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A New Fenchel Dual Problem in Vector Optimization

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Abstract

We introduce a new Fenchel dual problem in vector optimization inspired by the form of the Fenchel dual of the scalarized primal multiobjective problem. For the vector primal and dual problems we prove weak and strong duality. Furthermore, we recall two other Fenchel-type dual problems introduced in the past in the literature, in the vector case, and make a comparison among all three duals. Moreover, we show that their sets of maximal elements are equal.

Key Words. conjugate functions, Fenchel duality, vector optimization, weak and strong duality

AMS subject classification. 49N15, 90C29, 90C46

1 Introduction

Multiobjective optimization problems have generated a great deal of interest during the last years, not only from a theoretical point of view, but also from a practical one, due to their applicability in different fields, like economics and engineering. In general, when dealing with scalar optimization problems, the duality theory proves to be an important tool for giving some dual characterizations for the optimal solutions of a primal problem. Similar characterizations can also be given for multiobjective optimization problems, namely for problems having a vector function as objective function.

An overview on the literature dedicated to this field shows that the general interest was centered on multiobjective problems with inequality constraints. The

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duality theories developed for these problems are extensions of the classical Lagrange duality approach. We recall in this direction the concepts developed by Mond and Weir in [14], [15] (whose formulation is based on the optimality conditions which follow from Lagrange duality). Tanino, Nakayama and Sawaragi examined in [12] the duality for vector optimization in finite dimensional spaces using the perturbation approach, the duals obtained in this case being also Lagrange-type duals. They extended to the vector case the conjugate theory in scalar optimization (see for example [11]). In Jahn's paper [8] the Lagrange dual appears explicitly in the formulation of the feasible set of the multiobjective dual.

Another approach is due to Boţ and Wanka, who constructed a vector dual ([4]) using the Fenchel-Lagrange dual for scalar optimization problems. This is a combination of the classical Lagrange and Fenchel duals and was treated in papers like [1], [2] and [3].

With respect to vector duality based on Fenchel's duality concept, the bibliography is not very rich. We mention in this direction the works of Breckner and Kolumbán, [5] and [6] (see also Gerstewitz and Göpfert, [7] and Malivert, [10]).

The primal problem treated in this paper has as objective function the sum of a vector function, with another one, which is the composition of a vector function, with a linear operator. For it we propose a Fenchel-type dual which extends the well known Fenchel scalar dual from [11]. We prove weak and strong duality, and compare the new dual to two other from the literature.

The paper is organized as follows. In Section 2 we enumerate some elements of convex analysis which are used later, and we state the primal vector optimization problem along with the constraint qualification under which strong duality holds. In Section 3 we analyze the sclarized problem of the primal. For it, by means of the classical scalar Fenchel's duality theorem, we construct a dual, for which we prove both weak duality and, under a weak constraint qualification, strong duality. Optimality conditions for the scalarized primal problem are presented in the last part of the section.

Using the formulation of the scalarized dual, we define in Section 4 the new vector dual problem. For it, we prove weak and strong duality. In order to be able to understand the position of our dual among other duals given in the literature, we present in Section 5 two other Fenchel-type dual problems, one inspired by Breckner and Kolumban's [6] paper, while the other one is constructed by making a slight change in the feasible set of the first one (cf. [8]). For them we also give the weak and strong duality theorems.

The image sets of the three duals are closely connected, as it is proved in Section 6, where the existence of some relations of inclusion between these sets is proved. Moreover, we illustrate by some examples that in general these inclusions are strict. Finally, we show that even though this happens, the sets of the maximal elements of the image sets coincide.

2 Preliminary notions and results

In this section we present some notions and preliminary results used throughout the paper. We give some elements of convex analysis and introduce the primal vector optimization problem.

2.1 Elements of convex analysis

All the vectors considered are column vectors. For two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in $\mathbb{R}^n$, by $x^T y$ we denote the usual inner product, i.e. $x^T y = \sum_{i=1}^{n} x_i y_i$. Having a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ its *effective domain* is denoted by $\mathrm{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The function f is said to be *proper* if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\mathrm{dom}(f) \neq \emptyset$. Its *epigraph* is the set $\mathrm{epi}(f) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$ and its *conjugate function* is defined by $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$f^*(p) = \sup\left\{p^T x - f(x) : x \in \mathbb{R}^n\right\}.$$

Furthermore, we recall the well known Fenchel-Young's inequality:

$$f^*(p) + f(x) \geq p^T x \quad \forall x, p \in \mathbb{R}^n.$$

If $\lambda > 0$ and $p \in \mathbb{R}^n$, one has

$$(\lambda f)^*(p) = \lambda f^*\left(\frac{1}{\lambda} p\right). \tag{1}$$

The function f is *polyhedral* if $\mathrm{epi}(f)$ is a polyhedral set. Let us recall that a set is polyhedral if it can be written as the intersection of a finite family of closed half-spaces.

Having a nonempty subset C of $\mathbb{R}^n$, $\mathrm{int}(C)$ denotes its *interior*, meanwhile $\mathrm{ri}(C)$ denotes its *relative interior*.

For a linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$, its *adjoint* $A^* : \mathbb{R}^k \to \mathbb{R}^n$ is the linear operator defined by

$$(A^* y)^T x = y^T (Ax) \quad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^k.$$

Definition 1 *Let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i \in \{1, ..., m\}$, be proper convex functions. The function $f_1 \square ... \square f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by*

$$f_1 \square ... \square f_m(x) = \inf\left\{\sum_{i=1}^{m} f_i(x_i) : \sum_{i=1}^{m} x_i = x\right\}$$

is called the infimal convolution of $f_1, ..., f_m$.

The next result gives, under a weak regularity condition, a formula for the conjugate of the sum of a family of proper convex functions via the infimal convolution of their conjugates.

Theorem 1 *(cf. Theorem 20.1 in [11]) Let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i \in \{1, ..., k\}$, be proper convex functions and $f_j : \mathbb{R}^n \to \overline{\mathbb{R}}$, $j \in \{k+1, ..., m\}$, be proper polyhedral functions. Assume that $\bigcap\limits_{i=1}^{k} \mathrm{ri}\,(\mathrm{dom}(f_i)) \cap \bigcap\limits_{j=k+1}^{m} \mathrm{dom}(f_j) \neq \emptyset$. Then for all $p \in \mathbb{R}^n$ one has*

$$\left(\sum_{i=1}^{m} f_i\right)^*(p) = \inf\left\{\sum_{i=1}^{m} f_i^*(p_i) : \sum_{i=1}^{m} p_i = p\right\}$$

and the infimum is attained.

We state now a theorem which gives the formula for the conjugate of the composition of a convex function with a linear operator. Let us notice that the first part of the theorem is nothing else then Theorem 16.3 in [11]. The proof of the second part can be given as a direct application of Fenchel's duality theorem (cf. Theorem 31.1 in [11]).

Theorem 2 *Let $h : \mathbb{R}^k \to \overline{\mathbb{R}}$ be a proper function and $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear operator. Assume that one of the following conditions is fulfilled:*

a) h is convex and there exists $x' \in \mathbb{R}^n$ such that $Ax' \in \mathrm{ri}(\mathrm{dom}(h))$;

b) h is polyhedral and there exists $x' \in \mathbb{R}^n$ such that $Ax' \in \mathrm{dom}(h)$.

Then for all $p \in \mathbb{R}^n$ it holds

$$(h \circ A)^*(p) = \inf\left\{h^*(q) : q \in \mathbb{R}^k, A^*q = p\right\}$$

and the infimum is attained.

