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Impact of monotonicity in some model of inverse option pricing

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Abstract

The paper considers a problem of inverse option pricing aimed at the identification of a not directly observable volatility function. Although volatilities are in general assumed to be functions of time and the current price of the underlying asset the study of purely time-dependent volatility functions in combination with maturity-dependent option prices is of interest. In the latter situation an important aspect is the calibration of the antiderivative of the volatility. This inverse problem leads to an operator equation with a forward operator of Nemytskii type generated by a monotone function of two variables. In recent literature an analysis of this forward operator and several numerical case studies have been conducted which revealed certain instability effects. Consequently, the applicability of several regularization approaches has been discussed.

This paper supplements these results by answering some open questions. As the main advancement we show that the considered inverse problem is indeed *well-posed* in the Banach space of continuous functions over a certain time interval in the sense that the *inverse operator is continuous*. The proof is based on the special Nemytskii type of the forward operator and the monotonicity of its generating function. This result classifies the occuring instabilities as ill-conditioning phenomena.

In practice the above-mentioned ill-conditioning effects result in strongly oscillating solutions. Therefore, we study the stabilizing effect of apriori information concerning the monotonicity of the data and the searched solution. Using again the special structure of the forward operator we propose a numerically efficient algorithm for the computation of a strictly monotonically increasing approximate solution. Additionally, the situation of discrete stochastic noise is discussed. The described algorithms are illustrated by numerical case studies.

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1 Introduction

For the correct pricing of options the calibration of the parameters in the underlying asset price model is of key importance. One of the most well-known and widely accepted asset price models is the generalized Black-Scholes model. It assumes the price process P_t of an asset to satisfy the stochastic differential equation

$$dP_t = \mu P_t\, dt + \sigma(P_t, t) P_t\, dW_t \qquad t \in [0, \overline{T}]\,, \tag{1.1}$$

with a volatility function $\sigma(P_t, t) > 0$ depending on the current asset price P_t and the time t, a drift coefficient μ and a standard Wiener process W_t. Here, $\overline{T} > 0$ denotes the maximal time horizon of interest. Furthermore, the existence of a bond with constant interest rate r is assumed. Provided trading is frictionless and continuous and the asset price follows (1.1) it is well-known that at time $t = 0$ the price $C(P_0, 0, K, T)$ of a European call option with strike $K > 0$ and maturity $T \geq 0$ on this asset satisfies the Dupire equation (cf. e.g., [1])

$$\begin{aligned} C_T &= \tfrac{1}{2}\sigma^2(K,T)K^2 C_{KK} - rKC_K & (K,T) &\in \mathbb{R}_+ \times (0, \overline{T}], \\ C(P_0, 0; K, 0) &= \max(P_0 - K, 0) & K &\in \mathbb{R}_+\,. \end{aligned} \tag{1.2}$$

Note that this holds also true if the drift μ in (1.1) is not constant but an arbitrary function of P_t and other economic variables. Of course, $\mu(\cdot)$ must still satisfy some regularity conditions to ensure the existence of a solution of the stochastic differential equation (1.1).

As typically the available option price data do not suffice to recover a general volatility surface, in the paper [2] an ansatz of the form

$$\sigma(K,T) = \sigma_1\left(e^{-rT}K\right)\sigma_2(T) \tag{1.3}$$

with a volatility smile $\sigma_1(Y)$ and a term structure $\sigma_2(T)$ is suggested. In order to make this decomposition unique the function σ_2 is standardized by imposing the condition

$$\int_0^{\overline{T}} \sigma_2^2(t)\, dt = 1\,. \tag{1.4}$$

By introducing the notation $Y := Ke^{-rT}$, $S(T) := \int_0^T \sigma_2^2(t)\, dt$ and $U(Y,S) := C(P_0, 0, K, T)$ the Dupire equation (1.2) transforms into

$$\begin{aligned} U_S(Y,S) &= \tfrac{1}{2}\sigma_1^2(Y)Y^2 U_{YY}(Y,S) & (Y,S) &\in \mathbb{R}_+ \times (0,1] \\ U(Y,0) &= \max(P_0 - Y, 0) & Y &\in \mathbb{R}_+\,. \end{aligned} \tag{1.5}$$

Thus, the identification of $\sigma(K,T)$ decomposes into two separate subproblems: First the function $A(Y) := \frac{1}{2}\sigma_1^2(Y)$ has to be identified from option price data $C_1(Y) := C(K, \overline{T})$ $(K \in \mathbb{R}_+)$. Given this function $A(Y)$ the function $B(T) := \sigma_2^2(T)$ has to be identified from option price data $C_2(T) = C(K^*, T)$ with fixed strike K^* and varying maturity T. Both problems are studied in [2].

