{G}}(y),\ \breve{\beta} \text{ convex}\right\} . \tag{24}$$

Notice that $\breve{\beta}_{\bar{G}}$ is a piecewise linear function on its domain. We will see that given a bisection knapsack walk inequality (21) on some $V' \subseteq V$ and $\bar{V} \subseteq V \setminus V'$ subtracting any linear minorant $\sum_{e\in\bar{E}} c_e y_e$ of $\breve{\beta}_{\bar{G}}$, i.e.,

$$\sum_{e\in\bar{E}} c_e y_e \le \breve{\beta}_{\bar{G}}(y), \tag{25}$$

on the right-hand side of (21) yields again a valid inequality for P_{B}. It yields an improvement with respect to a given y if the minorant is positive for this y. For convenience, the next proposition states this for several disjoint subsets $\bar{V}$.

Proposition 31. *Let $\sum_{v\in V} a_v x_v \le a_0$ with $a_v \ge 0$ for all $v \in V$ be a valid inequality for the knapsack polytope P_{K}. Choose a non-empty $V' \subseteq V$ and subgraphs $(\bar V_l, \bar E_l) = \bar G_l \subset G$ with $\bar V_l \cap V' = \emptyset$, $\bar E_l \subseteq E(\bar V_l)$ for $l = 1, \dots, L$ and pairwise disjoint sets $\bar V_l$. Find for each l a linear minorant $\sum_{e\in \bar E_l} c_e y_e$ for the convex envelope $\check\beta_{\bar G_l}$ so that (25) holds for all y in P_{B}. Then the* capacity reduced bisection knapsack walk inequality

$$\sum_{v\in V'} a_v \left(1 - \sum_{e\in P_{rv}\setminus H_v} y_e - \sum_{e\in P_{rv}\cap H_v} (1-y_e)\right) \le a_0 - \sum_{l=1}^{L} \sum_{e\in \bar E_l} c_e y_e \tag{26}$$

is valid for P_{B}.

Proof. Let $y \in P_{\mathrm{B}}$ such that $y = \chi^{\delta(S_1)}$ with $S_1 \subseteq V$, $S_2 = V \setminus S_1$, i.e., $f(S_1) \le F$ and $f(S_2) \le F$. W.l.o.g. let $r \in S_1$. Recall $z_v^1 = 1$ for all $v \in S_1$. Then for all $l = 1, \dots, L$

$$\sum_{e\in \bar E_l} c_e y_e \le \check\beta_{\bar G_l}(y) = \min\left\{ \sum_{v\in \bar V_l \cap S_1} a_v, \sum_{v\in \bar V_l\cap S_2} a_v \right\} \le \sum_{\bar V_l \cap S_1} a_v z_v^1 \ .$$

Furthermore, by Lemma 23 we have $1 - \sum_{e\in P_{rv}\setminus H_v} y_e - \sum_{e\in H_v} (1-y_e) \le z_v^1$. for $v \in V'$. Thus

$$\begin{aligned} &\sum_{v\in V'} a_v \left(1 - \sum_{e\in P_{rv}\setminus H_v} y_e - \sum_{e\in H_v} (1-y_e)\right) + \sum_{l=1}^{L} \sum_{e\in \bar E_l} c_e y_e \\ &\quad\le \sum_{v\in V'} a_v z_v^1 + \sum_{l=1}^{L} \sum_{v\in \bar V_l} a_v z_v^1 \le \sum_{v\in V} a_v z_v^1 \le a_0 \ . \qquad \square \end{aligned}$$

Remark 32. *Note that it is possible that inequality (26) can be further strengthened using the strengthening of Lemma 3.*

Example 33. *For the graph G displayed in Figure 10 with $f_v = 1$ for all $v \in V$ the polytope P_{B} has 74 facets. Among these are 14 trivial facets, only 2 pure bisection knapsack walk facets, 19 truncated bisection knapsack walk facets, 16 capacity reduced bisection knapsack walk facets (some truncated), 4 capacity reduced odd bisection knapsack walk facets and 19 facets for which we are not yet able to recognize a construction rule. Here we want to give a first simple example for a capacity reduced bisection knapsack walk inequality. Two more involved examples will follow at the end of this section. We use the knapsack inequality $\sum_{v\in V} z_v \le 4$ in all three examples, thus $a_v = 1$ for all $v \in V$:*

(1) *For $V' = \{1, 3, 4, 5\}$, root node $r = 3$ and $H_v = \emptyset$ for all $v \in V'$ the bisection knapsack walk inequality is $1 + (1 - y_{13}) + (1 - y_{34}) + (1 - y_{34} - y_{45}) \le 4$. We choose $\bar G = (\bar V, \bar E)$ with $\bar V = \{6, 7\}$ and $\bar E = \{67\}$. We will see that the unique best minorizing function for $\check\beta_{\bar G}$ is y_{67}, thus the bisection knapsack walk inequality can be strengthened to $1 + (1 - y_{13}) + (1 - y_{34}) + (1 - y_{34} - y_{45}) \le 4 - y_{67}$. Now rewrite this inequality to $y_{13} + 2y_{34} + y_{45} - y_{67} \ge 0$ to see that we can use Lemma 3 to reduce the coefficient of y_{34} to 1 in order to find the facet $y_{13} + y_{34} + y_{45} - y_{67} \ge 0$ of P_{B}.*

To find inequalities (25) to apply in Proposition 31 we take a closer look at the lower envelope defined in (24). In certain cases, e.g., for the case of $\bar G = (\bar V, \bar E)$ being a star with $a(\bar V) \le a_0$, we are able to give a full description of $\check\beta_{\bar G}$ by giving a complete description of the cluster weight polytope defined below. This will provide the tightest improvement possible in (26).