2.2 Problem formulation

The primal problem, we deal with in this paper, is the following vector optimization problem

$$(P^A) \qquad v\text{-} \min_{x \in \mathbb{R}^n} (f(x) + (g \circ A)(x)),$$

where f and g are two vector functions such that

$$f = (f_1, f_2, ...f_m)^T \quad \text{and} \quad g = (g_1, g_2, ..., g_m)^T$$

with $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i : \mathbb{R}^k \to \overline{\mathbb{R}}$ for each $i \in \{1, ..., m\}$ and $A : \mathbb{R}^n \to \mathbb{R}^k$ is a linear operator.

Let I be the subset of $\{1, ..., m\}$ consisting of that indices i for which f_i is a proper polyhedral function and J be the subset of $\{1, ..., m\}$ consisting of that indices j for which g_j is also proper polyhedral. We work under the assumption that for each $l \in \{1, ..., m\} \backslash I$ and for each $t \in \{1, ..., m\} \backslash J$, f_l and g_t are proper convex functions, respectively. The constraint qualification we use in order to ensure strong duality, both in the scalar and vector case, is stated bellow:

$$(CQ^A) \quad \exists\ x' \in \bigcap_{i \in I} \mathrm{dom}(f_i) \quad \cap \bigcap_{l \in \{1,...,m\} \backslash I} \mathrm{ri}(\mathrm{dom}(f_l)) \quad \text{such that}$$

$$Ax' \in \bigcap_{j \in J} \mathrm{dom}\,(g_j) \quad \cap \bigcap_{t \in \{1,...,m\} \backslash J} \mathrm{ri}\,(\mathrm{dom}\,(g_t))\,.$$

On $\mathbb{R}^m$ we consider the partial ordering induced by the non-negative orthant $\mathbb{R}^m_+$. For $x, y \in \mathbb{R}^m$ one has

$$x \geqq y \quad \Leftrightarrow \quad x_i \geq y_i \text{ for all } i \in \{1, ..., m\}.$$

For the vector optimization problem (P^A) different notions of solutions have been introduced and studied in the literature. We use in this paper the so-called Pareto-efficient and properly efficient solutions, respectively. For the primal problem, which is a vector minimum one, these notions are defined bellow.

Definition 2 *An element $\overline{x} \in \mathbb{R}^n$ is said to be Pareto-efficient (or efficient, or minimal) with respect to problem (P^A) if from*

$$f(\overline{x}) + (g \circ A)(\overline{x}) \geqq f(x) + (g \circ A)(x) \text{ for } x \in \mathbb{R}^n$$

follows that

$$f(\overline{x}) + (g \circ A)(\overline{x}) = f(x)(g \circ A)(x).$$

Definition 3 *An element $\overline{x} \in \mathbb{R}^n$ is said to be properly efficient with respect to problem (P^A) if there exists $\lambda = (\lambda_1, ..., \lambda_m)^T$ from $\mathrm{int}(\mathbb{R}^m_+)$ such that*

$$\sum_{i=1}^{m} \lambda_i \Big(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \Big) \leq \sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big).$$

Remark 1. Any properly efficient element is efficient, too, but the reverse claim does not hold in general.

3 Duality for the scalarized problem

In order to be able to formulate a vector dual problem to (P^A), let us start by studying the duality theory for the following scalar optimization problem (motivated by the definition of a properly efficient solution), when $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$ is arbitrarily chosen

$$(P^A_\lambda) \quad \inf_{x\in\mathbb{R}^n} \sum_{i=1}^m \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big).$$

Consider now another scalar optimization problem

$$\left(D^A_\lambda\right) \quad \sup_{\substack{p_i\in\mathbb{R}^n, q_i\in\mathbb{R}^k \\ i=1,...m \\ \sum\limits_{i=1}^m \lambda_i(p_i+A^*q_i)=0}} \quad \sum_{i=1}^m \lambda_i \Big(-f_i^*(p_i) - g_i^*(q_i) \Big).$$

We prove that $\left(D^A_\lambda\right)$ is a dual problem of $\left(P^A_\lambda\right)$, namely, on the one hand the weak duality always holds, and on the other hand, using convexity assumptions and the fulfillment of a regularity condition, strong duality holds.

For the scalar problems (P^A_λ) and (D^A_λ) we denote by $v\left(P^A_\lambda\right)$ and $v\left(D^A_\lambda\right)$ their optimal objective values, respectively.

Theorem 3 *(scalar weak duality) It holds*

$$v\left(P^A_\lambda\right) \geq v\left(D^A_\lambda\right).$$

Proof. Let us consider $x \in \mathbb{R}^n$, $p = (p_1, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $q = (q_1, ..., q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$ such that x is feasible to $\left(P^A_\lambda\right)$ and (p, q) is feasible to $\left(D^A_\lambda\right)$. This means that $\sum\limits_{i=1}^m \lambda_i \left(p_i + A^*q_i\right) = 0$. From Fenchel-Young's inequality, we know that

$$f_i(x) + f_i^*(p_i) - p_i^T x \geq 0$$

and

$$(g_i \circ A)(x) + g_i^*(q_i) - (A^*q_i)^T x \geq 0,$$

for all $i \in \{1, ..., m\}$. Then

$$\sum_{i=1}^m \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big) \geq \sum_{i=1}^m \lambda_i \Big(-f_i^*(p_i) - g_i^*(q_i) \Big) +$$

$$+ \sum_{i=1}^m \lambda_i \Big(p_i + A^*q_i \Big)^T x = \sum_{i=1}^m \lambda_i \Big(-f_i^*(p_i) - g_i^*(q_i) \Big).$$

As x and (p, q) have been chosen arbitrarily, the conclusion follows. ■

Remark 2. One can notice that the weak duality holds without any convexity assumptions for the functions involved. But for the strong duality one needs this assumption to be fulfilled.

Theorem 4 *(scalar strong duality) Assume that* (CQ^A) *is fulfilled. Then*

$$v\left(P_\lambda^A\right) = v\left(D_\lambda^A\right)$$

and $\left(D_\lambda^A\right)$ *has an optimal solution.*

Proof. Let us start by noticing that

$$-v(P_\lambda^A) = \sup_{x\in\mathbb{R}^n}\left[-\sum_{i=1}^m \lambda_i\Big(f_i(x) + (g_i\circ A)(x)\Big)\right] = \left[\sum_{i=1}^m \lambda_i\Big(f_i + (g_i\circ A)\Big)\right]^*(0)$$

$$= \left[\sum_{i\in I}\lambda_i f_i + \sum_{j\in J}\lambda_j(g_j\circ A) + \sum_{l\in\{1,..,m\}\setminus\ I}^{m}\lambda_l f_l + \sum_{t\in\{1,...,m\}\setminus J}^{m}\lambda_t(g_t\circ A)\right]^*(0).$$

The functions $\lambda_l f_l$, $l\in\{1,...,m\}\setminus I$, and $\lambda_t(g_t\circ A)$, $t\in\{1,...,m\}\setminus J$, are proper convex, while the functions $\lambda_i f_i$, $i\in I$, and $\lambda_j(g_j\circ A)$, $j\in J$, are proper polyhedral. From (CQ^A) we have that there exists $x'\in\mathbb{R}^n$ such that

$$x' \in \bigcap_{i\in I}\operatorname{dom}(\lambda_i f_i) \ \cap\ \bigcap_{j\in J}\operatorname{dom}\big(\lambda_j(g_j\circ A)\big) \ \cap \bigcap_{l\in\{1,...,m\}\setminus I}\operatorname{ri}\Big(\operatorname{dom}(\lambda_l f_l)\Big)$$

and

$$Ax' \in \bigcap_{t\in\{1,...,m\}\setminus J}\operatorname{ri}\Big(\operatorname{dom}(g_t)\Big).$$