Let us now assume that the first step in this identification is done, i. e., the function $A(Y)$ and the solution $U(Y,S)$ of the transformed Dupire equation (1.5) are known. In this situation, the knowledge of $S(T)$ is necessary for pricing European Call and Put Options (or more generally claims for which the final payoff depends only on P_T). Therefore, the identification of $S(T)$ is of some importance. As data for this step one can use option prices

$$C_2(T) := C(P_0, 0; K^*, T) = U(K^* e^{-rT}, S(T)) \quad T \in (0, \overline{T}]\,, \tag{1.6}$$

for a fixed strike K^* and varying maturity T.

For a purely time-dependent volatility, i. e., the situation $\sigma_1 \equiv const$, the problem of identifying the integrals $S(T)$ on an interval $I = [0, \overline{T}]$ from observed option prices $C_2(T)$ has been discussed as outer problem in [5] and [4]. In [5], it has been emphasized, that this problem shows an instability, which leads in practice to approximate solutions S^δ having strong oscillations near the point $T = 0$. However, the question whether this instability is due to ill-posedness or due to ill-conditioning has been left open. In order to overcome the instability and suppress the oscillations, a descriptive regularization method, using the monotonicity of S as apriori information, was introduced and studied in $L^p(I)$ spaces. This approach chooses the regularized solution as minimizer of a least-squares problem over a compact subset $D_+^\kappa \subset L^p(I)$ containing monotonically increasing functions.

The present paper extends these studies with respect to the following three points. First of all, as *advancement* to [5, section 3], we show that the inverse problem is well-posed in the banach space $\mathcal{C}(I)$ of continuous functions over the interval I in the sense that the *inverse operator is continuous*. This result specifies the above mentioned instability near the point $T = 0$ as *ill-conditioning effect*. Secondly, for a given data function $C_2^\delta \in \mathcal{C}(I)$ with known noise level δ we propose a numerically efficient algorithm for the computation of a strictly monotonically increasing approximate solution $S^\delta \in \mathcal{C}(I)$. This algorithm relies on the one hand on the Nemytskii type of the forward operator, which means that $[N(S)](T)$ is uniquely determined by the value $S(T)$. On the other hand it relies on the monotonicity of the function $U(Y,T)$ with respect to Y and T. Last but not least, the situation of discrete stochastic noise is examined.

Admittedly, parameter estimation in a model where the volatility depends solely on time is much simpler than in the general situation (1.1). Nevertheless, it is an important benchmark to gain some insight on the nature of ill-posedness (cf. also [8]). Moreover, the following considerations show that the main analytic results can be extended straightforwardly to the identification of $S(T)$ from (1.6) in the model (1.1). Indeed, the result concerning the continuity of the inverse operator and the proposed algorithm are solely based on the continuity and the monotonicity of the function $U(Y,S)$, which can also be proven for the general case (1.1). To be precise, if σ_1 is such that $A(Y) \in \mathcal{C}^\lambda(\overline{\mathbb{R}_+})$ holds, the problem (1.5) has a unique solution $U \in \mathcal{C}^{2,1}(\mathbb{R}_+ \times (0,\overline{T}]) \cap \mathcal{C}^{\lambda,\lambda/2}(\mathbb{R}_+ \times I)$ (cf. e.g.[2, Proposition 2.1]). Furthermore, in the proof of [2, Proposition 4.1] the strict monotonicity of $U(Y,S)$ with respect to S is proven. By a similar consideration it can also be shown that $U(Y,S)$ is monotonically decreasing with respect to Y.

The paper is organized as follows: In the remaining part of the introduction we formulate the inverse problem under consideration. In Section 2 we specify the character of instability near $T = 0$ by proving the continuity of the inverse operator. In Section 3 we propose a fast algorithm for descriptive regularization of the inverse problem using the monotonicity of the searched function. After that we go on to study the situation of discrete data with (random) noise and illustrate the algorithm by a numerical case study.