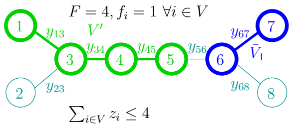

Figure 10: Graphs for Ex. 33

Definition 34. *Let a graph $G = (V, E)$ with non-negative node weights $a_v \in \mathbb{R}$ for all $v \in V$ be given. For a set $S \subseteq V$ we define the following point in $\mathbb{R}^{|E|+1}$*

$$h(S) = \begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix} .$$

With respect to a given non-negative $a_0 \in \mathbb{R}$ we define

$$P_{\mathrm{CW}} = \operatorname{conv}\{\, h(S) : \, S \subseteq V, \, a(S) \leq a_0, \, a(V \setminus S) \leq a_0 \,\}$$

and call this set the cluster weight polytope*.*

Proposition 35. *Let $\bar{G}$ be a subgraph of G. Then valid inequalities for $P_{\mathrm{CW}}(\bar{G})$ of the form $y_0 + \sum_{e \in \bar{E}} \gamma_e y_e \geq \gamma_0$ minorize $\check{\beta}_{\bar{G}}$ and the facets of $P_{\mathrm{CW}}(\bar{G})$ of this form correspond to supporting minorants of $\check{\beta}_{\bar{G}}$.*

Proof. Let $S \subseteq \bar{V}$ be the smaller cluster, i.e., $a(S) \leq a(\bar{V} \setminus S)$ and let $y = \chi^{\delta_{\bar{G}}(S)}$. Then $\check{\beta}_{\bar{G}}(y) = a(S)$ and y is an extreme point of the domain of $\check{\beta}_{\bar{G}}$. Therefore $\left(\check{\beta}_{\bar{G}}(y), y^T\right)^T = h(S)$ and any such point is in one to one correspondence to the “lower” facets of the polytope $P_{\mathrm{CW}}(\bar{G})$. ☐

For a star $\bar{G} = (\bar{V}, \bar{E})$ we are able to exhibit facets of $P_{\mathrm{CW}}(\bar{G})$, which in certain problems enable us to strengthen bisection knapsack walk inequalities of P_{B} to facet-defining inequalities of P_{B} (see Example 49 at the end of this section).

Let us first look at a symmetry of P_{CW} for general graphs $G = (V, E)$, a property which we will later use frequently to cut down our efforts in the proofs.

Proposition 36. *P_{CW} is symmetric to the hyperplane $\{\, y \in \mathbb{R}^{|E|} : \, 2y_0 = a(V) \,\}$.*

Proof. Observe that for any point $h(S)$ used in the definition of P_{CW} the point $h(V \setminus S)$ is contained in P_{CW}, too. Since $\chi^{\delta(S)} = \chi^{\delta(V \setminus S)}$ we have for all those pairs $(h(S), h(V \setminus S))$

$$\begin{pmatrix} \frac{1}{2} a(V) \\ \chi^{\delta(S)} \end{pmatrix} - h(S) = h(V \setminus S) - \begin{pmatrix} \frac{1}{2} a(V) \\ \chi^{\delta(S)} \end{pmatrix} .$$

☐

Another useful result for a star $G = (V, E)$ is the following

Lemma 37. *Let $G = (V, E)$ be a star with center $r \in V$, $a_v \geq 0$ for all $v \in V$ and $a_{v'} = a(V \setminus \{v'\})$ for at least one $v' \in V \setminus \{r\}$. Then $a(S) = a(V \setminus S)$ for all $S \subseteq V$ with $v' \in S$ and $r \in V \setminus S$ if and only if $a_{v'} = a_r$ and $a_v = 0$ for all $v \in V \setminus \{v', r\}$.*

Proof. The sufficiency is obvious. We will show necessity: Suppose $a(S) = a(V \setminus S)$ for all $S \subseteq V$ with $v' \in S$ and $r \in V \setminus S$. Then, in particular, this is true for $V \setminus S = \{r\}$, i.e.,

$a_r = a(V \setminus \{r\}) = a_{v'} + a(V \setminus \{v', r\}) = a(V \setminus \{v'\}) + a(V \setminus \{v', r\}) = a_r + 2a(V \setminus \{v', r\})$. Thus, $a_v = 0$ for all $v \in V \setminus \{v', r\}$ and $a_{v'} = a_r$. □

In the remaining part of the section we will look into P_{CW} for stars $G = (V, E)$ with center node $r \in V$ and the constraint $\sum_{v \in V} a_v x_v \leq a_0$. At first we determine the dimension of the polytope.

Proposition 38. *For a star $G = (V, E)$ with center $r \in V$ and $a(V) \leq a_0$ the polytope P_{CW} has full dimension $|E| + 1$ for $a \neq 0^{|E|}$ and dimension $|E| = |V|$ for $a = 0^{|E|}$.*

Proof. Since G is a star and by assumption $a(V) \leq a_0$, the $1 + |E|$ points $h(\emptyset)$ and $h(\{v\})$ for all $v \in V \setminus \{r\}$ are contained in P_{CW} and affinely independent. Thus the dimension of P_{CW} is at least $|E|$. If $a \neq 0^{|E|}$ then $h(V)$ is affinely independent from all points listed previously, thus P_{CW} is full-dimensional with dimension $|E| + 1$. For $a = 0^{|E|}$ all points lie on the hyperplane $y_0 = 0$. □

For $G = (V, E)$ a star with center $r \in V$, weights $a_v = 0$ for all $v \in V$ and $a_0 \geq 0$ it can easily be worked out that P_{CW} is completely described by the equality $y_0 = 0$ and the inequalities $0 \leq y_{rv} \leq 1$ for all $v \in V \setminus \{r\}$. So from now on we assume $a_v > 0$ for at least one $v \in V$. Let us first state trivial valid inequalities and facets of P_{CW}.

