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R. I. Boţ, N. Lorenz, G. Wanka

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Some formulas for the conjugate of convex risk measures

Radu Ioan Boț [*] Nicole Lorenz [†] Gert Wanka [‡]

Abstract. The aim of this paper is to give formulas for the conjugate functions of different convex risk measures. To this end we use, on the one hand, some classical results from convex analysis and, on the other hand, some tools from the conjugate duality theory. The characterizations of the so-called deviation measures recently given in the literature (see [8]) follow immediately from our results as natural consequences.

Keywords. conjugate functions, conjugate duality, convex risk measures, convex deviation measures.

1 Introduction

In many practical applications like those which appear in the portfolio optimization, the notion of "risk" plays an important role. It reflects the uncertainty of some processes and one challenge consists in quantifying it by an appropriate measure. Until now in the literature different formulations for such a so-called *risk measure* have been made. The classical application in financial mathematics is the portfolio optimization problem treated by Markowitz (cf. [7]), where the risk of a portfolio was measured by means of the standard deviation and variance, respectively.

In 1998 Artzner et al. (cf. [1]) first gave an axiomatic definition of what they called *coherent risk measure*. The properties in the definition of this class of measures seem to be common in many practical problems. In 2002 Rockafellar et al. (cf. [10]) introduced a new class of measures closely related to the coherent risk measures, called *deviation measures*. An important representative of this class of measures is the variance. It is remarkable that the coherent risk measures as well

[*]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de.

[†]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: nicole.lorenz@mathematik.tu-chemnitz.de.

[‡]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: gert.wanka@mathematik.tu-chemnitz.de

as the deviation measures fulfill some positive homogeneity and subadditivity properties. As many risk measures used in practice do not fulfill these properties, the class of coherent risk measures has been extended to the class of convex risk measures (see for example the paper of Föllmer and Schied, [5]), in which definition the sublinearity was replaced by convexity. Recent papers, where some theoretical results concerning convex risk measures have been given, are the works of Pflug (cf. [8]) and Ruszczynski and Shapiro (cf. [12]).

In [12] some necessary and sufficient conditions for the optimal solutions of optimization problems with convex risk measures as objective functions are given, whereas in [8] the author gives some dual representations for a number of convex risk and deviation measures with practical relevance.

In this paper we consider different convex risk and deviation measures, defined in analogy to Pflug's paper, and calculate their conjugate functions. To this end we use the powerful theory of conjugate functions from convex analysis as well as some duality results for convex optimization problems in locally convex spaces.

By using the Fenchel–Moreau theorem we succeed to give a dual representation for all measures we deal with. In this way we extend and improve the results obtained by Pflug ([8]).

The paper is organized as follows. In the next section we introduce some notations and preliminary results coming from convex analysis as well as from the theory of stochastics. Further, in Section 3, we introduce the notion of a convex risk measure and, closely connected with it, that of a convex deviation measures. Then we give some examples for these two classes of measures. Section 4 is devoted to the calculation of the conjugates of some classical convex risk and deviation measures. In Section 5 we deal with some elaborated convex risk and deviation measures and we calculate their conjugates by using the general formula of the conjugate of a composed convex function. Finally, in the last section, we derive some dual representations for the convex risk and deviation measures considered in the previous two sections and compare our results with the ones given by Pflug in [8].

2 Notations and preliminary results

In this section we introduce some notations and preliminary results used later in the paper.

Let $\mathcal{Z}$ be a nontrivial locally convex space and $\mathcal{Z}^*$ its topological dual space endowed with the weak* topology. We denote by $\langle x^*, x\rangle := x^*(x)$ the value of the linear continuous functional $x^* \in \mathcal{Z}^*$ at $x \in \mathcal{Z}$.

For a set $D \subseteq \mathcal{Z}$ we denote by $\mathrm{cl}(D)$ the *closure* of D, by $\mathrm{int}(D)$ its *interior* and by $\mathrm{core}(D) = \{d \in D : \forall x \in \mathcal{Z}\ \exists \varepsilon > 0 : \forall \lambda \in [-\varepsilon, \varepsilon]\ d + \lambda x \in D\}$ its *algebraic interior*. The *indicator function* $\delta_D : \mathcal{Z} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ of the set D is defined by

$$\delta_D(x) = \begin{cases} 0, & x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

By taking a function $f : \mathcal{Z} \to \overline{\mathbb{R}}$ we consider the *(Fenchel-Moreau) conjugate function of* f, $f^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in \mathcal{Z}} \{\langle x^*, x\rangle - f(x)\}.$$

Similarly, the *biconjugate function of* f, $f^{**} : \mathcal{Z} \to \overline{\mathbb{R}}$ is defined by

$$f^{**}(x) = \sup_{x^* \in \mathcal{Z}^*} \{\langle x^*, x\rangle - f^*(x^*)\}.$$

Further, for the function $f : \mathcal{Z} \to \overline{\mathbb{R}}$ we consider also its *epigraph* $\text{epi}(f) = \{(x, r) : x \in \mathcal{Z}, r \in \mathbb{R} : \ f(x) \leq r\}$ and its *effective domain* $\text{dom}(f) = \{x \in \mathcal{Z} : f(x) < +\infty\}$. We say that f is *proper* if $\text{dom}(f) \neq \emptyset$ and $f(x) > -\infty$, $\forall x \in \mathcal{Z}$. We can state now a very important result coming from convex analysis:

Theorem 2.1. (Fenchel-Moreau) *Let $f : \mathcal{Z} \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. Then it holds $f = f^{**}$.*

The following theorem gives a sufficient condition for the formula of the conjugate of the precomposition of a convex function with a linear continuous mapping (see [9]). In what follows $\mathcal{U}$ is another nontrivial locally convex space.
Let us mention that all around this paper we write min (max) instead of inf (sup) when the infimum (supremum) is attained.

Theorem 2.2. *Let $f : \mathcal{Z} \to \overline{\mathbb{R}}$ be a proper and convex function and $A : \mathcal{U} \to \mathcal{Z}$ a linear continuous mapping. Assume that there exists $x' \in A^{-1}(\text{dom}(f))$ such that f is continuous at Ax'. Then*

$$(f \circ A)^*(u^*) = \min\{f^*(z^*) : \ A^* z^* = u^*\}, \quad \forall u^* \in \mathcal{U}^*.$$

The next result we recall in this section deals with Lagrange duality for the optimization problem with cone constraints

$$(P) \qquad \inf_{\substack{x \in X, \\ g(x) \in -K}} f(x),$$

where $X \subseteq \mathcal{Z}$ is a non-empty convex set, $K \subseteq \mathcal{U}$ is a closed, convex cone, $f : \mathcal{Z} \to \mathbb{R}$ is a convex, continuous function and $g : \mathcal{Z} \to \mathcal{U}$ is a K-convex ($g(\mathcal{Z}) + K$ is convex), continuous function. For having strong duality between (P) and the Lagrange dual problem

$$(D_L) \quad \sup_{\lambda \in K^*} \inf_{x \in X} \{f(x) + \langle \lambda, g(x)\rangle\},$$

some weak closedness type constraint qualifications have been recently introduced in the literature (cf. [3], [4], [6]). In the definition of (D_L), by $K^* = \{\lambda \in \mathcal{U}^* : \langle \lambda, u \rangle \geq 0, \ \ \forall u \in K\}$ we denote the dual cone of K. Let further $\langle \lambda, g \rangle : \mathcal{Z} \to \mathbb{R}$ be the function defined by $\langle \lambda, g \rangle(x) := \langle \lambda, g(x) \rangle, \ \ \forall x \in X$.
In order to formulate the strong duality between (P) and (D_L) let us introduce first the following so-called closed cone constraint qualification (see [4], [6]):

$$(CCCQ) \qquad \bigcup_{\lambda \in K^*} \operatorname{epi}((\langle \lambda, g \rangle + \delta_X)^*) \quad \text{is a weak}^* \text{ closed set.}$$

For the optimization problem (P) we denote by $v(P)$ its optimal objective value.