In order to make notation easier we will replace the standardization (1.4) by the assumption that the constant σ_1 is equal to one. At time $t = 0$ we denote by $P := P_0 > 0$ the current asset price and consider a family of European vanilla call options with a fixed strike $K > 0$, a fixed risk-free interest rate $r \geq 0$ and maturities T varying through the whole interval I. Using the notation

$$a(t) := \sigma_2^2(t) \quad \text{and} \quad S(T) = \int_0^T a(\tau)\, d\tau\,, \tag{1.7}$$

introduced in [5], the fair price $u(T) = C(P, 0, K, T)$ of an option with maturity T and strike K is given by

$$u(T) = U(Ke^{-rT}, S(T)) =: M(T, S(T)),$$

where the function M is given by

$$M(T,s) := u_{BS}(P,K,r,T,s) = \begin{cases} P\Phi(d_1) - Ke^{-rT}\Phi(d_2) & \text{if } s > 0 \\ \\ \max\left(P - Ke^{-rT}, 0\right) & \text{if } s = 0 \end{cases} \tag{1.8}$$

in terms of the Black-Scholes function u_{BS} introduced in [5] and [6]. In this context, Φ denotes the cumulative density function of the standard normal distribution

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx \tag{1.9}$$

and

$$d_1 := \frac{\ln\left(\frac{P}{K}\right) + rT + \frac{s}{2}}{\sqrt{s}} \quad \text{and} \quad d_2 := d_1 - \sqrt{s}\,. \tag{1.10}$$

Now let σ_2^* denote the exact volatility term structure, a^* its square and S^* denote the corresponding antiderivative obtained from a^* via formula (1.7). Instead of the fair price function

$$u^*(T) := M(T, S^*(T)) \quad T \in I$$

we observe a nonnegative data function $u^\delta(T)$ $(T \in I)$ where we assume that

$$\|u^* - u^\delta\|_{\mathcal{C}(I)} \le \delta\,. \tag{1.11}$$

Our aim is to find an appropriate approximation $S^\delta \in \mathcal{C}(I)$ of the function S^*, where the accuracy of S^δ is measured in the norm of $\mathcal{C}(I)$.

Defining the Nemytskii operator $N : D(N) \subset \mathcal{C}(I) \to \mathcal{C}(I)$ with domain

$$D(N) = \mathcal{D}_{0,+} := \{f \in \mathcal{C}(I) : f(0) = 0, f(T) > 0\ \forall T \in (0, \overline{T}]\}$$

by

$$[N(S)](T) := u_{BS}(P,K,r,T,S(T)) \quad T \in I. \tag{1.12}$$

this task can be formulated in form of the nonlinear operator equation

$$N(S^\delta) = u^\delta \tag{1.13}$$

in the Banach space $\mathcal{C}(I)$.

2 Ill-conditioning effects

We start by summarizing some properties of the function u_{BS} defined in (1.8), which will be useful in the following. For a proof we refer to [4].

Lemma 2.1
Let the parameters P, $K > 0$ and $r \ge 0$ be fixed. Then the function $u_{BS}(P,K,r,T,s)$ is nonnegative and continuous for $(T,s) \in [0,\infty] \times [0,\infty)$. Moreover, for $T \ge 0$ and $s > 0$ the following properties hold.

1. The function u_{BS} is continuously differentiable with respect to s and it holds

$$\frac{\partial u_{BS}(P,K,r,T,s)}{\partial s} = \phi(d_1)\,P\,\frac{1}{2\sqrt{s}} > 0 \tag{2.1}$$

with d_1 defined by (1.10) and $\phi(z) = \Phi'(z) = \frac{1}{2\pi}e^{-\frac{z^2}{2}}$.

2. The function u_{BS} is continuously differentiable with respect to T, where we have

$$\frac{\partial u_{BS}(P,K,r,T,s)}{\partial T} = r\,K\,e^{-rT}\Phi(d_2) \geq 0\,.$$

with d_2 defined by (1.10).