Proposition 39. *For a star $G = (V, E)$ with center $r \in V$, $a \neq 0^{|E|}$ and $a(V) \leq a_0$ the trivial inequalities*

$$0 \leq y_{rv} \leq 1, \quad \forall\, v \in V \setminus \{r\} \tag{27}$$

are facet-inducing except for one particular case: if there is exactly one $v' \in V \setminus \{r\}$ with $a_{v'} = a_r = \frac{1}{2}a(V)$ then $y_{rv'} \leq 1$ does not induce a facet.

Proof. The validity of the inequalities $y_{rv'} \geq 0$ and $y_{rv'} \leq 1$ for all $v' \in V \setminus \{r\}$ follows from the definition of P_{CW}. In general, to prove that a valid inequality defines a facet of P_{CW} we have to find $\dim(P_{\mathrm{CW}})$ affinely independent points of P_{CW} which fulfill it with equality. From Proposition 38 we know that $\dim(P_{\mathrm{CW}}) = |V|$ if $a \neq 0^{|E|}$. For $y_{rv'} \geq 0$ we choose the $|V|$ points $h(\emptyset)$, $h(V)$ and $h(\{v\})$ for all $v \in V \setminus \{r, v'\}$. For $y_{rv'} \leq 1$ the accumulation of affinely independent points on the inequality is a bit more involved: If $a_{v'} \neq a(V \setminus \{v'\})$ we can choose the $|V|$ points $h(\{v'\})$, $h(V \setminus \{v'\})$ and $h(\{v', v\})$ for all $v \in V \setminus \{r\}$ with $v \neq v'$. If $a_{v'} = a(V \setminus \{v'\})$ we look at two cases:

1. $a_r \neq a_{v'}$: Then there is a $\tilde{v} \in V \setminus \{r, v'\}$ with $a_{\tilde{v}} > 0$. Furthermore, since $a_{v'} = a(V \setminus \{v'\})$, we have $a_{v'} = \frac{1}{2}a(V)$. Together with $a_{\tilde{v}} > 0$ this implies $a(\{v', \tilde{v}\}) \neq a(V \setminus \{v', \tilde{v}\})$, i.e., $h(\{v', \tilde{v}\}) \neq h(V \setminus \{v', \tilde{v}\})$. Thus we can choose the $|V|$ points $h(\{v'\})$, $h(\{v', v\})$ for all $v \in V \setminus \{r, v'\}$ and $h(V \setminus \{v', \tilde{v}\})$.

2. $a_r = a_{v'}$: The set of points contained in the definition of P_{CW} which fulfill $y_{rv'} = 1$ is $\{h(S), h(V \setminus S) : S \subseteq V, v' \in S, r \in V \setminus S\}$. Lemma 37 implies for every pair $(h(S), h(V \setminus S))$ in this set that $a(S) = a(V \setminus S)$. Since $a(S) + a(V \setminus S) = a(V)$ we get $a(S) = \frac{1}{2}a(V)$ for all S with $y = \chi^{\delta(S)}$ and $y_{rv} = 1$. Thus all vertices of P_{CW} fulfilling $y_{rv} = 1$ live in the hyperplane $\{y \in \mathbb{R}^{|E|+1} : y_0 = \frac{1}{2}a(V)\}$. Therefore, $y_{rv} \leq 1$ cannot induce a facet of P_{CW}. □

In the following two propositions we look into non-trivial facets of P_{CW}. Proposition 40 deals with the case $a(V \setminus \{r\}) > a_r$ and Proposition 41 with the case $a(V \setminus \{r\}) \leq a_r$.

Proposition 40. *Let $G = (V, E)$ be a star with center $r \in V$, $a \neq 0^{|E|}$, $a(V) \leq a_0$ and $a(V \setminus \{r\}) > a_r$. We call a triple $(V_p, \bar{v}, V_n)$ feasible if it fulfills $V = \{r, \bar{v}\} \mathbin{\dot{\cup}} V_p \mathbin{\dot{\cup}} V_n$ and $a(V_p) \leq \frac{1}{2}a(V) < a(V_p) + a_{\bar{v}}$. For all feasible triples $(V_p, \bar{v}, V_n)$ the inequalities*

$$y_0 + \sum_{v \in V_p} a_v y_{rv} + (a(V) - 2a(V_p) - a_{\bar{v}})\, y_{r\bar{v}} - \sum_{v \in V_n} a_v y_{rv} \leq a(V) \tag{28}$$

$$y_0 - \sum_{v \in V_p} a_v y_{rv} - (a(V) - 2a(V_p) - a_{\bar{v}})\, y_{r\bar{v}} + \sum_{v \in V_n} a_v y_{rv} \geq 0 \tag{29}$$

are facet-inducing for P_{CW}.

Note, that it is possible that either V_p or V_n of feasible triples $(V_p, \bar{v}, V_n)$ might be empty, but for $a(V \setminus \{r\}) > a_r$ there always is the special element $\bar{v}$.