For $t\in\{1,...,m\}\setminus J$, as $Ax'\in\operatorname{ri}\Big(\operatorname{dom}(g_t)\Big)$, by Theorem 6.7 in [11],

$$x'\in A^{-1}\Big(\operatorname{ri}(\operatorname{dom}(g_t))\Big) = \operatorname{ri}\left(A^{-1}(\operatorname{dom}(g_t))\right) = \operatorname{ri}(\operatorname{dom}(g_t\circ A)).$$

So

$$x' \in \bigcap_{i\in I}\operatorname{dom}(\lambda_i f_i) \ \cap\ \bigcap_{j\in J}\operatorname{dom}(\lambda_j(g_j\circ A)) \ \cap$$
$$\bigcap_{l\in\{1,...,m\}\setminus I}\operatorname{ri}(\operatorname{dom}(\lambda_l f_l)) \ \cap \bigcap_{t\in\{1,...,m\}\setminus J}\operatorname{ri}(\operatorname{dom}(\lambda_t(g_t\circ A))).$$

We can apply now Theorem 1, thus there exist $\overline{p}_i\in\mathbb{R}^n, \overline{v}_i\in\mathbb{R}^n, i\in\{1,...,m\}$, such that $\sum\limits_{i=1}^m(\bar{p}_i+\bar{v}_i) = 0$ and

$$
\begin{aligned}
-v(P_\lambda^A) &= \inf_{\substack{p_i\in\mathbb{R}^n, v_i\in\mathbb{R}^n,\\ i\in\{1,...,m\}\\ \sum_{i=1}^m (p_i+v_i)=0}} \left\{ \sum_{i\in I}(\lambda_i f_i)^*(p_i) + \sum_{j\in J}\Big(\lambda_j(g_j\circ A)\Big)^*(v_j) \right.\\
&+ \left. \sum_{l\in\{1,...,m\}\setminus I}(\lambda_l f_l)^*(p_l) + \sum_{t\in\{1,...,m\}\setminus J}\Big(\lambda_t(g_t\circ A)\Big)^*(v_t) \right\}\\
&= \sum_{i\in I}(\lambda_i f_i)^*(\overline{p}_i) + \sum_{j\in J}\Big(\lambda_j(g_j\circ A)\Big)^*(\overline{v}_j)\\
&+ \sum_{l\in\{1,...,m\}\setminus I}(\lambda_l f_l)^*(\overline{p}_l) + \sum_{t\in\{1,...,m\}\setminus J}\Big(\lambda_t(g_t\circ A)\Big)^*(\overline{v}_t).
\end{aligned}
$$

Applying now the statement *a)* of Theorem 2 for the proper convex functions $\lambda_t g_t, t\in\{1,...,m\}\setminus J$, and *b)* for the proper polyhedral functions $\lambda_j g_j \in \mathbb{R}^k, j\in J$, we obtain the existence of $\overline{q}_i\in\mathbb{R}^k, i\in\{1,...,m\}$, such that $A^*\overline{q}_i = \overline{v}_i$ and $\big(\lambda_i(g_i\circ A)\big)^*(\overline{v}_i) = \big(\lambda_i g_i\big)^*(\overline{q}_i)$. Then

$$
-v(P_\lambda^A) = \sum_{l=1}^m (\lambda_i f_i)^*(\overline{p}_i) + \sum_{i=1}^m (\lambda_i g_i)^*(\overline{q}_i) \text{ and } \sum_{i=1}^m (\overline{p}_i + A^*\overline{q}_i) = 0.
$$

As by (1)

$$
(\lambda_i f_i)^*(\overline{p}_i) = \lambda_i f_i^*\left(\frac{1}{\lambda_i}\overline{p}_i\right) \quad \text{and} \quad (\lambda_i g_i)^*(\overline{q}_i) = \lambda_i q_i^*\left(\frac{1}{\lambda_i}\overline{q}_i\right) \quad \forall i\in\{1,...,m\},
$$

by redenoting $\overline{p}_i := \frac{1}{\lambda_i}\overline{p}_i$ and $\overline{q}_i := \frac{1}{\lambda_i}\overline{q}_i, i\in\{1,...,m\}$, one has

$$
v(P_\lambda^A) = -\sum_{l=1}^m \lambda_i f_i^*(\overline{p}_i) - \sum_{i=1}^m \lambda_i g_i^*(\overline{q}_i), \text{ where } \sum_{i=1}^m \lambda_i(\overline{p}_i + A^*\overline{q}_i) = 0.
$$

By Theorem 3 we have that $v(P_\lambda^A) = v(D_\lambda^A)$, and $(\overline{p},\overline{q})$ with $\overline{p} = (\overline{p}_1,...,\overline{p}_m)$ and $\overline{q} = (\overline{q}_1,...,\overline{q}_m)$ is an optimal solution of the dual. ■

The next theorem states the optimality conditions that can be derived for (P_λ^A) and (D_λ^A).

Theorem 5 *a) If $\left(CQ^A\right)$ is fulfilled and $\overline{x}\in\mathbb{R}^n$ is an optimal solution of $\left(P_\lambda^A\right)$, than there exists $(\overline{p},\overline{q})$, $\overline{p} = (\overline{p}_1,...,\overline{p}_m)\in\mathbb{R}^n\times...\times\mathbb{R}^n$, $\overline{q} = (\overline{q}_1,...,\overline{q}_m)\in\mathbb{R}^k\times...\times\mathbb{R}^k$, an optimal solution of $\left(D_\lambda^A\right)$, such that*

$$
\begin{cases}
(i) & f_i(\overline{x}) + f_i^*(\overline{p}_i) = \overline{p}_i^T\overline{x}, \quad \forall i\in\{1,...,m\};\\
(ii) & (g_i\circ A)(\overline{x}) + g_i^*(\overline{q}_i) = (A^*q_i)^T\overline{x}, \quad \forall i\in\{1,...,m\};\\
(iii) & \sum_{i=1}^m \lambda_i(\overline{p}_i + A^*\overline{q}_i) = 0.
\end{cases}
$$

b) If $\overline{x} \in \mathbb{R}^n$, $\overline{p} = (\overline{p}_1, ..., \overline{p}_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $\overline{q} = (\overline{q}_1, ..., \overline{q}_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$ are such that (i), (ii) and (iii) are fulfilled, then it follows that they are optimal solutions to $\left(P_\lambda^A\right)$ and $\left(D_\lambda^A\right)$, respectively. Furthermore, the following equality holds

$$\sum_{i=1}^m \lambda_i \bigg(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \bigg) = \sum_{i=1}^m \lambda_i \bigg(- f_i^*(\overline{p}_i) - g_i^*(\overline{q}_i) \bigg).$$

Proof. a) Since $\overline{x}$ is an optimal solution of $\left(P_\lambda^A\right)$, which means that

$$v(P_\lambda^A) = \sum_{i=1}^m \lambda_i \bigg(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \bigg),$$

and $\left(CQ^A\right)$ is fulfilled, by Theorem 4, we obtain the existence of an optimal solution $(\overline{p}, \overline{q})$ to $\left(D_\lambda^A\right)$, $\overline{p} = (\overline{p}_1, ..., \overline{p}_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $\overline{q} = (\overline{q}_1, ..., \overline{q}_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$ fulfilling $\sum\limits_{i=1}^m \lambda_i(\overline{p}_i + A^*\overline{q}_i) = 0$ and

$$\sum_{i=1}^m \lambda_i \bigg(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \bigg) = \sum_{i=1}^m \lambda_i \bigg(- f_i^*(\overline{p}_i) - {g_i}^*(\overline{q}_i) \bigg).$$