3. Furthermore, we find the following limit conditions

$$\lim_{s\to 0}\frac{\partial u_{BS}(P,K,r,T,s)}{\partial s} = \begin{cases}\infty & \text{if } P = K\,e^{-rT}\\ 0, & \text{else}\end{cases} \tag{2.2}$$

as well as

$$\lim_{s\to\infty} u_{BS}(P,K,r,T,s) = P\,. \tag{2.3}$$

Concerning the function $S^*(T)$ we can formulate the following results.

Remark 2.2 *Combining Lemma 2.1 with Definitions* (1.7) *and* (1.8) *we get immediately the following properties of $u^*(T) = u_{BS}(P,K,r,T,S^*(T))$. We remark that properties 1 and 2 can also be shown on more general arbitrage-free markets (cf.[3, p. 94 ff.]).*

1. For all $T > 0$ we have

$$\max(P - Ke^{-rT}, 0) < u^*(T) < P\,.$$

2. The function $u^*(T)$ is continuous and strictly monotonically increasing. Furthermore it holds

$$u^*(0) = \max(P - K, 0)\,.$$

3. For a continuous volatility function, the term structure $u^*(T)$ is continuously differentiable, where we have

$$\begin{aligned}u^{*\prime}(T) &= \frac{\partial u_{BS}}{\partial s}(P,K,r,T,S^*(T))S^{*\prime}(T) + \frac{\partial u_{BS}}{\partial T}(P,K,r,T,S^*(T))\\ &= P\phi(d_1^*(T))\frac{a^*(T)}{2\sqrt{S^*(T)}} + Kre^{-rT}\Phi(d_2^*(T)) > 0\end{aligned}$$

with d_1^* and d_2^* defined by

$$d_1^*(T) := \frac{\ln\left(\frac{P}{K}\right) + rT + \frac{S^*(T)}{2}}{\sqrt{S^*(T)}} \quad \textit{and} \quad d_2^*(T) := d_1^*(T) - \sqrt{S^*(T)}\,.$$

Especially it holds

$$u^{*\prime}(T) \geq Kre^{-rT}\Phi(d_2^*(T))\,. \tag{2.4}$$

In order to guarantee the existence of a solution $S^\delta \in \mathcal{D}_{0,+}$ of the operator equation (1.13) it is necessary that the noisy data u^δ satisfy certain conditions. In fact, in [5, p. 1325] it is shown that for all noisy data u^δ satisfying the first and second condition of Remark 2.2 with u^* replaced by u^δ there exists a uniquely defined function $S^\delta(T) \quad (T \in I)$, which satisfies the equation

$$u_{BS}(P, K, r, T, S^\delta(T)) = u^\delta(T) \quad T \in I\,. \tag{2.5}$$

Furthermore the solution S^δ is an element of $\mathcal{D}_{0,+}$ and S^δ is bounded by $0 < S^\delta \leq \overline{S}\ T \in I$ where $\overline{S}$ satisfies the equation

$$u_{BS}(P, K, r, 0, \overline{S}) = \max_{T \in I} u^\delta(T)\,. \tag{2.6}$$

Noting that the proof of this result does not make use of the strict monotonicity of u^δ one can omit this condition and formulate the following assumption guaranteeing the existence of a solution $S^\delta \in \mathcal{D}_{0,+}$ of the operator equation (1.13).

Assumption 2.3 The data function $u^\delta(T)$ is assumed to be continuous and to satisfy

$$u^\delta(0) = \max(P-K, 0) \quad \text{and} \quad \max(P-Ke^{-rT}, 0) < u^\delta(T) < P \quad \text{for all } T \in (0, \overline{T}]\,. \tag{2.7}$$

Besides, the properties of the function u_{BS} imply that the conditions (2.7) are also satisfied for any function $u \in N(\mathcal{D}_{0,+})$. Together these considerations show that Assumption 2.3 is necessary and sufficient for the existence of a function S^δ satisfying (2.5). In other words, the range of N is equal to the set

$$\begin{aligned} \mathcal{C}_{A1'} := \{f \in \mathcal{C}(I) : f(0) &= \max(P-K, 0) \text{ and} \\ &\max(P - Ke^{-rT}, 0) < f(T) < P \;\forall T \in (0, \overline{T}]\}\,. \end{aligned}$$