Proof of Proposition 40. To cut down our efforts in this proof and the ones to follow observe that for each feasible triple $(V_p, \bar{v}, V_n)$ the corresponding pair of inequalities (28) and (29) is symmetric to the hyperplane $\{\, y \in \mathbb{R}^{|E|} : 2y_0 = a(V) \,\}$. To see this, subtract the equation $2y_0 = a(V)$ from (28) to yield (29). Thus it suffices to show that (28) is valid and facet-defining. Furthermore, to show the validity of (28) it is sufficient to only look at the "upper" points defining P_{CW}, i.e., if w.l.o.g. $S \subseteq V$ such that $a(S) \geq a(V \setminus S)$ then we only need to check validity of (28) for $h(S) = (a(S), \chi^{\delta(S)})^T$.
Consider an arbitrary $S \subseteq V$ such that $a(S) \geq a(V \setminus S)$. Let $V^1 = \{v \in V : rv \in \delta(S)\}$. Recall that $a(S) + a(V \setminus S) = a(V)$. We discern the following four cases:

1. $\bar{v} \in V^1 = S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (28) equals

   $$\begin{aligned} a(V^1) + a(V_p \cap V^1) + a(V) - 2a(V_p) - a_{\bar{v}} - a(V_n \cap V^1) &= \\ 2a(V_p \cap V^1) + a(V) - 2a(V_p) &= \\ a(V) - 2a(V_p \setminus V^1) &\leq a(V) \end{aligned}$$

   where the first equality uses $a(V^1) = a(V_p \cap V^1) + a_{\bar{v}} + a(V_n \cap V^1)$ and the inequality is due to $a(V_p \setminus V^1) \geq 0$.

2. $\bar{v} \notin V^1 = S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (28) equals

   $$a(V^1) + a(V_p \cap V^1) - a(V_n \cap V^1) = 2a(V_p \cap V^1) \leq 2a(V_p) \leq a(V)$$

   where the equality uses $a(V^1) = a(V_p \cap V^1) + a(V_n \cap V^1)$ and the last inequality is due to $a(V_p) \leq \frac{1}{2}a(V)$ by the definition of V_p.

3. $\bar{v} \in V^1 = V \setminus S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (28) equals

   $$\begin{aligned} a(V) - a(V^1) + a(V_p \cap V^1) + a(V) - 2a(V_p) - a_{\bar{v}} - a(V_n \cap V^1) &= \\ 2a(V) - 2a(V_p) - 2a_{\bar{v}} - 2a(V_n \cap V^1) &< \\ 2a(V) - a(V) - 2a(V_n \cap V^1) &\leq a(V) \end{aligned}$$

   where the first equality uses $a(V^1) = a(V_p \cap V^1) + a_{\bar{v}} + a(V_n \cap V^1)$, the strict inequality is due to $a(V_p) + a_{\bar{v}} > \frac{1}{2}a(V)$ by the definition of V_p and $\bar{v}$ and the inequality holds since $a(V_n \cap V^1) \geq 0$.

4. $\bar{v} \notin V^1 = V \setminus S$: For $\begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ the left-hand side of (28) equals

$$a(V) - a(V^1) + a(V_p \cap V^1) - a(V_n \cap V^1) = a(V) - 2a(V_n \cap V^1) \leq a(V)$$

where the first equality uses $a(V^1) = a(V_p \cap V^1) + a(V_n \cap V^1)$ and the inequality is due $a(V_n \cap V^1) \geq 0$.

In order to show that (28) is also facet-defining, let $V_p = \{v_1^p, \ldots, v_{|V_p|}^p\}$ and $V_n = \{v_1^n, \ldots, v_{|V_n|}^n\}$. Then the $|V|$ points

$$\begin{array}{l} h(V) \\ h(V \setminus \{v_1^p\}) \\ \ldots \\ h(V \setminus \{v_1^p, \ldots, v_{|V_p|}^p\}) \\ h(\{v_1^p, \ldots, v_{|V_p|}^p, \bar{v}\}) \\ h(\{v_1^p, \ldots, v_{|V_p|}^p, \bar{v}, v_1^n\}) \\ \ldots \\ h(\{v_1^p, \ldots, v_{|V_p|}^p, \bar{v}, v_1^n, \ldots, v_{|V_n|}^n\}) \end{array}$$

fulfill the inequality (28) with equality and are affinely independent, thus (28) is a facet-inducing inequality. □

In the case of $a(V \setminus \{r\}) \leq a_r$ the set V_n is empty, there is no $\bar{v}$ and the inequalities (28) and (29) take the following form.

Proposition 41. *For a star $G = (V, E)$ with root $r \in V$, $a \neq 0^{|E|}$, $a(V) \leq a_0$ and $a(V \setminus \{r\}) \leq a_r$ the inequalities*

$$y_0 + \sum_{v \in V \setminus \{r\}} a_v y_{e_v} \leq a(V) \tag{30}$$

$$y_0 - \sum_{v \in V \setminus \{r\}} a_v y_{e_v} \geq 0 \tag{31}$$

are facet-inducing for P_{CW}.

Proof. We start again by observing the symmetry of the inequalities (30) and (31) to the hyperplane $\{ y \in \mathbb{R}^{|E|} : 2y_0 = a(V) \}$. To see this, subtract the equation $2y_0 = a(V)$ from inequality (30) to yield inequality (31). Thus we only have to prove the validity and facet-induction of the inequality (30) or (31). We choose (30). Take an $S \subseteq V$ with $a(S) \geq a(V \setminus S)$. Then $h(S) = \begin{pmatrix} a(S) \\ \chi^{\delta(S)} \end{pmatrix}$ is one of the points defining P_{CW}. We see that $V \setminus S = \{v \in V : rv \in \delta(S)\}$. Now plug $h(S)$ into the left-hand side of (30) to get

$$a(S) + a(V \setminus S) = a(V) \; . \tag{32}$$

The point $h(V \setminus S) = \begin{pmatrix} a(V \setminus S) \\ \chi^{\delta(V \setminus S)} \end{pmatrix}$ can also not violate (30) since $a(V \setminus S) \leq a(S)$, thus (30) is valid for P_{CW}.