Theorem 2.3. ([4], [6]) *Under the assumptions made above for (P), if $(CCCQ)$ is fulfilled, then $v(P) = v(D_L)$ and the Lagrange dual has an optimal solution.*

Consider now the *probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is a basic space, $\mathfrak{F}$ a σ-algebra on Ω and $\mathbb{P}$ a probability measure on the measureable space $(\Omega, \mathfrak{F})$. For a measurable random variable $x : \Omega \to \mathbb{R}$ the expectation value is defined with respect to $\mathbb{P}$ by

$$\mathbb{E}(x) = \int_{\Omega} x(\omega) d\mathbb{P}(\omega).$$

The *essential supremum* of x is

$$\operatorname{essup} x = \inf\{a \in \mathbb{R} : \mathbb{P}(\omega : x(\omega) > a) = 0\}.$$

Furthermore for $p \in [1, +\infty)$ let L_p be the following linear space of random variables:

$$L_p := L_p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \left\{ x : \Omega \to \mathbb{R}, \ x \ \text{ measureable}, \int_{\Omega} |x(\omega)|^p d\mathbb{P}(\omega) < +\infty \right\}.$$

The space L_p equipped with the norm $||x||_p = (\mathbb{E}(|x|^p))^{\frac{1}{p}}$ for $x \in L_p$ is a Banach space. It is well-known that the dual space of L_p is $L_q := L_q(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R})$, where $q \in (1, +\infty]$ fulfills $\frac{1}{p} + \frac{1}{q} = 1$. The space

$$L_\infty := L_\infty(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \left\{ x : \Omega \to \mathbb{R}, \ x \ \text{ measureable}, \operatorname{essup} |x| < +\infty \right\}$$

is considered to be equipped with the essential supremum norm.
In order to make these spaces paired we consider on L_p the norm topology and on L_q the weak* topology. If $p \in (1, +\infty)$ then L_p and L_q are reflexive Banach spaces and they are paired spaces equipped with the norm topologies. The closed unit ball in L_q is denoted by $B_q(0, 1)$.

For $x \in L_p$, $x^* \in L_q$ and $x^*x : \Omega \to \mathbb{R}$, defined by $(x^*x)(\omega) := x^*(\omega) \cdot x(\omega)$, one can define now

$$\langle x^*, x\rangle := \mathbb{E}(x^*x) = \int_\Omega x^*(\omega)x(\omega)d\mathbb{P}(\omega).$$

Equalities and inequalities between random variables are to be viewed in the sense of holding almost surely (a.s.). Thus for $x, y : \Omega \to \mathbb{R}$ when we write "$x = y$" or "$x \geq y$" we mean "$x = y\ a.s.$" or "$x \geq y\ a.s.$", respectively. For $p \in [1, +\infty)$, the cone $(L_p)_+ = \{x \in L_p :\ x \geq 0\ a.s.\}$ is inducing the partial ordering denoted by "$\geq$". The dual cone of $(L_p)_+$ is $(L_q)_+$, where $q \in (1, +\infty]$ fulfills $\frac{1}{p} + \frac{1}{q} = 1$. The partial ordering induced by $(L_q)_+$ is also denoted by "$\geq$". As these orderings are given in different linear spaces, no confusion is possible.
Having a random variable $x : \Omega \to \mathbb{R}$ which takes the constant value $c \in \mathbb{R}$, i.e. $x = c\ a.s.$, we identify it with the real number $c \in \mathbb{R}$.
For an arbitrary random variable $x : \Omega \to \mathbb{R}$ we also define $x_-, x_+ : \Omega \to \mathbb{R}$ in the following way:

$$x_-(\omega) := \max(-x(\omega), 0) \qquad \forall \omega \in \Omega$$
$$\text{and} \qquad x_+(\omega) := \max(x(\omega), 0) \qquad \forall \omega \in \Omega.$$

One can easily see that $x = x_+ - x_-$, $x_+ = (-x)_-$ and $x_- = (-x)_+$.

3 Risk measures and deviation measures

In this section we give some formal definitions of convex risk and deviation measures. In 2002 Föllmer and Schied (cf. [5]) first introduced the convex risk measures as an extension of the well-known coherent risk measures. The latter have been introduced in [1], where for the first time an axiomatic way for defining risk measures has been given. Rockafellar and his coauthors (see [10]) introduced along the coherent risk measures the so-called deviation measures and studied the relation between these concepts.
In this paper we deal with the broad class of convex risk measures as done by Ruszczynski and Shapiro (cf. [12]) and Pflug (see [8]), respectively. We want to notice that a large number of risk functions mentioned in the literatur do not have the sublinearity properties asked by the axioms of a coherent risk measure, however they fulfill the properties in the definition of a convex risk measure. In the following definition we introduce the notion of a *convex risk measure* as done in [8].

Definition 3.1. *The function $\rho : L_p \to \overline{\mathbb{R}}$ is called a convex risk measure if the following properties are fulfilled:*

(R1) Translation equivariance: $\rho(x + b) = \rho(x) - b, \quad \forall x \in L_p,\ \forall b \in \mathbb{R}$;
(R2) Strictness: $\rho(x) \geq -\mathbb{E}(x), \quad \forall x \in L_p$;
(R3) Convexity: $\rho(\lambda x+(1-\lambda)y) \leq \lambda\rho(x)+(1-\lambda)\rho(y), \quad \forall \lambda \in [0, 1], \forall x, y \in L_p$.

For certain applications it can be usefull to postulate some monotonicity properties of the risk measure, like, for example, the monotonicity with respect to the pointwise ordering:

$$x \geq y \quad \Rightarrow \quad \rho(x) \geq \rho(y), \qquad \forall x, y \in L_p.$$

Closely related to the risk measure we can define the so-called *convex deviation measure*.

Definition 3.2. *The function $d : L_p \to \overline{\mathbb{R}}$ is called a convex deviation measure if the following properties are fulfilled:*
(R1) Translation invariance: $d(x+b) = d(x), \quad \forall x \in L_p, \ \forall b \in \mathbb{R}$*;*
(R2) Strictness: $d(x) \geq 0, \quad \forall x \in L_p$*;*
(R3) Convexity: $d(\lambda x + (1-\lambda)y) \leq \lambda d(x) + (1-\lambda) d(y), \quad \forall \lambda \in [0,1], \forall x, y \in L_p.$

The following theorem states the connection between convex risk and convex deviation measures (see [8],[10], [11]).