Thus

$$\sum_{i=1}^m \lambda_i \bigg(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) + f_i^*(\overline{p}_i) + {g_i}^*(\overline{q}_i) \bigg) = 0 \iff$$

$$\begin{aligned} 0 \; = \; & \sum_{i=1}^m \lambda_i \bigg(f_i(\overline{x}) + f_i^*(\overline{p}_i) - \overline{p}_i^T \overline{x} \bigg) + \\ & \sum_{i=1}^m \lambda_i \bigg((g_i \circ A)(\overline{x}) + {g_i}^*(\overline{q}_i) - (A^*\overline{q}_i)^T \overline{x} \bigg). \end{aligned}$$

But, for each $i \in \{1, ..., m\}$, $f_i(\overline{x}) + f_i^*(\overline{p}_i) - \overline{p}_i^T \overline{x} \geq 0$ and $(g_i \circ A)(\overline{x}) + {g_i}^*(\overline{q}_i) - (A^*\overline{q}_i)^T \overline{x} \geq 0$ due to Fenchel-Young's inequality. Thus we have obtained that a sum of terms, each greater than or equal to zero is zero. Therefore each of them must be zero. Hence for all $i \in \{1, ..., m\}$, (i), (ii) and (iii) hold.

b) All the calculations and transformations done within part a) may be carried out backwards, starting from the conditions (i), (ii) and (iii). ■

4 The new vector dual problem

By using the results obtained in the previous section, we are now able to formulate a multiobjective dual to (P^A). The dual (D^A) will be a vector maximum problem, therefore efficient solutions in the sense of the maximum are considered for it.

The aim of this section is to introduce the new vector dual problem (D^A) and to prove weak and strong duality between the two problems.

Let us define (D^A) by

$$\left(D^A\right) \quad v - \max_{(p,q,\lambda,t)\in\mathcal{B}} h\left(p,q,\lambda,t\right)$$

where

$$\mathcal{B} = \left\{ \begin{array}{ll} (p,q,\lambda,t): & p = (p_1,...,p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \\ & q = (q_1,...,q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k, \\ & \lambda = (\lambda_1,...,\lambda_m)^T \in \operatorname{int}(\mathbb{R}^m_+), \\ & t = (t_1,...,t_m)^T \in \mathbb{R}^m, \\ & \sum\limits_{i=1}^m \lambda_i\left(p_i + A^* q_i\right) = 0, \sum\limits_{i=1}^m \lambda_i t_i = 0 \end{array} \right\},$$

and h is defined by

$$h\left(p,q,\lambda,t\right) = \begin{pmatrix} h_1\left(p,q,\lambda,t\right) \\ ... \\ h_m\left(p,q,\lambda,t\right) \end{pmatrix},$$

with

$$h_i\left(p,q,\lambda,t\right) = -f_i^*\left(p_i\right) - g_i^*\left(q_i\right) + t_i \text{ for all } i \in \{1,...,m\}.$$

Definition 4 *An element $\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) \in \mathcal{B}$ is said to be Pareto-efficient (or efficient, or maximal) with respect to the problem $\left(D^A\right)$ if from*

$$h\left(p,q,\lambda,t\right) \geqq h\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) \textit{ for } \left(p,q,\lambda,t\right) \in \mathcal{B}$$

follows that $h\left(p,q,\lambda,t\right) = h\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)$.

Between $\left(P^A\right)$ and $\left(D^A\right)$ the following weak duality assertion holds.

Theorem 6 *(vector weak duality) There exist no $x \in \mathbb{R}^n$ and no $(p,q,\lambda,t) \in \mathcal{B}$ such that*

$$h\left(p,q,\lambda,t\right) \geqq f\left(x\right) + \left(g \circ A\right)\left(x\right) \textit{ and } h\left(p,q,\lambda,t\right) \neq f\left(x\right) + (g \circ A)(x).$$

Proof. We proceed by contradiction, assuming that there exist $x \in \mathbb{R}^n$ and $(p,q,\lambda,t) \in \mathcal{B}$ such that

$$h\left(p,q,\lambda,t\right) \geqq f(x) + \left(g \circ A\right)\left(x\right)$$

and $h\left(p,q,\lambda,t\right) \neq f\left(x\right) + \left(g \circ A\right)\left(x\right)$. This means that

$$h_i\left(p,q,\lambda,t\right) \geq f_i(x) + \left(g_i \circ A\right)\left(x\right) \quad \forall i \in \{1,...,m\}$$

and that there exists at least one $j \in \{1, ..., m\}$ such that

$$h_j(p, q, \lambda, t) > f_j(x) + (g_j \circ A)(x).$$

Therefore

$$\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) > \sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big). \tag{2}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) &= \sum_{i=1}^{m} \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) + t_i \Big) \\
&= \sum_{i=1}^{m} \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) \Big) + \sum_{i=1}^{m} \lambda_i t_i \\
&= \sum_{i=1}^{m} \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) \Big).
\end{aligned}$$

Applying again the previously mentioned Fenchel-Young's inequality, which ensures that for each $i \in \{1, ..., m\}$, $f_i^*(p_i) \geq p_i^T x - f_i(x)$ and $g_i^*(q_i) \geq (A^* q_i)^T x - (g_i \circ A)(x)$, we obtain

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i h_i(p, q, \lambda, t) &\leq \sum_{i=1}^{m} \lambda_i \Big(f_i(x) - p_i^T x + (g_i \circ A)(x) - (A^* q_i)^T x \Big) \\
&= \sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big) + \sum_{i=1}^{m} \lambda_i \left(-p_i^T x - (A^* q_i)^T x \right) \\
&= \sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big),
\end{aligned}$$

which is a contradiction to (2). This concludes the proof. ■

Remark 3. As in the scalar case, the just verified weak duality holds without any convexity assumptions.

Theorem 7 *(vector strong duality) If (CQ^A) is fulfilled and $\overline{x}$ is a properly efficient solution to (P^A), then there exists an efficient solution $(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t})$ to (D^A) and*

$$h(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) = f(\overline{x}) + (g \circ A)(\overline{x})$$

holds.

Proof. Let $\overline{x}$ be a properly efficient solution to (P^A). Then, according to Definition 3, there exists $\overline{\lambda} \in \text{int}(\mathbb{R}_+^m)$ such that $\overline{x}$ is an optimal solution to the scalar optimization problem $\left(P_{\overline{\lambda}}^{A}\right)$. As we are working under the assumption that $\left(CQ^{A}\right)$ holds, Theorem 4 ensures the existence of an optimal solution to $\left(D_{\overline{\lambda}}^{A}\right)$, $(\overline{p},\overline{q})$ and, further, Theorem 5 affirms that the optimality conditions (i), (ii) and (iii) are satisfied. Let us define

$$\overline{t}_i := \left(\overline{p}_i + A^*\overline{q}_i\right)^T \overline{x} \in \mathbb{R} \quad \text{for all} \quad i \in \{1,...,m\}.$$

Since

$$\sum_{i=1}^{m} \overline{\lambda}_i \left(\overline{p}_i + A^*\overline{q}_i\right) = 0 \text{ and } \sum_{i=1}^{m} \overline{\lambda}_i \overline{t}_i = \sum_{i=1}^{m} \lambda_i \left(\overline{p}_i + A^*\overline{q}_i\right)^T \overline{x} = 0,$$

there is $\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) \in \mathcal{B}$, which means that it is feasible to $\left(D_{\overline{\lambda}}^{A}\right)$.