In order to show that (30) is facet-inducing let $v_1, \ldots, v_{|V|-1}$ be an arbitrary ordering of the nodes in $V \setminus \{r\}$. Then by (32) the $\dim(P_{\mathrm{CW}}) = |V|$ points

$$h(V),\, h(V \setminus \{v_1\}),\, \ldots,\, h\left(V \setminus \{v_1, \ldots, v_{|V|-1}\}\right)$$

fulfill the inequality (30) with equality and are affinely independent. □

All possible facets of P_{CW} fall into one of the following three classes:

$$y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v y_{rv} \le \gamma_0 \tag{33}$$

$$\sum_{v \in V\setminus\{r\}} \gamma_v y_{rv} \le \gamma_0 \tag{34}$$

$$-y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v y_{rv} \le \gamma_0 \tag{35}$$

In the next two lemmas we will look closer into coefficients of facets of the form (33). The following three propositions state that we have found all facets of P_{CW} of the forms (33), (34) and (35), respectively. Finally, Theorem 47 summarizes the results. The section is accompanied by two small examples on how to apply the inequalities to derive capacity reduced bisection knapsack walk inequalities.

Lemma 42. *For an arbitrary facet of P_{CW} of the form (33) we have for all $v \in V \setminus \{r\}$*

$$-a_v \le \gamma_v \le a_v \; .$$

Proof. Let $\gamma_{\tilde v} > 0$. The facet has a root $\left(\hat y_0, \hat y^T\right)^T$ with $\hat y_{r\tilde v} = 0$, because otherwise all roots $\hat y$ would lie on the equation $\hat y_{r\tilde v} = 1$, thus (33) could not induce a facet. Let $\hat y = \chi^{\delta(S)}$ for an $S \subseteq V$ with $a(S) \ge a(V \setminus S)$, i.e., $\hat y_0 = a(S)$. To bound $\gamma_{\tilde v}$ we look at $\bar y = \hat y + e_{r\tilde v}$, i.e., the cut $\delta(S) \cup \{r\tilde v\}$. We discern three cases concerning the location of node $\tilde v$ and the size of the bigger cluster:

1. $\tilde v \in V \setminus S$: By assumption $a(S) \ge a(V \setminus S)$, thus $a(S \cup \{\tilde v\}) \ge a(V \setminus (S \cup \{\tilde v\}))$. Set $\bar y_0 = a(S \cup \{\tilde v\})$, i.e., $\left(\bar y_0, \bar y^T\right)^T = h(S \cup \{\tilde v\}) \in P_{\mathrm{CW}}$. In order for (33) to be feasible for $\left(\bar y_0, \bar y^T\right)^T$ we need $\gamma_0 \ge \bar y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \bar y_{rv}$. Since $\left(\hat y_0, \hat y^T\right)^T$ is a root of (33) we have $\gamma_0 = \hat y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \hat y_{rv}$. Thus, $\hat y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \hat y_{rv} \ge \bar y_0 + \sum_{v \in V\setminus\{r\}} \gamma_v \bar y_{rv}$, i.e., $\hat y_0 \ge \bar y_0 + \gamma_{\tilde v}$. Therefore, $\gamma_{\tilde v} \le \hat y_0 - \bar y_0 = -a_{\tilde v}$. This contradicts our assumption $\gamma_{\tilde v} > 0$, thus the case $\tilde v \in V \setminus S$ is not possible.

2. $\tilde v \in S$ and $a(S \setminus \{\tilde v\}) \ge a((V \setminus S) \cup \{\tilde v\})$: Set $\bar y_0 = a(S \setminus \{\tilde v\})$, i.e., $\left(\bar y_0, \bar y^T\right)^T = h(S \setminus \{\tilde v\}) \in P_{\mathrm{CW}}$. As $\left(\bar y_0, \bar y^T\right)^T$ is feasible for (33) we derive, as in the previous case, $\hat y_0 \ge \bar y_0 + \gamma_{\tilde v}$, hence $\gamma_{\tilde v} \le \hat y_0 - \bar y_0 = a(S) - a(S \setminus \{\tilde v\}) = a_{\tilde v}$.

3. $\tilde v \in S$ and $a(S \setminus \{\tilde v\}) < a((V \setminus S) \cup \{\tilde v\})$: This implies $a(S \setminus \{\tilde v\}) < \frac{1}{2} a(V)$. Set $\bar y_0 = a((V \setminus S) \cup \{\tilde v\})$, i.e., $\left(\bar y_0, \bar y^T\right)^T = h((V \setminus S) \cup \{\tilde v\}) \in P_{\mathrm{CW}}$. From the feasibility of (33) we conclude $\hat y_0 \ge \bar y_0 + \gamma_{\tilde v}$. Therefore, $\gamma_{\tilde v} \le \hat y_0 - \bar y_0 = a(S) - a((V \setminus S) \cup \{\tilde v\}) = a_{\tilde v} + 2a(S \setminus \{\tilde v\}) - a(V) < a_{\tilde v}$, where the last inequality uses $2a(S \setminus \{\tilde v\}) - \frac{1}{2} a(V) < 0$.

An analogous argumentation yields $-a_{\tilde{v}} \leq \gamma_{\tilde{v}}$ in case $\gamma_{\tilde{v}} < 0$ if we choose $\hat{y}$ as a root of (33) with $\hat{y}_{rv} = 1$ and construct $\bar{y} = \hat{y} - e_{rv}$. □

Lemma 43. *For an arbitrary facet of P_{CW} of the form (33) we have $\gamma_0 = a(V)$ and $\sum_{v \in V \setminus \{r\}} \gamma_v \leq a_r$.*

Proof. In order for (33) to be valid for $h(V) = \left(a(V), (\chi^{\delta(V)})^T\right)^T \in P_{\text{CW}}$ we get $\gamma_0 \geq a(V)$. We discern two cases regarding the weight of the root node r.