Theorem 3.1. *The function $\rho : L_p \to \overline{\mathbb{R}}$ is a convex risk measure if and only if $d : L_p \to \overline{\mathbb{R}}$, $d(x) = \rho(x) + \mathbb{E}(x)$, $\forall x \in L_p$, is a convex deviation measure.*

Next we give some examples for convex risk measures and for the corresponding deviation measures.

Example 3.1.
First we consider, for $p = 2$, $\rho : L_2 \to \mathbb{R}$ defined by

$$\rho(x) = ||x - \mathbb{E}(x)||_2^2 - \mathbb{E}(x), \quad x \in L_2.$$

This is a convex risk measure and it is closely related to the classical *variance* $\sigma^2(x)$ which is the corresponding deviation measure:

$$d(x) = \sigma^2(x) = ||x - \mathbb{E}(x)||_2^2, \quad x \in L_2.$$

Example 3.2.
Let be again $p = 2$ and $\rho : L_2 \to \mathbb{R}$ defined by

$$\rho(x) = ||x - \mathbb{E}(x)||_2 - \mathbb{E}(x), \quad x \in L_2.$$

The related convex deviation measure is the *standard deviation* $\sigma(x)$

$$d(x) = \sigma(x) = ||x - \mathbb{E}(x)||_2, \quad x \in L_2.$$

The convex risk and deviation measures in the previous examples are special cases of some general classes of risk and deviation measures, respectively, which are described in the following.

Example 3.3.
For $p \in [1, +\infty)$ and $a \geq 1$ we define the convex risk measure $\rho : L_p \to \mathbb{R}$,

$$\rho(x) = ||x - \mathbb{E}(x)||_p^a - \mathbb{E}(x), \ \ x \in L_p.$$

The corresponding convex deviation measure is $d : L_p \to \mathbb{R}$,

$$d(x) = ||x - \mathbb{E}(x)||_p^a, \ \ x \in L_p.$$

In case $p = a = 1$, d is called *mean absolute deviation*.

Example 3.4.
Similar to Example 3.3, for $p \in [1, +\infty)$ and $a \geq 1$ we consider the following pairs of convex risk and deviation measures, $\rho : L_p \to \mathbb{R}$ and $d : L_p \to \mathbb{R}$ defined by

$$\rho(x) = ||(x - \mathbb{E}(x))_-||_p^a - \mathbb{E}(x), \quad d(x) = ||(x - \mathbb{E}(x))_-||_p^a$$

and

$$\rho(x) = ||(x - \mathbb{E}(x))_+||_p^a - \mathbb{E}(x), \quad d(x) = ||(x - \mathbb{E}(x))_+||_p^a,$$

respectively.
The deviation measures we get by taking $a = p = 1$ are the so-called *lower* and *upper semideviation*, respectively. For $p = 2$ and $a = 1$ we get the *standard lower* and *upper semideviation*, respectively.

4 Conjugates of convex deviation measures: the case a=1

In this section we deal with the formulas of the conjugate functions of some convex deviation measures, including those in Example 3.3 and 3.4, for $p \in [1, +\infty)$ and $a = 1$. Having these formulas we can easily calculate the formulas of the conjugate functions of the corresponding risk measures. The following relation proves this, as for $x^* \in L_q$ it holds

$$\begin{aligned} \rho^*(x^*) &= \sup_{x \in L_p} \{\langle x^*, x\rangle - \rho(x)\} = \sup_{x \in L_p} \{\langle x^*, x\rangle - d(x) + \mathbb{E}(x)\} \\ &= \sup_{x \in L_p} \{\langle x^*, x\rangle - d(x) + \langle 1, x\rangle\} = \sup_{x \in L_p} \{\langle x^* + 1, x\rangle - d(x)\} = d^*(x^* + 1). \end{aligned} \tag{1}$$

In order to derive the formulas for the conjugates of the convex deviation measures we need the following preliminary results.

Example 4.1.

Let be $f_1 : L_p \to \mathbb{R},\ f_1(x) = ||x||_p$. It is well-known that the conjugate function of f_1 is $f_1^* : L_q \to \overline{\mathbb{R}}$,

$$f_1^*(x^*) = \begin{cases} 0, & ||x^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Example 4.2.

We consider now $f_2 : L_p \to \mathbb{R},\ f_2(x) = ||x_-||_p$. For $x^* \in L_q$ one obtains the following formula for the conjugate function of f_2, $f_2^* : L_q \to \overline{\mathbb{R}}$:

$$\begin{aligned} -f_2^*(x^*) &= \inf_{x \in L_p} \{||x_-||_p - \langle x^*, x\rangle\} \\ &= \inf_{x \in L_p} \{|| \max(-x, 0)||_p - \langle x^*, x\rangle\}. \end{aligned}$$

Having for an arbitrary $z \in L_p$ with the property $z \geq \max(-x, 0) \geq 0$ that $||z||_p \geq || \max(-x, 0)||_p$, one gets further

$$-f_2^*(x^*) = \inf_{\substack{x \in L_p, z \in L_p, \\ z \geq \max(-x,0)}} \{||z||_p - \langle x^*, x\rangle\} = \inf_{\substack{(x,z) \in L_p \times L_p, \\ -x-z \leq 0, \\ -z \leq 0}} \{||z||_p - \langle x^*, x\rangle\}.$$

Let (P) be the following convex optimization problem:

$$(P) \qquad \inf_{\substack{(x,z) \in L_p \times L_p, \\ -x-z \leq 0, \\ -z \leq 0}} \{||z||_p - \langle x^*, x\rangle\}.$$

The Lagrange dual problem of (P) looks like

$$\begin{aligned} (D_L) \qquad & \sup_{\lambda_1, \lambda_2 \in (L_q)_+} \inf_{(x,z) \in L_p \times L_p} \{||z||_p - \langle x^*, x\rangle - \langle \lambda_1, z\rangle - \langle \lambda_2, x + z\rangle\} \\ &= \sup_{\lambda_1, \lambda_2 \in (L_q)_+} \left\{ \inf_{x \in L_p} \{-\langle x^*, x\rangle - \langle \lambda_2, x\rangle\} + \inf_{z \in L_p} \{||z||_p - \langle \lambda_1, z\rangle - \langle \lambda_2, z\rangle\} \right\} \\ &= \sup_{\lambda_1, \lambda_2 \in (L_q)_+} \left\{ \inf_{x \in L_p} \{-\langle x^* + \lambda_2, x\rangle\} + \inf_{z \in L_p} \{||z||_p - \langle \lambda_1 + \lambda_2, z\rangle\} \right\} \\ &= \sup_{\lambda_1, \lambda_2 \in (L_q)_+} \left\{ -\delta_{L_p}^*(x^* + \lambda_2) - (|| \cdot ||_p)^*(\lambda_1 + \lambda_2) \right\}. \end{aligned}$$