We proceed by reviewing results of [5] concerning pointwise estimates of the absolute error $|S^*(T) - S^\delta(T)|$. For fixed $T \in (0, \overline{T}]$ the function u_{BS} can be expanded in a Taylor series at the point $S^*(T)$. This gives

$$\begin{aligned} u^\delta(T) &= u_{BS}(P, K, r, T, S^\delta(T)) \\ &= u_{BS}(P, K, r, T, S^*(T)) + \frac{\partial u_{BS}(P, K, r, T, S_{im}(T))}{\partial s} \left(S^\delta(T) - S^*(T)\right)\,, \end{aligned}$$

where S_{im} denotes a positive intermediate function such that

$$S_{im}(T) \in [\min(S^\delta(T), S^*(T)), \max(S^\delta(T), S^*(T))]\,.$$

Because of (2.1) we can subtract $u^*(T) = u_{BS}(P, K, r, T, S^*(T))$ on both sides and divide by $\frac{\partial u_{BS}(P,K,r,T,S_{im}(T))}{\partial s}$. Taking on both sides the absolute value gives

$$|S^\delta(T) - S^*(T)| = \left(\frac{\partial u_{BS}(P, K, r, T, S_{im}(T))}{\partial s}\right)^{-1} |u^\delta(T) - u^*(T)|\,. \tag{2.8}$$

Thus, for fixed S^* and S^δ the function

$$h(T) := \left(\frac{\partial u_{BS}(P, K, r, T, S_{im}(T))}{\partial s}\right)^{-1} > 0 \quad 0 < T \leq \overline{T}$$

can be defined and interpreted as error amplification factor. From the definition of $\mathcal{D}_{0,+}$ follows

$$\lim_{T\to 0} S^*(T) = 0 \quad \text{and} \quad \lim_{T\to 0} S^\delta(T) = 0$$

and therefore also $\lim_{T\to 0} S_{im}(T) = 0$. Now, for the situation $P \neq K$ the limit condition (2.2) shows that $h(T)$ may become arbitrarily large for small T. Thus, in this situation for $T \approx 0$ small errors in the data can lead to huge errors in the approximate solution.

These considerations show also that there cannot exist any constant L, possibly depending on S^0, such that an estimate of the form

$$\|S^\delta - S^0\|_{C(I)} \leq L\|u^\delta - u^0\|_{C(I)}$$

holds. If the operator N were linear, we could now conclude that the inverse operator

$$N^{-1} : \mathcal{C}_{A1'} \subset \mathcal{C}(I) \to D(N) \subset \mathcal{C}(I)$$

is unbounded and thus not continuous. However, for the nonlinear problem the unboundedness of the error amplification factor might either be a sign of ill-conditioning or of ill-posedness effects in a neighbourhood of $T = 0$.

Thus, it remains the question whether the convergence of a sequence $u_n = N(S_n)$ to some function $u^0 = N(S^0)$ in $\mathcal{C}(I)$ implies convergence of the corresponding functions S_n to S^0 in $\mathcal{C}(I)$. In [4] pointwise convergence of $S_n(T)$ to $S^0(T)$ has already been proven. In the following we will show the convergence in $\mathcal{C}(I)$. Together with the injectivity of N this implies that the inverse operator N^{-1}, considered as mapping from the $\mathcal{C}(I)$ to $\mathcal{C}(I)$, is continuous, i. e., the inverse problem

$$N(S^\delta) = u^\delta \qquad (S^\delta \in D(N), u^\delta \in \mathcal{C}_{A1'}) \tag{2.9}$$

is well-posed.

Theorem 2.4
Let $P, K > 0$ and $r \geq 0$. Then the operator $N^{-1} : \mathcal{C}_{A1'} \subset \mathcal{C}(I) \to \mathcal{C}(I)$ is continuous, i. e.,

$$\|N(S_n) - N(S^0)\|_{\mathcal{C}(I)} \to 0 \quad \textit{implies} \quad \|S_n - S^0\|_{\mathcal{C}(I)} \to 0.$$