1. $a_r < a(V \setminus \{r\})$: (33) has to be valid for $\left(a(V \setminus \{r\}), (\chi^{\delta(V \setminus \{r\})})^T\right)^T = h(V \setminus \{r\}) \in P_{\text{CW}}$, thus $\sum_{v \in V \setminus \{r\}} \gamma_v \leq \gamma_0 - a(V \setminus \{r\})$.
2. $a_r \geq a(V \setminus \{r\})$: (33) has to be valid for $\left(a_r, (\chi^{\delta(\{r\})})^T\right)^T = h(\{r\}) \in P_{\text{CW}}$, thus $\sum_{v \in V \setminus \{r\}} \gamma_v \leq \gamma_0 - a_r \leq \gamma_0 - a(V \setminus \{r\})$.

Thus in any case we have

$$\sum_{v \in V \setminus \{r\}} (a_v + \gamma_v) \leq \gamma_0 \; . \tag{36}$$

We can now use $a_v + \gamma_v \geq 0$ (by Lemma 42) and $y_{rv} \in [0, 1]$ for all $\left(y_0, y^T\right)^T \in P_{\text{CW}}$ to conclude that

$$\sum_{v \in V \setminus \{r\}} (a_v + \gamma_v) y_{rv} \leq \gamma_0 \tag{37}$$

is a valid inequality for P_{CW}. Additionally, $a_r = a(V) - a(V \setminus \{r\})$, thus it is sufficient to show that $\gamma_0 = a(V)$ if (33) induces a facet of P_{CW}, because then (36) implies $\sum_{v \in V \setminus \{r\}} \gamma_v \leq a_r$. To show that (33) cannot define a facet if $\gamma_0 > a(V)$, we study which points $h(S) \in P_{\text{CW}}$ could fulfill (33) with equality if $\gamma_0 > a(V)$. For this purpose let $S \subseteq V$ such that $\tilde{y}_0 = a(S) \geq a(V \setminus S)$ and let $\tilde{y} = \chi^{\delta(S)}$. First we prove that no points $h(S)$ with $r \in S$ can lie on such an inequality. Indeed, if $r \in S$ then $\tilde{y}_0 + \sum_{v \in V \setminus \{r\}} \gamma_v \tilde{y}_{rv} = a(S) + \sum_{v \in V \setminus S} \gamma_v \leq a(V) < \gamma_0$, where the $\leq$-inequality is due to $\gamma_v \leq a_v$ by Lemma 42. Therefore, $\left(\tilde{y}_0, \tilde{y}^T\right)^T$ cannot lie on the facet. For points $h(S)$ with $r \in V \setminus S$ lying on the inequality we show that they also satisfy (37) with equality. Indeed, let $\left(\tilde{y}_0, \tilde{y}^T\right)^T$ as defined above satisfy $\tilde{y}_0 + \sum_{v \in V \setminus \{r\}} \gamma_v \tilde{y}_{rv} = \gamma_0$. Since $\tilde{y}_0 = \sum_{v \in V \setminus \{r\}} a_v \tilde{y}_{rv}$ we obtain $\gamma_0 = \sum_{v \in V \setminus \{r\}} (a_v + \gamma_v) \tilde{y}_{rv}$. Thus, we have proved that all points of P_{CW} which lie on (33) also fulfill another valid inequality, which is not a scalar multiple of (33), with equality. Therefore, (33) cannot be a facet of P_{CW}. □

Proposition 44. *For a star $G = (V, E)$ with root $r \in V$, $a \neq 0^{|E|}$ and $a(V) \leq a_0$ all facets of the form (33) for P_{CW} are defined by (28) if $a(V \setminus \{r\}) > a_r$ and (30) if $a(V \setminus \{r\}) \leq a_r$.*

Proof. We have shown in Lemma 42 that each coefficient γ_v for all $v \in V \setminus \{r\}$ in all facets of P_{CW} of the form (33) fulfills

$$-a_v \leq \gamma_v \leq a_v \; . \tag{38}$$

Lemma 43 tells us that for each individual facet of P_{CW} of the form (33) the coefficients fulfill

$$\sum_{v \in V \setminus \{r\}} \gamma_v \leq a_r \tag{39}$$

and

$$\gamma_0 = a(V) \ . \tag{40}$$

For any given $y \in [0,1]^{|E|}$ we will now determine the best γ_0 and γ subject to the constraints (38), (39) and (40) so that $y_0 \leq \gamma_0 - \sum_{v\in V\setminus\{r\}} \gamma_v y_{rv}$ is as small as possible. If we can always exhibit an optimal solution γ_0^*, γ^* that corresponds to the coefficients of (28) if $a(V \setminus \{r\}) > a_r$ or (30) if $a(V \setminus \{r\}) \leq a_r$ then the proof is complete. At first note that (40) directly fixes γ_0 to $a(V)$ which corresponds to the right-hand sides of (28) and (30). Now look at the problem

$$\begin{aligned} \min \quad & a(V) - \sum_{v\in V\setminus\{r\}} y_{rv}\gamma_v \\ \text{s.t.} \quad & \sum_{v\in V\setminus\{r\}} \gamma_v \leq a_r \\ & -a_v \leq \gamma_v \leq a_v \ \forall v \in V \setminus \{r\} \ . \end{aligned} \tag{41}$$

Noting that a maximal y_0 will always equal $\gamma_0 - \sum_{v\in V\setminus\{r\}} \gamma_v y_{rv}$ and using the variable transformation $\tilde{\gamma}_v = \gamma_v + a_v$ we see that problem (41) is equivalent to

$$\begin{aligned} \max \quad & \sum_{v\in V\setminus\{r\}} y_{rv}\tilde{\gamma}_v - \sum_{v\in V\setminus\{r\}} y_{rv} a_v \\ \text{s.t.} \quad & \sum_{v\in V\setminus\{r\}} \tilde{\gamma}_v \leq a(V) \\ & 0 \leq \tilde{\gamma}_v \leq 2a_v \ \forall v \in V \setminus \{r\} \ . \end{aligned} \tag{42}$$