Since

$$\delta_{L_p}^*(x^* + \lambda_2) = \begin{cases} 0, & x^* = -\lambda_2, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(|| \cdot ||_p)^*(\lambda_1 + \lambda_2) = \begin{cases} 0, & ||\lambda_1 + \lambda_2||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

the optimal objective value of the Lagrange dual (D_L) can be written as

$$
\begin{aligned}
v(D_L) &= \begin{cases} 0, & \exists \lambda_1 \in (L_q)_+ : \quad ||\lambda_1 - x^*||_q \leq 1 \quad \text{and} \quad x^* \in -(L_q)_+, \\ -\infty, & \text{otherwise,} \end{cases} \\
&= \begin{cases} 0, & x^* \in \Big(B_q(0,1) + (L_q)_+ \Big) \cap -(L_q)_+, \\ -\infty, & \text{otherwise,} \end{cases} \\
&= \begin{cases} 0, & x^* \in B_q(0,1) \cap -(L_q)_+, \\ -\infty, & \text{otherwise.} \end{cases}
\end{aligned}
$$

The equivalence of the last two formulas comes from the equality of the sets $\Big(B_q(0,1) + (L_q)_+ \Big) \cap -(L_q)_+$ and $B_q(0,1) \cap -(L_q)_+$. As the inclusion $B_q(0,1) \cap -(L_q)_+ \subseteq \Big(B_q(0,1) + (L_q)_+ \Big) \cap -(L_q)_+$ is trivial, we prove the opposite one.

Let be $x^* \in \Big(B_q(0,1) + (L_q)_+ \Big) \cap -(L_q)_+$. Then $x^* = t^* + z^* \leq 0$, where $t^* \in B_q(0,1)$ and $z^* \in (L_q)_+$. Since $-x^* \geq 0$ and $-z^* \leq 0$ we have $0 \leq -x^* = -t^* - z^* \leq -t^*$ and so $||x^*||_q = ||-x^*||_q \leq ||-t^*||_q = ||t^*||_q \leq 1$. Thus $x^* \in B_q(0,1) \cap -(L_q)_+$.

In order to identify $-f_2^*(x^*)$ with the optimal objective value of (D_L) we have to prove that between (P) and (D_L) strong duality holds. As (P) is a convex optimization problem one needs a constraint qualification for closing the gap between these duals. Let us notice that, since $\operatorname{int}((L_p)_+) = \operatorname{core}((L_p)_+) = \emptyset$, one cannot use the generalized interior-point constraint qualifications given in the literature for convex optimization problems with cone inequality constraints. But, what we prove now is that $(CCCQ)$ is fulfilled. Let be $g : L_p \times L_p \to \mathcal{Z}$, $g(x,z) = (-x-z, -z)$ and $\lambda = (\lambda_1, \lambda_2) \in (L_q)_+ \times (L_q)_+$. One has

$$
\begin{aligned}
(x^*, z^*, r) \in \operatorname{epi}((\langle \lambda, g \rangle + \delta_{L_p \times L_p})^*) = \operatorname{epi}(\langle \lambda, g \rangle^*) \\
\Leftrightarrow \qquad & \langle \lambda, g \rangle^*(x^*, z^*) \leq r \\
\Leftrightarrow \qquad & \sup_{x \in L_p, z \in L_p} \{ \langle x^*, x \rangle + \langle z^*, z \rangle - \langle \lambda_1, -x-z \rangle - \langle \lambda_2, -z \rangle \} \leq r \\
\Leftrightarrow \qquad & \sup_{x \in L_p} \{ \langle x^*, x \rangle - \langle \lambda_1, -x \rangle \} + \sup_{z \in L_p} \{ \langle z^*, z \rangle - \langle \lambda_1 + \lambda_2, -z \rangle \} \leq r \\
\Leftrightarrow \qquad & \sup_{x \in L_p} \{ \langle x^* + \lambda_1, x \rangle \} + \sup_{z \in L_p} \{ \langle z^* + \lambda_1 + \lambda_2, z \rangle \} \leq r \\
\Leftrightarrow \qquad & x^* = -\lambda_1, \quad z^* = -\lambda_1 - \lambda_2, \quad r \in [0, +\infty).
\end{aligned}
$$

In order to prove that $(CCCQ)$ is fulfilled we have to show that

$$
M := \bigcup_{\lambda_1, \lambda_2 \in (L_q)_+} \{-\lambda_1\} \times \{-\lambda_1 - \lambda_2\} \times [0, +\infty)
$$

is closed in $L_q \times L_q \times \mathbb{R}$ with respect to the product topology induced by the norm topology on L_q and the Euclidean topology on $\mathbb{R}$. One can notice that since M is convex this is the same with M is weak* closed.
Take $(x^*, z^*, r) \in \text{cl}(M)$ and $(x_n^*, z_n^*, r_n) \in M$ such that $(x_n^*, z_n^*, r_n) \to (x^*, z^*, r)$ $(n \to +\infty)$. Then there exists $\forall n \in \mathbb{N}$, $(\lambda_1^n), (\lambda_2^n) \in (L_q)_+$ and $r_n \geq 0$ such that $x_n^* = -\lambda_1^n$ and $z_n^* = -\lambda_1^n - \lambda_2^n$. It follows $-x^* \in (L_q)_+$, $x^* - z^* = \lim\limits_{n\to\infty}(-\lambda_1^n - z_n^*) = \lim\limits_{n\to\infty}(\lambda_2^n) \in (L_q)_+$ and $r \geq 0$. Thus $(x^*, z^*, r) = (-(-x^*), -(-x^*) - (x^* - z^*), r) \in M$ and the set M turns out to be closed.
By Theorem 2.3 it follows that

$$f_2^*(x^*) = -v(P) = -v(D_L) = \begin{cases} 0, & ||x^*||_q \leq 1,\ x^* \leq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

Finally we give an equivalent formulation since the domain of the conjugate f_2^* can be restricted to the set of those $x^* \in L_q$ which fulfill $x^* \leq 0$ and $-1 \leq \mathbb{E}(x^*) \leq 0$. (Note that $x^* \leq 0$ implies $\mathbb{E}(x^*) \leq 0$ and that $||x^*||_q \leq 1$ implies $|\mathbb{E}(x^*)| \leq 1$.) This leads to the following formula for the conjugate of f_2 (see also [8]):

$$f_2^*(x^*) = \begin{cases} 0, & x^* \leq 0,\ \ ||x^*||_q \leq 1,\ \ -1 \leq \mathbb{E}(x^*) \leq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

In the next example we deal with the conjugate function of the deviation measure $d_1(x) = ||x - \mathbb{E}(x)||_p$, which will be derived from a more general formula.