Moreover, by the optimality conditions (i), (ii) and (iii) from Theorem 5, for each $i \in \{1,...,m\}$ one has

$$\begin{aligned} h_i\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) &= -f_i^*\left(\overline{p}_i\right) - g_i^*\left(\overline{q}_i\right) + \overline{t}_i \\ &= f_i\left(\overline{x}\right) - \overline{p}_i^T\overline{x} + \left(g_i \circ A\right)\left(\overline{x}\right) - \left(A^*\overline{q}_i\right)^T \overline{x} + \overline{t}_i \\ &= f_i\left(\overline{x}\right) + \left(g_i \circ A\right)\left(\overline{x}\right). \end{aligned}$$

We prove now that $\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)$ is efficient. If this were not the case, then there would exist $(p,q,\lambda,t) \in \mathcal{B}$ such that $h\left(p,q,\lambda,t\right) \geqq h\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)$ and $h\left(p,q,\lambda,t\right) \neq h\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) = f(\overline{x}) + (g \circ A)(\overline{x})$. But this is a contradiction to the weak duality theorem (Theorem 6). ■

Remark 4. In the particular case when $n = 1$ (we denote f_1 and g_1 by f and g, respectively) our dual proves to be exactly the classical Fenchel dual problem (cf. [11]) to the primal scalar problem

$$\inf_{x\in\mathbb{R}^n} (f(x) + g(Ax)).$$

In this case $\lambda_1 > 0$, $t_1 = 0$ and denoting $p := p_1$ and $q := q_1$, the dual becomes

$$\sup_{\substack{p\in\mathbb{R}^n,\, q\in\mathbb{R}^k \\ p+A^*q=0}} \left\{ -f^*(p) - g^*(q) \right\},$$

which is nothing else than

$$\sup_{q\in\mathbb{R}^k} \left\{ -f^*(-A^*q) - g^*(q) \right\}.$$

This means that we have obtained here a natural generalization of the classical Fenchel duality for multiobjective optimization problems.

5 Other two Fenchel-type vector dual problems

In this section we introduce two other Fenchel-type dual problems for the problem (P^A), when $A : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be the identical operator. The first vector dual problem, denoted by (D_1), is a particular case of the one introduced by Breckner and Kolumbán in [6], while the second, denoted by (D_2), is constructed by making a slight change in the feasible set of (D_1). The same idea was used by Jahn in [8] when introducing a Lagrange-type vector dual for the multiobjective optimization problem with inequality constraints. As previously mentioned, we consider the primal problem

$$(P) \qquad v\text{-}\min_{x \in \mathbb{R}^n} (f(x) + g(x))$$

in the framework presented in Section 2, when $A : \mathbb{R}^n \to \mathbb{R}^n$ is the identical operator. The constraint qualification (CQ^A) becomes

$$(CQ) \qquad \bigcap_{i \in I} \text{dom}(f_i) \quad \cap \quad \bigcap_{j \in J} \text{dom}(g_j) \quad \cap$$

$$\bigcap_{l \in \{1,\dots,m\} \setminus I} \text{ri}\,(\text{dom}\,(f_l)) \quad \cap \quad \bigcap_{t \in \{1,\dots,m\} \setminus J} \text{ri}\,(\text{dom}\,(g_t)) \neq \emptyset.$$

The vector dual problem of (P), introduced in [6], is nothing else than

$$(D_1) \qquad v - \max_{(\lambda,p,d) \in \mathcal{B}_1} h^1(\lambda, p, d)$$

with the objective function $h^1(\lambda, p, d) = d$, and the feasible set

$$\mathcal{B}_1 = \left\{ \begin{array}{c} (\lambda, p, d) \in \text{int}(\mathbb{R}^m_+) \times \mathbb{R}^n \times \mathbb{R}^m : \\ \\ \lambda^T d = -\left(\sum\limits_{i=1}^m \lambda_i f_i \right)^* (p) - \left(\sum\limits_{i=1}^m \lambda_i g_i \right)^* (-p) \end{array} \right\}.$$

The second vector dual problem we introduce in this section is

$$(D_2) \qquad v - \max_{(\lambda,p,d) \in \mathcal{B}_2} h^2(\lambda, p, d)$$

with the objective function $h^2(\lambda, p, d) = d$, and the feasible set

$$\mathcal{B}_2 = \left\{ \begin{array}{c} (\lambda, p, d) \in \text{int}(\mathbb{R}^m_+) \times \mathbb{R}^n \times \mathbb{R}^m : \\ \\ \lambda^T d \leq -\left(\sum\limits_{i=1}^m \lambda_i f_i \right)^* (p) - \left(\sum\limits_{i=1}^m \lambda_i g_i \right)^* (-p) \end{array} \right\}.$$

The weak and strong duality theorems for the vector dual problem (D_1) are particular cases of Proposition 2.1 and Theorem 3.1 in [6], respectively.

Theorem 8 *(vector weak duality for (D_1)) There exist no $x \in \mathbb{R}^n$ and no $(\lambda, p, d) \in \mathcal{B}_1$ such that $d \geqq f(x) + g(x)$ and $d \neq f(x) + g(x)$.*

Theorem 9 *(vector strong duality for (D_1)) If (CQ) is fulfilled and $\overline{x} \in \mathbb{R}^n$ is a properly efficient solution to (P), then there exists $\left(\overline{\lambda}, \overline{p}, \overline{d}\right) \in \mathcal{B}_1$, efficient solution to (D_1), and*

$$f(\overline{x}) + g(\overline{x}) = \overline{d}.$$

holds.

Let us now prove the weak and strong duality theorems for the multiobjective dual (D_2).

Theorem 10 *(vector weak duality for (D_2)) There exist no $x \in \mathbb{R}^n$ and no $(\lambda, p, d) \in \mathcal{B}_2$ such that $d \geqq f(x) + g(x)$ and $d \neq f(x) + g(x)$.*

Proof. We proceed by contradiction, supposing that there exist $x \in \mathbb{R}^n$ and $(\lambda, p, t) \in \mathcal{B}_2$ such that $d \geqq f(x) + g(x)$ and $d \neq f(x) + g(x)$. This means that for all $i \in \{1, ..., m\}$, $d_i \geq f_i(x) + g_i(x)$, and that for at least one $j \in \{1, ..., m\}$, $d_j > f_j(x) + g_j(x)$. Thus

$$\lambda^T d > \sum_{i=1}^{m} \lambda_i \bigg(f_i(x) + g_i(x) \bigg).$$

On the other hand, due to the fact that $(\lambda, p, t) \in \mathcal{B}_2$,

$$\lambda^T d \leq -\left(\sum_{i=1}^{m} \lambda_i f_i\right)^* (p) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^* (-p).$$

From the two inequalities above we obtain

$$-\left(\sum_{i=1}^{m} \lambda_i f_i\right)^* (p) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^* (-p) > \left(\sum_{i=1}^{m} \lambda_i f_i\right)(x) + \left(\sum_{i=1}^{m} \lambda_i g_i\right)(x),$$

thus

$$-\left(\sum_{i=1}^{m} \lambda_i f_i\right)^* (p) - \left(\sum_{i=1}^{m} \lambda_i f_i\right)(x) + p^T x -$$

$$-\left(\sum_{i=1}^{m} \lambda_i g_i\right)^* (-p) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)(x) + (-p)^T x > 0.$$

But Fenchel-Young's inequality states that

$$-\left(\sum_{i=1}^{m} \lambda_i f_i\right)^* (p) - \left(\sum_{i=1}^{m} \lambda_i f_i\right)(x) + p^T x \leq 0$$

and

$$-\left(\sum_{i=1}^{m}\lambda_i g_i\right)^{*}(-p)-\left(\sum_{i=1}^{m}\lambda_i g_i\right)(x)+(-p)^{T}x\leq 0,$$

thus their sum must be less than or equal to zero, whereas, from our assumption it is greater than zero. In this way we have reached a contradiction. ■

Theorem 11 *(vector strong duality for (D_2)) If (CQ) is fulfilled and $\overline{x}\in\mathbb{R}^n$ is a properly efficient solution to (P), then there exists $\left(\overline{\lambda},\overline{p},\overline{d}\right)\in\mathcal{B}_2$, efficient solution to (D_2), and*

$$f\left(\overline{x}\right)+g\left(\overline{x}\right)=\overline{d}$$

holds.