We recognize (42) as the continuous bounded knapsack problem (see Sections 3.2 and 3.3.1 in [13]) with continuous variables γ_v, profits y_{rv}, weights 1 and upper bound $2a_v$ for all items $v \in V \setminus \{r\}$ and knapsack capacity $a(V)$. An optimal solution can be found by sorting the items v with respect to non-increasing profit-to-weight ratios $y_{rv}/1$, w.l.o.g. let this ordering be $1, 2, \ldots, |V|-1$, and using this ordering to pack the knapsack in the following way: $\tilde{\gamma}_v = 2a_v$ for all $v = 1, \ldots, \bar{v}-1$ with $2a(\{1, \ldots, \bar{v}-1\}) \leq a(V)$ and $2a(\{1, \ldots, \bar{v}-1\}) + 2a_{\bar{v}} > a(V)$, $\tilde{\gamma}_{\bar{v}} = a(V) - 2a(\{1, \ldots, \bar{v}-1\})$, and $\tilde{\gamma}_v = 0$ for all $v = \bar{v}+1, \ldots, |V|-1$. The item $\bar{v}$ is called the critical item. Note that if one $\bar{v}$ can be chosen as the critical item then so can all $v \neq \bar{v}$ with $y_{rv} = y_{r\bar{v}}$.

Now we can substitute again $\tilde{\gamma}_v = \gamma_v + a_v$ and obtain the optimal solution of problem (41): $\gamma_v = a_v$ for all $v = 1, \ldots, \bar{v}-1$ with $a(\{1, \ldots, \bar{v}-1\}) \leq \frac{1}{2}a(V)$ and $a(\{1, \ldots, \bar{v}-1\}) + a_{\bar{v}} > \frac{1}{2}a(V)$, $\gamma_{\bar{v}} = a(V) - 2a(\{1, \ldots, \bar{v}-1\}) - a_{\bar{v}}$, and $\gamma_v = -a_v$ for all $v = \bar{v}+1, \ldots, |V|-1$. Finally we observe that we have determined a feasible triple $(V_p = \{1, \ldots, \bar{v}-1\}, \bar{v}, V_n = \{\bar{v}+1, \ldots, |V|-1\})$, i.e., we have found an inequality of (28) if $a(V \setminus \{r\}) > a_r$, because in this case the capacity restriction $\sum_{v\in V\setminus\{r\}} \gamma_v \leq a_r$ of (41) is a bottleneck, i.e., there must be a critical item $\bar{v} \in V \setminus \{r\}$. In case $a(V \setminus \{r\}) \leq a_r$ there is no critical item $\bar{v} \in V \setminus \{r\}$, i.e., all items can be packed with their full availability of a_v into the knapsack (41), thus $\gamma_v = a_v$ for all $v \in V \setminus \{r\}$ and we have determined inequality (30). □

Proposition 45. *For a star $G = (V, E)$ with root $r \in V$, $a \neq 0^{|E|}$ and $a(V) \leq a_0$ all facets of the form (35) for P_{CW} are defined by (29) if $a(V \setminus \{r\}) > a_r$ and (31) if $a(V \setminus \{r\}) \leq a_r$.*

Proof. Use the symmetry of P_{CW}, of pairs (33) and (35) with the same γ_v and γ_0, of pairs (28) and (29) and of pairs (30) and (31) to the hyperplane $\{\, y \in \mathbb{R}^{|E|} : 2y_0 = a(V) \,\}$ and apply Proposition 44. □

Proposition 46. *For a star $G = (V, E)$ with root $r \in V$, $a \neq 0^{|E|}$ and $a(V) \leq a_0$ all facets of the form (34) for P_{CW} are defined by (27).*

Proof. It is trivial to show that facets of a polytope with coefficient zero for a fixed variable are also facets of the projection of this polytope if one projects out this variable. Since the hyperplanes defined by inequalities of the form (34) have coefficient zero for variable y_0 we have to look at the projection of P_{CW} onto the space $\mathbb{R}^{|E|}$ and have to show that this projection only has facets of the form (27). A point $\big(a(S), (\chi^{\delta(S)})^T\big)^T \in \mathbb{R}^{|E|+1}$ used to define P_{CW} is projected to $\chi^{\delta(S)} \in \mathbb{R}^{|E|}$, and since $a(V) \leq a_0$ the polytope P_{CW} contains the points $\big(a(S), (\chi^{\delta(S)})^T\big)^T \in \mathbb{R}^{|E|+1}$ for all $S \subseteq V$, thus its projection contains all possible points $\{0,1\}^{|E|}$. Furthermore, the projection of any other point of P_{CW} can be written as the convex combination of points $\{0,1\}^{|E|}$. Thus the projection of P_{CW} is exactly the $|E|$-dimensional hypercube. To finish the proof we note that the $|E|$-dimensional hypercube is completely described by the projection of the inequalities (27). $\square$

Theorem 47. *For a star $G = (V, E)$ with root $r \in V$, $a \neq 0^{|E|}$ and $a(V) \leq a_0$ we have*

$$P_{\mathrm{CW}} = \{\, y \in \mathbb{R}^{|E|+1} : y \ \textit{ fulfills } \ (27), (28) \ \textit{and} \ (29) \,\} =: Y, \ \ \textit{if} \ a(V \setminus \{r\}) > a_r, \ \textit{and}$$
$$P_{\mathrm{CW}} = \{\, y \in \mathbb{R}^{|E|+1} : y \ \textit{ fulfills } \ (27), (30) \ \textit{and} \ (31) \,\} =: Y^r, \ \ \textit{if} \ a(V \setminus \{r\}) \leq a_r.$$