Example 4.3.
Consider $d_1 : L_p \to \mathbb{R},\ \ d_1(x) = ||x - \mathbb{E}(x)||_p$ and $A : L_p \to L_p,\ \ Ax = x - \mathbb{E}(x)$. Here we have to interpret $\mathbb{E}(x) \in \mathbb{R}$ as a (constant) element of L_p. Denoting for $C \in \mathfrak{F}$ by $\mathbb{1}_C : \Omega \to \mathbb{R}$ the *random indicator function*

$$\mathbb{1}_C(\omega) = \begin{cases} 1, & \omega \in C, \\ 0, & \text{otherwise,} \end{cases}$$

the linear continuous mapping A can be written as $Ax = x - \mathbb{E}(x)\mathbb{1}_\Omega$.
Having now $d_1(x) = ||Ax||_p,\ \ \forall x \in L_p$, in order to calculate d_1^*, we can use Theorem 2.2. Since $||\cdot||_p$ is continuous on L_p, the regularity condition is fulfilled and we have $\forall x^* \in L_q$:

$$\begin{aligned} d_1^*(x^*) &= \min\{(||\cdot||_p)^*(y^*) : \quad A^*y^* = x^*\} \\ &= \begin{cases} 0, & \exists y^* \in L_q : A^*y^* = x^* \text{ and } ||y^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

From the calculation above one can see that we need the adjoint operator of A. In the following we show that A is self-adjoint, i.e. $A = A^*$. For $x \in L_p$ and $x^* \in L_q$ it holds:

$$\langle x^*, Ax\rangle = \langle x^*, x - \mathbb{E}(x)\rangle = \langle x^*, x\rangle - \langle x^*, \mathbb{E}(x)\mathbb{1}_\Omega\rangle.$$

The second term can be written as follows (we apply here the Theorem of Fubini):

$$\langle x^*, \mathbb{E}(x)\mathbb{1}_\Omega\rangle = \int_\Omega x^*(\omega)\mathbb{E}(x)d\mathbb{P}(\omega) = \int_\Omega x^*(\omega)\left(\int_\Omega x(\tau)d\mathbb{P}(\tau)\right)d\mathbb{P}(\omega)$$
$$= \int_\Omega x(\tau)\left(\int_\Omega x^*(\omega)d\mathbb{P}(\omega)\right)d\mathbb{P}(\tau) = \int_\Omega x(\tau)\mathbb{E}(x^*)d\mathbb{P}(\tau) = \langle \mathbb{E}(x^*)\mathbb{1}_\Omega, x\rangle.$$

So we have $\langle x^*, Ax\rangle = \langle x^* - \mathbb{E}(x^*)\mathbb{1}_\Omega, x\rangle, \ \ \forall x^* \in L_q, x \in L_p$ and, in conclusion, $A^*x^* = x^* - \mathbb{E}(x^*)\mathbb{1}_\Omega = x^* - \mathbb{E}(x^*)$.
Thus the conjugate function of d_1 becomes $\forall x^* \in L_q$,

$$d_1^*(x^*) = \begin{cases} 0, & \exists y^* \in L_q : y^* - \mathbb{E}(y^*) = x^* \text{ and } ||y^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

We prove now that $\exists y^* \in L_q$ such that $y^* - \mathbb{E}(y^*) = x^*$ and $||y^*||_q \leq 1$ if and only if $\mathbb{E}(x^*) = 0$ and $\exists c \in \mathbb{R}$ such that $||x^* - c||_q \leq 1$. Let be an $y^* \in L_q$ fulfilling $y^* - \mathbb{E}(y^*) = x^*$ and $||y^*||_q \leq 1$. Then $\mathbb{E}(x^*) = \mathbb{E}(y^* - \mathbb{E}(y^*)) = 0$ and for $c := -\mathbb{E}(y^*)$ one has $||x^* - c||_q \leq 1$. On the other hand, assume that $\mathbb{E}(x^*) = 0$ and that $\exists c \in \mathbb{R}$ with the property $||x^* - c||_q \leq 1$. Defining $y^* := x^* - c \in L_q$ one has $||y^*||_q \leq 1$ and $y^* - \mathbb{E}(y^*) = x^*$.
So the conjugate of d_1 turns out to be $\forall x^* \in L_q$,

$$d_1^*(x^*) = \begin{cases} 0, & \mathbb{E}(x^*) = 0 \text{ and } \exists c \in \mathbb{R} : ||x^* - c||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} 0, & \mathbb{E}(x^*) = 0 \text{ and } \min\limits_{c\in\mathbb{R}} ||x^* - c||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Considering the convex risk measure $\rho_1 : L_p \to \mathbb{R}, \ \rho_1(x) = d_1(x) - \mathbb{E}(x) = ||x - \mathbb{E}(x)||_p - \mathbb{E}(x)$, by (1), one can easily deduce the formula for the conjugate of $\rho_1 : L_p \to \overline{\mathbb{R}}$. This looks like

$$\rho_1^*(x^*) = d_1^*(x^* + 1) = \begin{cases} 0, & \mathbb{E}(x^*) = -1 \text{ and } \min\limits_{c\in\mathbb{R}} ||x^* - c||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

In the last example we consider in this section, we calculate the conjugate function of the convex deviation measure know also as *lower semideviation*. After that, we derive the formula for the conjugate of the corresponding convex risk measure.

Example 4.4.
Let be $d_2 : L_p \to \mathbb{R}, \ \ d_2(x) = ||(x - \mathbb{E}(x))_-||_p$. Denoting again by $A : L_p \to L_p$ the linear continuous mapping defined by $Ax = x - \mathbb{E}(x)$, we have that $d_2 = f_2 \circ A$.

Since f_2 is a convex continuous function with real values, by Theorem 2.2, one has for all $x^* \in L_q$

$$d_2^*(x^*) = \min\{f_2^*(y^*) : \quad A^*y^* = x^*\},$$

which can be further written as (see (3))

$$d_2^*(x^*) \\ = \begin{cases} 0, & \exists y^* \in L_q : \ A^*y^* = x^*, \ y^* \leq 0, \ ||y^*||_q \leq 1, \ -1 \leq \mathbb{E}(y^*) \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $A^*y^* = y^* - \mathbb{E}(y^*)$ (see Example 4.3) we obtain $\forall x^* \in L_q$

$$d_2^*(x^*) \\ = \begin{cases} 0, & \exists y^* \in L_q : y^* - \mathbb{E}(y^*) = x^*, y^* \leq 0, ||y^*||_q \leq 1, -1 \leq \mathbb{E}(y^*) \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Like in Example 4.3 one can show that there exists $y^* \in L_q$ such that $y^* - \mathbb{E}(y^*) = x^*, \ y^* \leq 0, \ ||y^*||_q \leq 1$ and $-1 \leq \mathbb{E}(y^*) \leq 0$ if and only if $\mathbb{E}(x^*) = 0$ and there exists $c \in \mathbb{R}$ fulfilling $0 \leq c \leq 1, \ \ ||x^* - c||_q \leq 1$ and $x^* \leq c$. Thus

$$d_2^*(x^*) = \begin{cases} 0, & \mathbb{E}(x^*) = 0 \text{ and } \exists c \in \mathbb{R} : \ 0 \leq c \leq 1, \ ||x^* - c||_q \leq 1, \ \ x^* \leq c, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us prove now that for $x^* \in L_q$ the relations

$$\mathbb{E}(x^*) = 0 \text{ and } \exists c \in \mathbb{R} : \ 0 \leq c \leq 1, \ ||x^* - c||_q \leq 1, \ \ x^* \leq c \tag{5}$$

and

$$\mathbb{E}(x^*) = 0, \ \ x^* \leq 1, \ \ ||\operatorname{essup} x^* - x^*||_q \leq 1 \tag{6}$$

are equivalent. Assuming that (5) holds, one has $x^* \leq c \leq 1$. Further we have $\operatorname{essup} x^* \leq c$ and this means that $c - x^* \geq \operatorname{essup} x^* - x^* \geq 0$, implying $1 \geq ||c - x^*||_q \geq ||\operatorname{essup} x^* - x^*||_q$. Relation (6) is so proved.
On the other hand, if (6) holds, one can take $c = \operatorname{essup} x^*$. That $c \leq 1$ and $x^* \leq c$ is obvious. Assuming now that $c < 0$, this would mean that $\mathbb{E}(x^*) < 0$. In conclusion, relation (5) must also hold.
This lead us to the following formula for d_2^* $\forall x^* \in L_q$,

$$d_2^*(x^*) = \begin{cases} 0, & \mathbb{E}(x^*) = 0, \ x^* \leq 1, \ ||\operatorname{essup} x^* - x^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