Proof. Let $\overline{x}$ be a properly efficient solution to (P). Then, according to Definition 3, there exists $\overline{\lambda}\in\operatorname{int}(\mathbb{R}^m_+)$ such that $\overline{x}$ is an efficient solution to the scalar optimization problem

$$(P_{\overline{\lambda}})\qquad \inf_{x\in\mathbb{R}^n}\sum_{i=1}^{m}\overline{\lambda}_i\Big(f_i(x)+g_i(x)\Big).$$

As we are working under the assumption that (CQ) holds, Theorem 4 ensures the existence of an efficient solution to $(D_{\overline{\lambda}})$ (we denote by $(D_{\overline{\lambda}})$ the dual $\left(D^{A}_{\overline{\lambda}}\right)$ in case A is the identity of $\mathbb{R}^n$) $(\overline{p},\overline{q})$, $\overline{p}=(\overline{p}_1,...,\overline{p}_m)\in\mathbb{R}^n\times...\times\mathbb{R}^n$, $\overline{q}=(\overline{q}_1,...,\overline{q}_m)\in\mathbb{R}^n\times...\times\mathbb{R}^n$, such that $\sum\limits_{i=1}^{m}\bar{\lambda}_i(\bar{p}_i+\bar{q}_i)=0$ and

$$\sum_{i=1}^{m}\bar{\lambda}_i\Big(f_i\left(\overline{x}\right)+g_i\left(\overline{x}\right)\Big)=\sum_{i=1}^{m}\bar{\lambda}_i\Big(-f_i^{*}\left(\overline{p}_i\right)-g_i^{*}\left(\overline{q}_i\right)\Big)$$

$$\leq-\left(\sum_{i=1}^{m}\overline{\lambda}_i f_i\right)^{*}\left(\sum_{i=1}^{m}\overline{\lambda}_i\overline{p}_i\right)-\left(\sum_{i=1}^{m}\overline{\lambda}_i g_i\right)^{*}\left(\sum_{i=1}^{m}\overline{\lambda}_i\overline{q}_i\right).$$

Defining

$$\overline{d}_i:=f_i(\overline{x})+g_i(\overline{x})\qquad\forall i\in\{1,...,m\},$$

we have

$$\sum_{i=1}^{m}\overline{\lambda}_i\overline{d}\leq-\left(\sum_{i=1}^{m}\overline{\lambda}_i f_i\right)^{*}\left(\sum_{i=1}^{m}\overline{\lambda}_i\overline{p}_i\right)-\left(\sum_{i=1}^{m}\overline{\lambda}_i g_i\right)^{*}\left(-\sum_{i=1}^{m}\overline{\lambda}_i\overline{p}_i\right),$$

therefore $\left(\overline{\lambda},\sum\limits_{i=1}^{m}\overline{\lambda}_i\overline{p}_i,\overline{d}\right)\in\mathcal{B}_2$. Since $\overline{d}=f(\overline{x})+g(\overline{x})$, the efficiency of $\overline{d}$ follows from Theorem 10. ■

6 A comparison of the image sets of the duals

Throughout this section we assume that for $i \in I$ and $j \in J$ the functions f_i and g_j are proper polyhedral, respectively, while for $l \in \{1, ..., m\} \backslash I$ and $t \in \{1, ..., m\} \backslash J$ the functions f_l, g_t are proper convex, respectively. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the identical operator. The vector dual which follows from (D^A) will be denoted by (D). Furthermore, we assume that the constraint qualification (CQ) is satisfied.

Proposition 12 *The following relations among the image sets of the three duals hold:*

$$h^1(\mathcal{B}_1) \subseteq h(\mathcal{B}) \cap \mathbb{R}^m \subseteq h^2(\mathcal{B}_2).$$

Proof. We start with the first relation of inclusion. Let $d \in h^1(\mathcal{B}_1)$. Then there exists $\lambda \in \text{int}(\mathbb{R}^m_+)$ and $p \in \mathbb{R}^n$ such that $(\lambda, p, d) \in \mathcal{B}_1$. Furthermore,

$$\sum_{i=1}^m \lambda_i d_i = -\left(\sum_{i=1}^m \lambda_i f_i\right)^* (p) - \left(\sum_{i=1}^m \lambda_i g_i\right)^* (-p).$$

Since (CQ) is fulfilled, we can apply Theorem 1 and relation (1), obtaining thus the existence of $p_i \in \mathbb{R}^n, q_i \in \mathbb{R}^n$, $i \in \{1, ..., m\}$, such that $\sum\limits_{i=1}^m \lambda_i p_i = p$, $\sum\limits_{i=1}^m \lambda_i q_i = -p$ and

$$\left(\sum_{i=1}^m \lambda_i f_i\right)^* (p) = \sum_{i=1}^m \lambda_i f_i^*(p_i) \text{ and } \left(\sum_{i=1}^m \lambda_i g_i\right)^* (-p) = \sum_{i=1}^m \lambda_i g_i^*(q_i).$$

Therefore

$$\lambda^T d = -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \sum_{i=1}^m \lambda_i g_i^*(q_i).$$

For

$$t_i := d_i + f_i^*(p_i) + g_i^*(q_i) \quad \forall i \in \{1, ..., m\},$$

we have that $\sum\limits_{i=1}^m \lambda_i (p_i + q_i) = 0$ and

$$\sum_{i=1}^m \lambda_i t_i = \lambda^T d + \sum_{i=1}^m \lambda_i f_i^*(p_i) + \sum_{i=1}^m \lambda_i g_i^*(q_i) = 0.$$

Then $(p, q, \lambda, t) \in \mathcal{B}$ and for all $i \in \{1, ..., m\}$, $h_i(p, q, \lambda, t) = d_i$, thus $d = h(p, q, \lambda, t) \in h(\mathcal{B}) \cap \mathbb{R}^m$. Hence $h^1(\mathcal{B}_1) \subseteq h(\mathcal{B}) \cap \mathbb{R}^m$.

We come now to the second relation of inclusion. Let $(p, q, \lambda, t) \in \mathcal{B}$ be such that $h(p, q, \lambda, t) \in h(\mathcal{B}) \cap \mathbb{R}^m$. For $\overline{p} := \sum\limits_{i=1}^m \lambda_i p_i$ and $d := h(p, q, \lambda, t)$ we have

$$\begin{aligned}
\lambda^T d &= \lambda^T h(p,q,\lambda,t) = \sum_{i=1}^m \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) + t_i \Big) \\
&= \sum_{i=1}^m \lambda_i \left(-f_i^*(p_i) - g_i^*(q_i) \right) \leq \sup \left\{ - \sum_{i=1}^m \lambda_i f_i^*(p_i) : \sum_{i=1}^m \lambda_i p_i = \overline{p} \right\} \\
&+ \sup \left\{ - \sum_{i=1}^m \lambda_i g_i^*(q_i) : \sum_{i=1}^m \lambda_i q_i = -\overline{p} \right\} \\
&= - \left(\sum_{i=1}^m \lambda_i f_i \right)^* (\overline{p}) - \left(\sum_{i=1}^m \lambda_i g_i \right)^* (-\overline{p}),
\end{aligned}$$

where the last equalities follow from Theorem 1. Hence $(\lambda, \overline{p}, d) \in \mathcal{B}_2$ and $h(p,q,\lambda,t) = d \in h^2(\mathcal{B}_2)$. Thus $h(\mathcal{B}) \cap \mathbb{R}^m \subseteq h^2(\mathcal{B}_2)$ ∎

In the following we give some examples which prove that the inclusions among the image sets in Proposition 12 are in general strict, i.e.

$$h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2).$$

Example 13 Consider the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x) = (x-1, -x-1)^T \quad \text{and} \quad g(x) = (x, -x)^T \text{ for all } x \in \mathbb{R}.$$

We prove that $h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2)$.