Proof: We will prove the result indirectly, i. e., we assume that $u_n := N(S_n)$ $(n \in \mathbb{N})$ converge to $u^0 := N(S^0)$,

$$\delta_n := \max_{T\in I} \left|u_n(T) - u^0(T)\right| \to 0\,,$$

but the sequence $\{S_n\}_{n=1}^\infty$ does not converges to S^0. Defining the distances

$$\alpha_n := \max_{T\in I} \left|S_n(T) - S^0(T)\right|$$

then there exists a constant $\varepsilon > 0$ and a strictly monotonically increasing sequence $\{n_k\}_{k=1}^\infty$, $n_k \in \mathbb{N}$ such that $\alpha_{n_k} \geq \varepsilon$ holds for all $k \geq 0$. For $k \in \mathbb{N}$ we choose $T_k \in I$ such that

$$\left|S_{n_k}(T_k) - S^0(T_k)\right| = \max_{T\in I} \left|S_{n_k}(T) - S^0(T)\right| = \alpha_{n_k} \geq \varepsilon$$

holds. For each $k \in \mathbb{N}$ two cases are possible. Either we have

$$S_{n_k}(T_k) \geq S^0(T_k) + \varepsilon \tag{2.10a}$$

or

$$0 \le S_{n_k}(T_k) \le S^0(T_k) - \varepsilon \,. \tag{2.10b}$$

As the function S^0 is continuous, the set

$$I_\varepsilon := \left\{T \in I : S^0(T) \ge \varepsilon\right\} \subset I \subset \mathbb{R}$$

is closed, bounded and thus compact.

We consider now the functions $f_1 : I \to \mathbb{R}$ and $f_2 : I_\varepsilon \to \mathbb{R}$ defined by

$$f_1(T) := u_{BS}(P, K, r, T, S^0(T) + \varepsilon) - u^0(T) \quad T \in I$$

and

$$f_2(T) := u^0(T) - u_{BS}(P, K, r, T, S^0(T) - \varepsilon) \quad T \in I_\varepsilon \,.$$

From the continuity of S^0 and the continuity of the function $u_{BS}(P, K, r, T, s)$ in T and s it follows the continuity of f_1 and f_2.

As every continuous function attains its minimum over a compact set, the values

$$m_1 := \min_{T \in I} f_1(T) \quad \text{and} \quad m_2 := \min_{T \in I_\varepsilon} f_2(T) \tag{2.11}$$

are well defined. Besides, the strict monotonicity of $u_{BS}(P, K, r, T, s)$ with respect to s implies $m_1 > 0$ and $m_2 > 0$.

For all $k \in \mathbb{N}$ satisfying (2.10a) it holds

$$\begin{aligned} \delta_{n_k} &\ge u_{BS}\left(P, K, r, T_k, S_{n_k}(T_k)\right) - u^0(T_k) \\ &\ge u_{BS}(P, K, r, T_k, S^0(T_k) + \varepsilon) - u^0(T_k) \ge m_1 > 0 \,. \end{aligned}$$

Analogously, for all $k \in \mathbb{N}$ satisfying (2.10b) it holds

$$\begin{aligned} \delta_{n_k} &\ge u^0(T_k) - u_{BS}\left(P, K, r, T_k, S_{n_k}(T_k)\right) \\ &\ge u^0(T_k) - u_{BS}(P, K, r, T_k, S^0(T_k) - \varepsilon) \ge m_2 > 0 \,. \end{aligned}$$

Thus we have shown $\delta_{n_k} \ge \min\{m_1, m_2\} > 0$ for all $k \in \mathbb{N}$, which contradicts the assumption $\delta_{n_k} \to 0$.

■

Remark 2.5 An inspection of the proof shows, that the result presented here can be generalized as follows. Let the function $k(t, s)$ satisfy the following two conditions.

- $k : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous
- For all $t \in [0, 1]$ it holds

$$k(t, s_1) < k(t, s_2) \qquad \text{whenever } s_1 < s_2 \,.$$

Then the Nemytskii operator

$$N : D(N) \subset \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$$

defined by

$$[N(S)](t) = k(t, S(t))$$

is injective and its inverse $N^{-1} : N(D(N)) \subset \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ is continuous.