As in the previous example, the formula of the conjugate function of the corresponding convex risk measure $\rho_2 : L_p \to \overline{\mathbb{R}}, \ \rho_2(x) = ||(x - \mathbb{E}(x))_-||_p - \mathbb{E}(x)$ can be also calculated. By (1) we have $\forall x^* \in L_q$,

$$\rho_2^*(x^*) = d_2^*(x^* + 1) = \begin{cases} 0, & \mathbb{E}(x^*) = -1, \ x^* \leq 0, \ ||\operatorname{essup} x^* - x^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 4.1.
One can notice that the formulas for the conjugates of f_2 and d_2 allow us to calculate the formulas for the conjugates of the functions $x \mapsto ||x_+||_p$ and $x \mapsto ||(x - \mathbb{E}(x))_+||_p$, as these are nothing but $f_2(-x)$ and $d_2(-x)$, respectively. In general, having $h : \mathcal{Z} \to \mathbb{R},\ h(x) = f(-x)$ it holds $\forall x^* \in \mathcal{Z}^*$

$$\begin{aligned} h^*(x^*) &= \sup_{x\in\mathcal{Z}}\{\langle x^*, x\rangle - h(x)\} = \sup_{x\in\mathcal{Z}}\{\langle x^*, x\rangle - f(-x)\} \\ &= \sup_{x\in\mathcal{Z}}\{\langle -x^*, x\rangle - f(x)\} = f^*(-x^*). \end{aligned}$$

5 Conjugates of convex deviation measures: the case a>1

In this section we extend our investigations to the conjugate functions of convex deviation measures given in Example 3.3 and Example 3.4, but in the case $a > 1$ (like before, $p \in [1, +\infty)$). We use relation (1) in order to calculate the conjugate functions of the corresponding convex risk measures.
In our approach we use the very well-developed calculus existing in the theory of conjugate functions. The functions considered in this section will be viewed as compositions of a convex increasing function with a convex function. The conjugates will be obtained by using a formula existing in the literature for the conjugate of a composed convex function (see [2] for more on this subject). Let us state now the main theorem used in this section:

Theorem 5.1. *Let $\mathcal{Z}$ be a nontrivial locally convex space and $f : \mathcal{Z} \to \mathbb{R},\ \ g : \mathbb{R} \to \overline{\mathbb{R}}$ be convex functions such that g is increasing on $f(\mathcal{Z}) + [0, +\infty)$. We assume that there exists $x' \in \mathcal{Z}$ such that $f(x') \in \mathrm{dom}(g)$ and g is continuous at $f(x')$. Then for all $x^* \in \mathcal{Z}^*$ one has*

$$(g \circ f)^*(x^*) = \min_{\beta\in\mathbb{R}_+}\{g^*(\beta) + (\beta f)^*(x^*)\}. \tag{8}$$

We apply Theorem 5.1, by taking for f the convex deviation measures considered in the previous section and for $g : \mathbb{R} \to \overline{\mathbb{R}}$ the function defined for $a > 1$ by

$$g(x) = \begin{cases} x^a, & x \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The set $f(L_p) + [0, +\infty)$ is equal $[0, +\infty)$ and one can see that both f and g are convex functions and that g is increasing on $[0, +\infty)$. The regularity condition is also fulfilled, so formula (8) will hold.
The following lemma provides the formula for the conjugate of the function g.

Lemma 5.1. *The conjugate function of g, $g^* : \mathbb{R} \to \overline{\mathbb{R}}$ is*

$$g^*(\beta) = \begin{cases} (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}, & \beta \geq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By definition it holds

$$g^*(\beta) = \sup_{x\in\mathbb{R}} (x\beta - g(x)) = \sup_{x\geq 0} (x\beta - x^a).$$

In the case $\beta \leq 0$, one gets $g^*(\beta) = 0$. In case $\beta > 0$, we consider $h : [0, +\infty) \to \mathbb{R}$, $h(x) = x\beta - x^a$. One has $h'(x) = 0 \Leftrightarrow x = \left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} > 0$. Since h is concave it attains its maximum at $x = \left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} > 0$ and so

$$g^*(\beta) = \beta\left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} - \left(\left(\frac{\beta}{a}\right)^{\frac{1}{a-1}}\right)^a = \beta\left(\frac{\beta}{a}\right)^{\frac{1}{a-1}}\left[1 - \frac{1}{a}\right] = (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}.$$

In conclusion, we get

$$g^*(\beta) = \begin{cases} (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}, & \beta \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

□

In order to calculate the formulas for the conjugate functions of the convex deviation measures in Example 3.3 and Example 3.4 we need the following intermediate formulas.

Example 5.1.
Let be $f_3(x) = ||x||_p^a$. For $x \in L_p$ we have $f_3(x) = (g \circ f_1)(x)$. Let us for $\beta \in \mathbb{R}_+$ first calculate the formula for $(\beta f_1)^*(x^*)$.
Since for $\beta > 0$ and $x^* \in L_q$ it holds (see Example 4.1)

$$\begin{aligned}(\beta f_1)^*(x^*) &= \beta f_1^*\left(\frac{1}{\beta}x^*\right) = \beta(||\cdot||_p)^*\left(\frac{1}{\beta}x^*\right) \\ &= \begin{cases} 0, & ||\frac{x^*}{\beta}||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases} = \begin{cases} 0, & ||x^*||_q \leq \beta, \\ +\infty, & \text{otherwise,} \end{cases}\end{aligned}$$

and for $\beta = 0$ and $x^* \in L_q$ one has

$$(\beta f_1)^*(x^*) = \begin{cases} 0, & x^* = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

we finally get $\forall \beta \geq 0,\ \forall x^* \in L_q$,

$$(\beta f_1)^*(x^*) = \begin{cases} 0, & ||x^*||_q \leq \beta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus with Theorem 5.1 the conjugate of f_3 becomes $\forall x^* \in L_q$,

$$f_3^*(x^*) = \min_{\substack{\beta \geq 0, \\ ||x^*||_q \leq \beta}} g^*(\beta) = \min_{\substack{\beta \geq 0, \\ ||x^*||_q \leq \beta}} (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}} = (a-1)\left|\left|\frac{1}{a}x^*\right|\right|_q^{\frac{a}{a-1}}.$$