Since

$$f_1^*(p) = \begin{cases} 1, & \text{if } p = 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad f_2^*(p) = \begin{cases} 1, & \text{if } p = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$g_1^*(p) = \begin{cases} 0, & \text{if } p = 1, \\ +\infty, & \text{otherwise,} \end{cases}, \quad g_2^*(p) = \begin{cases} 0, & \text{if } p = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

one has

$$(f_1 + f_2)^*(p) = \inf \{ f_1^*(p_1) + f_2^*(p_2) : p_1 + p_2 = p \} = \begin{cases} 2, & \text{if } p = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$(g_1 + g_2)^*(p) = \inf \{ g_1^*(p_1) + g_2^*(p_2) : p_1 + p_2 = p \} = \begin{cases} 0, & \text{if } p = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $\lambda = (1,1)^T$, $p = 0$ and $d = (-2,-2)^T$, there is $(\lambda, p, d) \in \mathcal{B}_2$ and $d \in h^2(\mathcal{B}_2)$ since

$$\lambda^T d = -2 - 2 = -4 < -2 = -(f_1 + f_2)^*(p) - (g_1 + g_2)^*(-p).$$

We show now that $d \notin h(\mathcal{B})$. Let us suppose by contradiction that there exists $(p', q', \lambda', t') \in \mathcal{B}$ such that $h(p', q', \lambda', t') = d$. This means

$$h_i(p', q', \lambda', t') = -f_i^*(p_i') - g_i^*(q_i') + t_i' = -2 \text{ for } i \in \{1,2\}.$$

Taking into account the values we got for the conjugate of the functions involved, the equalities above hold only if

$$p_1' = 1,\ p_2' = -1,\ q_1' = 1 \text{ and } q_2' = -1.$$

Moreover, $\sum_{i=1}^{2} \lambda_i'(p_i' + q_i') = 0$, which means that $\lambda_1' - \lambda_2' = 0$. We obtain thus

$$-f_i^*(p_i') - g_i^*(q_i') + t_i' = -1 + t_i' = -2 \text{ for } i \in \{1,2\}, \text{ meaning that } t_1' = t_2' = -1.$$

Since we have supposed that $(p', q', \lambda', t') \in \mathcal{B}$, $\sum_{i=1}^{2} \lambda_i' t_i' = -\lambda_1' - \lambda_2' = -2\lambda_1'$ must hold. This is a contradiction due to the fact that $\lambda' \in \text{int}(\mathbb{R}_+^2)$.

Thus, for $d = (-2,-2)^T \in h^2(\mathcal{B}_2)$, there exists no $(p', q', \lambda', t') \in \mathcal{B}$ such that $h(p', q', \lambda', t') = d$, which shows that $h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2)$. □

Example 14 Consider now the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x) = \left(2x^2 - 1, x^2\right)^T \quad \text{and} \quad g(x) = (-2x, -x+1)^T \text{ for all } x \in \mathbb{R}.$$

We prove that $h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m$.

For $p = (3,0), q = (-2,-1), \lambda = (1,1)^T$ and $t = \left(\frac{3}{8}, -\frac{3}{8}\right)^T$ we have both relations $\sum_{i=1}^{2} \lambda_i(p_i + q_i) = 0$ and $\sum_{i=1}^{2} \lambda_i t_i = 0$ fulfilled. Thus $(p, q, \lambda, t) \in \mathcal{B}$.

$$f_1^*(3) = \sup_{x \in \mathbb{R}} \left\{3x - 2x^2 + 1\right\} = \frac{17}{8},\ f_2^*(0) = \sup_{x \in \mathbb{R}} \left\{-x^2\right\} = 0,$$

$$g_1^*(-2) = \sup_{x \in \mathbb{R}} \{-2x + 2x\} = 0,\ g_2^*(-1) = \sup_{x \in \mathbb{R}} \{-x + x - 1\} = -1.$$

Hence

$$h_1(p, q, \lambda, t) = -\frac{17}{8} - 0 + \frac{3}{8} = -\frac{14}{8},\ h_2(p, q, \lambda, t) = 0 + 1 - \frac{3}{8} = \frac{5}{8}.$$

Suppose now that there exists $(\lambda', p', d') \in \mathcal{B}_1$ such that $d' = h(p, q, \lambda, t) = \left(-\frac{14}{8}, \frac{5}{8}\right)^T$. Then

$$\lambda'^T d' = -\left(\sum_{i=1}^{2} \lambda_i' f_i\right)^* (p') - \left(\sum_{i=1}^{2} \lambda_i' g_i\right)^* (-p'). \tag{3}$$

But

$$-\left(\sum_{i=1}^{2}\lambda_i' f_i\right)^*(p')-\left(\sum_{i=1}^{2}\lambda_i' g_i\right)^*(-p')$$

$$=\inf_{x\in\mathbb{R}}\left\{-p'x+x^2\left(2\lambda_1'+\lambda_2'\right)-\lambda_1'\right\}+\inf_{x\in\mathbb{R}}\left\{x\left(p'-2\lambda_1'-\lambda_2'\right)+\lambda_2'\right\}.$$

Since

$$\inf_{x\in\mathbb{R}^n}\left\{x\left(p'-2\lambda_1'-\lambda_2'\right)\right\}=\begin{cases}0, & \text{if } p'-2\lambda_1'-\lambda_2'=0,\\ -\infty, & \text{otherwise,}\end{cases}$$

and because of $\lambda'^T d'$ is finite, there must be $p'=2\lambda_1'-\lambda_2'$, implying

$$-\left(\sum_{i=1}^{2}\lambda_i' f_i\right)^*(p')-\left(\sum_{i=1}^{2}\lambda_i' g_i\right)^*(-p')=$$

$$=\inf_{x\in\mathbb{R}}\left\{-\left(2\lambda_1'+\lambda_2'\right)x+x^2\left(2\lambda_1'+\lambda_2'\right)\right\}-\lambda_1'+\lambda_2'=-\frac{2\lambda_1'+\lambda_2'}{4}-\lambda_1'+\lambda_2'.$$

By (3) we obtain that

$$-\frac{14}{8}\lambda_1'+\frac{5}{8}\lambda_2'=-\frac{2\lambda_1'+\lambda_2'}{4}-\lambda_1'+\lambda_2'$$

which is equivalent to

$$-\frac{3\left(2\lambda_1'+\lambda_2'\right)}{8}=-\frac{2\lambda_1'+\lambda_2'}{4},\ \text{i.e. } 2\lambda_1'+\lambda_2'=0,$$

obviously a contradiction to $\lambda'\in\operatorname{int}(\mathbb{R}^2_+)$. Therefore, for $(p,q,\lambda,t)\in\mathcal{B}$ chosen as above, there exists no $(\lambda',p',d')\in\mathcal{B}_1$ such that $d'=h(p,q,\lambda,t)$. Hence $h^1(\mathcal{B}_1)\underset{\neq}{\subset}h(\mathcal{B})\cap\mathbb{R}^m$. □

In what follows, we study the relations among the sets of maximal elements of the image sets. They are defined as

$$v-\max h(\mathcal{B})=\left\{\begin{array}{c} d\in\mathbb{R}^m:\exists(p,q,\lambda,t)\in\mathcal{B}\text{ efficient to }(D),\\ \text{such that}\quad d=h(p,q,\lambda,t)\end{array}\right\}$$

for the problem (D), while $v-\max h^1(\mathcal{B}_1)$ and $v-\max h^2(\mathcal{B}_2)$, respectively, are defined analogously.