Proof. If $a(V \setminus \{r\}) > a_r$ Propositions 39 and 40 show that $Y \supseteq P_{\mathrm{CW}}$, and to show $Y \subseteq P_{\mathrm{CW}}$ we can use Propositions 44, 45 and 46. If $a(V \setminus \{r\}) \leq a_r$ Propositions 39 and 41 show that $Y^r \supseteq P_{\mathrm{CW}}$ and to prove $Y^r \subseteq P_{\mathrm{CW}}$ we can use again Propositions 44, 45 and 46. $\square$

Remark 48. *Note that in all assertions of this section we have assumed $a(V) \leq a_0$. This assumption guarantees that every $S \subseteq V$ contributes its point $h(S)$ to P_{CW}. If we reduce a_0 below $a(V)$ the facial structure of P_{CW} becomes much more complicated, because suddenly the whole complexity of the knapsack polytope P_{K} comes into play. So far a complete description of P_{CW} with $a(V) > a_0$ seems out of reach, even if we assume $a_v = 1$ for all $v \in V$.*

Example 49. *We continue Example 33. For the choice of the subgraphs $\bar{G}_l$ compare Figure 11.*

(2) The bisection knapsack walk inequality on $V' = \{1,2,3\}$ with root node $r = 3$ and $H_v = \emptyset$ for all $v \in V'$ is $1 + (1 - y_{13}) + (1 - y_{23}) \leq 4$. With $\bar{G}_1$ and $\bar{G}_2$ such that $\bar{V}_1 = \{4,5\}$, $\bar{V}_2 = \{6,7\}$, $\bar{E}_1 = \{45\}$ and $\bar{E}_2 = \{67\}$ the capacity reduced bisection knapsack walk inequality reads $1 + (1 - y_{13}) + (1 - y_{23}) \leq 4 - y_{45} - y_{67}$ and is a facet of P_{B}.

(3) For $V' = \{1,2,3,4\}$, $r = 3$ and $H_v = \emptyset$ for all $v \in V'$ the bisection knapsack walk inequality is $1 + (1 - y_{13}) + (1 - y_{23}) + (1 - y_{34}) \leq 4$. Proposition 40 establishes that for $\bar{G}$ with $\bar{V} = \{5,6,7,8\}$ and $\bar{E} = \{56, 67, 68\}$ one of the best minorizing functions for $\check{\beta}_{\bar{G}}$ is $y_{56} + y_{67} - y_{68}$. Thus the resulting capacity reduced bisection knapsack walk inequality reads $1 + (1 - y_{13}) + (1 - y_{23}) + (1 - y_{34}) \leq 4 - y_{56} - y_{67} + y_{68}$. It is a facet of P_{B}.

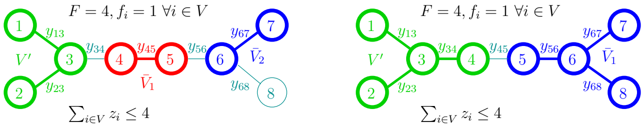

Figure 11: Graphs for Ex. 49

References

[1] Armbruster, M. (2007): *tba*, PhD-Thesis, Chemnitz University of Technology, Chemnitz 2007.

[2] Barahona, F. (1983): *The max-cut problem in graphs not contractible to* K_5, Operations Research letters, 2, pp. 107-111.

[3] Barahona, F., Mahjoub, A.R. (1986): *On the cut polytope*, Mathematical Programming, 36, pp. 157-173.

[4] Conforti, M., Rao, M.R., Sassano, A. (1990): *The equipartition polytope I*, Mathematical Programming, 49, pp. 49-70.

[5] Conforti, M., Rao, M.R., Sassano, A. (1990): *The equipartition polytope II*, Mathematical Programming, 49, pp. 71-90.

[6] de Souza, C.C. (1993): *The graph equipartition problem: Optimal solutions, extensions and applications*, PhD-Thesis, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

[7] Deza, M., Laurent, M. (1997): *Geometry of Cuts and Metrics*, volume 15 of Algorithms and Combinatorics, Springer.

[8] Eisenblätter, A. (2001): *Frequency Assignment in GSM Networks*, PhD-Thesis, Technische Universität Berlin, Berlin 2001.

[9] Ferreira, C.E., Martin, A., de Souza, C.C., Weismantel, R., Wolsey, L.A. (1996): *Formulations and valid inequalities for the node capacitated graph partitioning problem*, Mathematical Programming, 74, pp. 247-266.

[10] Fügenschuh, M. (2007): *tba*, PhD-Thesis, Darmstadt University of Technology, Darmstadt 2007.

[11] Garey, M. R., Johnson D.S. (1979): *Computers and Intractability*, W.H. Freeman and Company, New York.

[12] Gilbert, J.R., Tarjan R.E. (1987): *The analysis of a nested dissection algorithm*, Numer. Math., 50, pp. 377-404.

[13] Martello, S., Toth, P. (1990): *Knapsack Problems – Algorithms and Computer Implementations*, John Wiley and Sons Ltd, Chichester.

[14] Laurent, M., de Souza, C.C. (1995): *Some new classes of facets for the equicut polytope*, Discrete Applied Mathematics, 62, pp. 167-191.

[15] Johnson E., Mehrotra, A., Nemhauser, G. (1993): *Min-cut clustering*, Mathematical Programming, 62, pp. 133-152.

[16] Lengauer, T. (1990): *Combinatorial Algorithms for Integrated Circuit Layout*, John Wiley and Sons Ltd, Chichester.

[17] Schrijver, A. (2003): *Combinatorial Optimization – Polyhedra and Efficiency*, Springer-Verlag, Berlin.

[18] Weismantel, R. (1997): *On the 0/1 Knapsack polytope*, Mathematical Programming, 77, pp. 49-68.