Example 5.2.
Let be $f_4 : L_p \to \mathbb{R},\ f_4(x) = ||x_-||_p^a,\ x \in L_p$. One can see that in this case $f_4 = g \circ f_2$. In order to use the relation in (8), we have to calculate $(\beta f_2)^*$ for $\beta \geq 0$. If $\beta = 0$ one has again

$$(\beta f_2)^*(x^*) = \begin{cases} 0, & x^* = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

while if $\beta > 0$, by (2), it holds

$$\begin{aligned} (\beta f_2)^*(x^*) &= \beta f_2^*\left(\frac{1}{\beta}x^*\right) \\ &= \begin{cases} 0, & \left|\left|\frac{1}{\beta}x^*\right|\right|_q \leq 1,\ \frac{1}{\beta}x^* \leq 0, \\ +\infty, & \text{otherwise,} \end{cases} = \begin{cases} 0, & ||x^*||_q \leq \beta,\ x^* \leq 0, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

By (8) we obtain that for all $x^* \in L_q$ such that $x^* \in -(L_q)_+$,

$$\begin{aligned} f_4^*(x^*) &= \min_{\beta \geq 0}\{g^*(\beta) + (\beta f_2)^*(x^*)\} = \min_{\substack{\beta \geq 0, \\ ||x^*||_q \leq \beta}} \left\{(a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}\right\} \\ &= (a-1)\left|\left|\frac{1}{a}x^*\right|\right|_q^{\frac{a}{a-1}}. \end{aligned}$$

If $x^* \notin -(L_q)_+,\ f_4^*(x^*) = +\infty$, so one has $\forall x^* \in L_q$,

$$f_4^*(x^*) = \begin{cases} (a-1)\left|\left|\frac{1}{a}x^*\right|\right|_q^{\frac{a}{a-1}}, & x^* \in -(L_q)_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

In the next example, we come to the convex deviation measure considered in Example 3.3, $d_3(x) = ||x - \mathbb{E}(x)||_p^a,\ x \in L_p$.

Example 5.3.
The convex deviation measure $d_3 : L_p \to \mathbb{R}$, $d_3(x) = ||x - \mathbb{E}(x)||_p^a$ can be written as $d_3 = g \circ d_1$. Let be $x^* \in L_q$. For $\beta = 0$ one has

$$(\beta d_1)^*(x^*) = \begin{cases} 0, & x^* = 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

while if $\beta > 0$ it holds,

$$\begin{aligned}(\beta d_1)^*(x^*) = \beta d_1^*\left(\frac{1}{\beta}x^*\right) &= \begin{cases} 0, & \mathbb{E}\left(\frac{1}{\beta}x^*\right) = 0 \text{ and } \min\limits_{c\in\mathbb{R}} \left|\left|\frac{1}{\beta}x^* - c\right|\right|_q \leq 1, \\ +\infty, & \text{otherwise}, \end{cases} \\ &= \begin{cases} 0, & \mathbb{E}(x^*) = 0 \text{ and } \min\limits_{c\in\mathbb{R}} ||x^* - c||_q \leq \beta, \\ +\infty, & \text{otherwise}. \end{cases}\end{aligned}$$

Then by (8), we have $\forall x^* \in L_q$ such that $\mathbb{E}(x^*) = 0$,

$$d_3^*(x^*) = \min_{\substack{\beta\geq 0, \\ \beta \geq \min\limits_{c\in\mathbb{R}} ||x^*-c||_q}} \{g^*(\beta)\} = \min_{c\in\mathbb{R}}\left\{(a-1)\left|\left|\frac{1}{a}(x^*-c)\right|\right|_q^{\frac{a}{a-1}}\right\}$$

and $d_3^*(x^*) = +\infty$, if $\mathbb{E}(x^*) \neq 0$. We conclude that

$$d_3^*(x^*) = \begin{cases} \min\limits_{c\in\mathbb{R}}\left\{(a-1)\left|\left|\frac{1}{a}(x^*-c)\right|\right|_q^{\frac{a}{a-1}}\right\}, & \mathbb{E}(x^*) = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

The conjugate function of the corresponding convex risk measure $\rho_3 : L_p \to \mathbb{R}$, $\rho_3(x) = d_3(x) - \mathbb{E}(x) = ||x - \mathbb{E}(x)||_p^a - \mathbb{E}(x)$, turns out to be $\forall x^* \in L_p$ (cf. (1)),

$$\rho_3^*(x^*) = d_3^*(x^*+1) = \begin{cases} \min\limits_{c\in\mathbb{R}}\left\{(a-1)\left|\left|\frac{1}{a}(x^*-c)\right|\right|_q^{\frac{a}{a-1}}\right\}, & \mathbb{E}(x^*) = -1, \\ +\infty, & \text{otherwise}. \end{cases}$$

The last conjugate function we derive now is that of the convex deviation measure given in Example 3.4.

Example 5.4.
Considering $d_4 : L_p \to \mathbb{R}$, $d_4(x) = ||(x - \mathbb{E}(x))_-||_p^a$, $x \in L_p$, one can see that $d_4 = g \circ d_2$. Let us calculate now for all $\beta \geq 0$, $(\beta d_2)^*$.
We fix an $x^* \in L_q$. If $\beta = 0$, then

$$(\beta d_2)^*(x^*) = \begin{cases} 0, & x^* = 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

while for $\beta > 0$ one has (see (7))

$$\begin{aligned}(\beta d_2)^*(x^*) &= \beta d_2^*\left(\frac{1}{\beta}x^*\right)\\ &= \begin{cases} 0, & \mathbb{E}\left(\frac{1}{\beta}x^*\right) = 0, \ \ \frac{1}{\beta}x^* \leq 1, \ \ \left|\left|\operatorname{essup}\left(\frac{1}{\beta}x^*\right) - \frac{1}{\beta}x^*\right|\right|_q \leq 1,\\ +\infty, & \text{otherwise,}\end{cases}\\ &= \begin{cases} 0, & \mathbb{E}(x^*) = 0, \ \ x^* \leq \beta, \ \ ||\operatorname{essup} x^* - x^*||_q \leq \beta,\\ +\infty, & \text{otherwise.}\end{cases}\end{aligned}$$

Let us notice that for $\mathbb{E}(x^*) = 0$ if $\beta \geq ||\operatorname{essup} x^* - x^*||_q$, then $\beta \geq \mathbb{E}(\operatorname{essup} x^* - x^*) = \operatorname{essup} x^*$, which implies that $(\beta d_2)^*(x^*)$ is nothing else than

$$(\beta d_2)^*(x^*) = \begin{cases} 0, & \mathbb{E}(x^*) = 0, \ \ ||\operatorname{essup} x^* - x^*||_q \leq \beta,\\ +\infty, & \text{otherwise.}\end{cases}$$