Theorem 15 *It holds*

$$v-\max h^1\left(\mathcal{B}_1\right)=v-\max h^2\left(\mathcal{B}_2\right).$$

Proof. "$v-\max h^1(\mathcal{B}_1)\subseteq v-\max h^2(\mathcal{B}_2)$" Let $\left(\overline{\lambda},\overline{p},\overline{d}\right)\in\mathcal{B}_1$ be such that $\overline{d}\in v-\max h^1(\mathcal{B}_1)$. We suppose that $\overline{d}\notin v-\max h^2(\mathcal{B}_2)$. Since $\overline{d}\in h^2(\mathcal{B}_2)$, there exists $(\lambda, p, d) \in \mathcal{B}_2$ such that $d \geqq \overline{d}$ and $d \neq \overline{d}$. Further, $(\lambda, p, d) \in \mathcal{B}_1$ would contradict that $\overline{d} \in v - \max h^1(\mathcal{B}_1)$, i.e. $(\lambda, p, d) \notin \mathcal{B}_1$ which means that

$$\lambda^T d < -\left(\sum_{i=1}^{m} \lambda_i f_i\right)^*(p) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^*(-p).$$

Thus there exists $\widetilde{d} \in d + \mathbb{R}^m_+ \backslash \{0\}$ such that

$$\lambda^T \widetilde{d} = -\left(\sum_{i=1}^{m} \lambda_i f_i\right)^*(p) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^*(-p).$$

It follows that $\left(\lambda, p, \widetilde{d}\right) \in \mathcal{B}_1$. But, in this case $\widetilde{d} \geqq \overline{d}$ and $\widetilde{d} \neq \overline{d}$, which is a contradiction to the maximality of $\overline{d}$ in $h^1(\mathcal{B}_1)$. Therefore we must have

$$v - \max h^1(\mathcal{B}_1) \subseteq v - \max h^2(\mathcal{B}_2).$$

"$v - \max h^2(\mathcal{B}_2) \subseteq v - \max h^1(\mathcal{B}_1)$". Let $\left(\overline{\lambda}, \overline{p}, \overline{d}\right) \in \mathcal{B}_2$ be such that $\overline{d} \in v - \max h^2(\mathcal{B}_2)$. We start by proving that $\left(\overline{\lambda}, \overline{p}, \overline{d}\right) \in \mathcal{B}_1$. Assuming the contrary, one has $\overline{\lambda}^T \overline{d} < -\left(\sum\limits_{i=1}^{m} \overline{\lambda}_i f_i\right)^*(\overline{p}) - \left(\sum\limits_{i=1}^{m} \overline{\lambda}_i g_i\right)^*(-\overline{p})$. But in this case there exists $\widetilde{d} \in \overline{d} + \mathbb{R}^m_+ \backslash \{0\}$, such that

$$\overline{\lambda}^T \overline{d} < \overline{\lambda}^T \widetilde{d} = -\left(\sum_{i=1}^{m} \overline{\lambda}_i f_i\right)^*(\overline{p}) - \left(\sum_{i=1}^{m} \overline{\lambda}_i g_i\right)^*(-\overline{p}).$$

As $\left(\overline{\lambda}, \overline{p}, \widetilde{d}\right) \in \mathcal{B}_2$ and $\widetilde{d} \in \overline{d} + \mathbb{R}^m_+ \backslash \{0\}$ we have obtained a contradiction to the maximality of $\overline{d}$ in $h^2(\mathcal{B}_2)$ for (D_2). Therefore $\overline{d} \in h_1(\mathcal{B}_1)$.

Let us suppose now that $\overline{d} \notin v - \max h^1(\mathcal{B}_1)$. Then there exists $(\lambda, p, d) \in \mathcal{B}_1$ such that $d \geqq \overline{d}$ and $d \neq \overline{d}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}_2$, $d \in h^2(\mathcal{B}_2)$ and we obtain a contradiction to the maximality of $\overline{d}$ in $h^2(\mathcal{B}_2)$. Thus

$$v - \max h^2(\mathcal{B}_2) \subseteq v - \max h^1(\mathcal{B}_1).$$

■

Remark 5. Let us emphasize the fact that in the proof of Theorem 15 the convexity assumptions on the functions involved and the fulfillment of the constraint qualification (CQ) are not used. Thus the sets of maximal elements of the problems (D_1) and (D_2) are always identical.

Theorem 16 It holds

$$v - \max h(\mathcal{B}) = v - \max h^2(\mathcal{B}_2).$$

Proof. "$v - \max h(\mathcal{B}) \subseteq v - \max h^2(\mathcal{B}_2)$" Let $\overline{d} \in v - \max h(\mathcal{B})$. Since $h(\mathcal{B}) \cap \mathbb{R}^m \subseteq h^2(\mathcal{B}_2)$, one has $\overline{d} \in h^2(\mathcal{B}_2)$. Let us suppose by contradiction that $\overline{d} \notin v - \max h^2(\mathcal{B}_2)$. Then there exists $d \in h^2(\mathcal{B}_2)$, with $(\lambda, p, d) \in \mathcal{B}_2$, such that $d \geqq \overline{d}$ and $d \neq \overline{d}$. Then we have

$$\lambda^T \overline{d} < \lambda^T d \leq -\left(\sum_{i=1}^m \lambda_i f_i\right)^*(p) - \left(\sum_{i=1}^m \lambda_i g_i\right)^*(-p).$$

So, there exists $\widetilde{d}$ such that $\widetilde{d} \geqq d$ (obviously, $\widetilde{d} \geqq \overline{d}$ and $\widetilde{d} \neq \overline{d}$), for which

$$\lambda^T \widetilde{d} = -\left(\sum_{i=1}^m \lambda_i f_i\right)^*(p) - \left(\sum_{i=1}^m \lambda_i g_i\right)^*(-p).$$

Thus we have obtained an element $\left(\lambda, p, \widetilde{d}\right) \in \mathcal{B}_1$. Since $h^1(\mathcal{B}_1) \subseteq h(\mathcal{B})$, it follows that $\widetilde{d} \in h(\mathcal{B})$ which contradicts the maximality of $\overline{d}$ in $h(\mathcal{B})$. Therefore,

$$v - \max h(\mathcal{B}) \subseteq v - \max h^2(\mathcal{B}_2).$$

"$v - \max h^2(\mathcal{B}_2) \subseteq v - \max h(\mathcal{B})$" Let $\overline{d} \in v - \max h^2(\mathcal{B}_2)$. By Theorem 15 follows that $\overline{d} \in v - \max h^1(\mathcal{B}_1)$. Since $h^1(\mathcal{B}_1) \subseteq h(\mathcal{B})$, we have further $\overline{d} \in h(\mathcal{B})$. Let us suppose that there exists $(p, q, \lambda, t) \in \mathcal{B}$ such that $h(p, q, \lambda, t) = d \geqq \overline{d}$ and $h(p, q, \lambda, t) \neq \overline{d}$. Since $h(\mathcal{B}) \cap \mathbb{R}^m \subseteq h^2(\mathcal{B}_2)$, one has $d \in h^2(\mathcal{B}_2)$, $d \geqq \overline{d}$ and $d \neq \overline{d}$, which is a contradiction to the maximality of $\overline{d}$ in $h^2(\mathcal{B}_2)$. Therefore

$$v - \max h^2(\mathcal{B}_2) \subseteq v - \max h(\mathcal{B}).$$

■

From the theorems 15 and 16 we can conclude that under the assumptions we made at the beginning of this section the sets of maximal elements of the image sets of the three duals (D), (D_1) and (D_2) coincide. Thus

$$v - \max h^1(\mathcal{B}_1) = v - \max h(\mathcal{B}) = v - \max h^2(\mathcal{B}_2)$$

even though

$$h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2).$$

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