By (8) we get $\forall x^* \in L_q$ such that $\mathbb{E}(x^*) = 0$,

$$d_4^*(x^*) = \inf_{\substack{\beta \geq 0,\\ \beta \geq ||\operatorname{essup} x^* - x^*||_q}} \left\{(a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}\right\} = (a-1)\left|\left|\frac{1}{a}(\operatorname{essup} x^* - x^*)\right|\right|_q^{\frac{a}{a-1}},$$

while if $\mathbb{E}(x^*) \neq 0$, $d_4^*(x^*) = +\infty$. Thus

$$d_4^*(x^*) = \begin{cases} (a-1)\left|\left|\frac{1}{a}(\operatorname{essup} x^* - x^*)\right|\right|_q^{\frac{a}{a-1}}, & \mathbb{E}(x^*) = 0,\\ +\infty, & \text{otherwise.}\end{cases}$$

The conjugate function of the corresponding convex risk measure $\rho_4 : L_p \to \mathbb{R}$, $\rho_4(x) = d_4(x) - \mathbb{E}(x) = ||(x - \mathbb{E}(x))_-||_p^a - \mathbb{E}(x)$ follows $\forall x^* \in L_q$(cf. (1)),

$$\rho_4^*(x^*) = d_4^*(x^* + 1) = \begin{cases} (a-1)\left|\left|\frac{1}{a}(\operatorname{essup} x^* - x^*)\right|\right|_q^{\frac{a}{a-1}}, & \mathbb{E}(x^*) = -1,\\ +\infty, & \text{otherwise.}\end{cases}$$

6 Dual representation of convex risk measures

In this section we give for the convex risk and deviation measures considered in this paper some dual representations which will follow by applying the Fenchel-Moreau theorem (Theorem 2.1). For $p \in [1, +\infty)$ and $f : L_p \to \overline{\mathbb{R}}$ a proper, convex and lower-semicontinuous function we have $\forall x \in L_p$,

$$f(x) = f^{**}(x) = \sup_{x^* \in L_q} \{\langle x^*, x\rangle - f^*(x^*)\} = \sup_{x^* \in L_q} \{\mathbb{E}(x^* x) - f^*(x)\}. \qquad (9)$$

As all convex risk and deviation measures fulfill the hypotheses of Theorem 2.1, by using the formulas of the conjugates derived in the previous sections, we obtain in a very natural way the desired dual representations. Our representations turn out to be generalizations of the recently published results by Pflug (cf. [8]). More than that, we show the usefulness of the powerful theory of conjugate functions from the convex analysis in this field as well.

Example 6.1.
The first convex deviation measure we treat is $d_1 : L_p \to \mathbb{R}$, $d_1(x) = ||x - \mathbb{E}(x)||_p$. We proved that $\forall x^* \in L_q$,

$$d_1^*(x^*) = \begin{cases} 0, & \mathbb{E}(x^*) = 0 \text{ and } \min\limits_{c \in \mathbb{R}} ||x^* - c||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and so, by (9), $\forall x \in L_p$,

$$d_1(x) = \sup\{\mathbb{E}(x^* x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = 0 \text{ and } \min_{c \in \mathbb{R}} ||x^* - c||_q \leq 1\}.$$

Analogously, by (4), we obtain $\forall x \in L_p$,

$$\rho_1(x) = \sup\left\{\mathbb{E}(x^* x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = -1 \text{ and } \min_{c \in \mathbb{R}} ||x^* - c||_q \leq 1\right\}.$$

Pflug (cf. [8, Proposition 3]) also gives for $p \in (1, +\infty)$ representations for these convex risk and deviation measures, which are actually generalizations of the standard deviation. The formulas given by Pflug are not quite accurate, as he considers inf instead of min. But, as we have seen in the previous chapters, the existence of a $c \in \mathbb{R}$, such that $||x^* - c||_q \leq 1$, is indispensable. Let us also notice that Pflug uses instead of convex risk measures so-called *acceptability functionals* (we denote them like in [8] by A). They are linked to the convex risk measures in our paper by the relation $A(x) = -\rho(x),\ x \in L_p$.

Example 6.2.
Take now $d_2 : L_p \to \mathbb{R}$, $d_2(x) = ||(x - \mathbb{E}(x))_-||_p$, as in Example 4.4. For the conjugate function d_2^* it holds $\forall x^* \in L_q$ (see (7)),

$$d_2^*(x^*) = \begin{cases} 0, & \mathbb{E}(x^*) = 0,\ x^* \leq 1,\ ||\operatorname{essup} x^* - x^*||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and so one gets the following dual representation $\forall x \in L_p$,

$$d_2(x) = \sup\{\mathbb{E}(x^* x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = 0,\ x^* \leq 1,\ ||\operatorname{essup} x^* - x^*||_q \leq 1\}.$$

Similary, we get $\forall x \in L_p$,

$$\rho_2(x) = \sup\{\mathbb{E}(x^* x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = -1,\ x^* \leq 0,\ ||\operatorname{essup} x^* - x^*||_q \leq 1\}.$$

The last two equalities are actually the formulas proved in Proposition 5 in [8].

Example 6.3.
Let be $d_3(x) = ||x - \mathbb{E}(x)||_p^a$, $x \in L_p$, the convex risk measure considered in Example 5.3 for $a > 1$. The conjugate funtion $d_3^* : L_q \to \overline{\mathbb{R}}$ is $\forall x^* \in L_q$,

$$d_3^*(x^*) = \begin{cases} \min\limits_{c\in\mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\}, & \mathbb{E}(x^*) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

By (9) we get the following dual representation $\forall x \in L_p$,

$$d_3(x) = \sup \left\{ \mathbb{E}(x^* x) - \min_{c\in\mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\} : \ x^* \in L_q, \mathbb{E}(x^*) = 0 \right\}.$$

Similarly, we get the representation of the convex risk measure $\forall x \in L_p$,

$$\rho_3(x) = \sup \left\{ \mathbb{E}(x^* x) - \min_{c\in\mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\} : \ x^* \in L_q, \mathbb{E}(x^*) = -1 \right\}.$$

Pflug gives in Proposition 2 in [8] the formula just for the special case when $a = p$ and $p \in (1, +\infty)$. More than that, his formula may be improved by mentioning that the inner infimum is attained.

Example 6.4.
Finally let be $d_4 : L_p \to \mathbb{R}$, $d_4(x) = ||(x - \mathbb{E}(x))_-||_p^a$, $x \in L_p$, where $a > 1$. From Example 5.4, one has $\forall x^* \in L_q$,

$$d_4^*(x^*) = \begin{cases} (a-1) \left|\left| \frac{1}{a}(\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}}, & \mathbb{E}(x^*) = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and so, $\forall x \in L_p$, d_4 can be represented as

$$d_4(x) = \sup \left\{ \mathbb{E}(x^* x) - (a-1) \left|\left| \frac{1}{a}(\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}} : x^* \in L_q, \mathbb{E}(x^*) = 0 \right\}.$$

Again, for the convex risk measure $\rho_4(x) = ||(x - \mathbb{E}(x))_-||_p^a - \mathbb{E}(x)$ we get $\forall x \in L_p$,

$$\rho_4(x) = \sup \left\{ \mathbb{E}(x^* x) - (a-1) \left|\left| \frac{1}{a}(\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}} : x^* \in L_q, \ \mathbb{E}(x^*) = -1 \right\}.$$

These assertions generalize Proposition 4 in [8] where the formulas have been given just in the case $a = p$ and $p \in (1, +\infty)$.

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