In conclusion, the above analytic considerations have shown that the inverse problem is well-posed but it shows ill-conditioning effects in a neighbourhood of $T = 0$. Now we are going to illustrate these effects by a numerical case study with synthetic data. For a fixed asset price $P = 100$, a strike $K = 85$, a riskless interest rate $r = 0.05$, discrete maturities $T_i = i\frac{0.2}{100}$ $(i = 0, \ldots, N = 100)$ and a volatility function

$$a(t) = 0.04\left[0.5 + \frac{0.9}{1 + 100(2.05t - 0.2)^2}\right]$$

we computed the corresponding fair option prices $u^*(T_i)$. After that we collected these values in a vector $\mathbf{u}^* = (u^*(T_0), \ldots, u^*(T_N))$ and added a (random) noise vector to obtain a noisy data vector $\mathbf{u}^\delta$, which satisfies

$$\max_{0 \le i \le N} \left|u^*(T_i) - \mathbf{u}^\delta[i]\right| \le \delta \quad \text{with } \delta = 0.0126\,.$$

We remark that the random noise vector had been chosen in such a way that the condition

$$\max(P - Ke^{-rT_i}, 0) < \mathbf{u}^\delta[i] < P\,, \tag{2.12}$$

which can be interpreted as discrete analogue of Assumption 2.3, holds.

Next, we computed

$$\underline{u}^\delta(T) := \max(u^*(T) - \delta, P - Ke^{-rT}, 0) \quad \text{and} \quad \overline{u}^\delta(T) := u^*(T) + \delta \qquad T \in I\,.$$

In this context, it should be noted that $u^*(\overline{T}) < P - \delta$ and thus $\overline{u}^\delta(T) < P$ holds. Clearly, any function u^δ with noise level δ which satisfies Assumption 2.3 varies between $\underline{u}^\delta$ and $\overline{u}^\delta$. Figure 2.1 shows the exact data function $u^*(T)$, the noisy data $\mathbf{u}^\delta$ and the bounds $\underline{u}^\delta(T)$ and $\overline{u}^\delta(T)$. Note that we have already zoomed into the picture and displayed the functions only in the interval $[0.07, 0.13]$, otherwise the plotted functions could not be visually distinguished.

Next, we used $\mathbf{u}^\delta$ to compute $\mathbf{S}^\delta$ as solution of $u_{BS}(P, K, r, T_i, \mathbf{S}^\delta[i]) = \mathbf{u}^\delta[i]$. From the monotonicity of u_{BS} follows that the inequality $\underline{u}^\delta \le u^\delta \le \overline{u}^\delta$ implies $\underline{S}^\delta \le S^\delta \le \overline{S}^\delta$, where $\underline{S}^\delta$ and $\overline{S}^\delta$ are defined by

$$u_{BS}(P, K, r, T, \underline{S}^\delta(T)) = \underline{u}^\delta(T) \quad \text{and} \quad u_{BS}(P, K, r, T, \overline{S}^\delta(T)) = \overline{u}^\delta(T) \qquad T \in I\,. \tag{2.13}$$

Figure 2.2 compares the exact function $S(T)$ with the approximation $\mathbf{S}^\delta$ and the bounds $\underline{S}^\delta(T)$, $\overline{S}^\delta(T)$. The strong oscillations of the solution $\mathbf{S}^\delta$ and the large distance between $\underline{S}^\delta$ and $\overline{S}^\delta$ in the interval $[0, 0.1]$ are a result of the ill-conditioning effects described above.

We will now illustrate the continuity of the inverse operator, i. e., for δ tending to zero the corresponding bounds $\underline{S}^\delta$ and $\overline{S}^\delta$ converge (slowly) to S^*. To do this, we compute $\underline{u}^{\delta_k}(T) := \max(u^*(T) - \delta_k, P - Ke^{-rT}, 0)$ and $\overline{u}^{\delta_k}(T) := u^*(T) + \delta_k$ and the corresponding $\underline{S}^{\delta_k}$ and $\overline{S}^{\delta_k}$ for $\delta_k = \frac{10^{-3}}{4^k}(k = 1, \ldots, 16)$. The results are shown in Figure 2.3.

Each pointed line corresponds to one $\underline{S}^{\delta_k}$ or $\overline{S}^{\delta_k}$, respectively. Although the maximal errors $\|\underline{u}^{\delta_k} - u^*\|_{C(I)}$ and $\|\overline{u}^{\delta_k} - u^*\|_{C(I)}$ decrease by a factor $\frac{1}{4}$ the errors $\|\underline{S}^{\delta_k} - S^*\|_{C(I)}$ as well as $\|\overline{S}^{\delta_k} - S^*\|_{C(I)}$ decrease very slowly. This can also be seen in Figure 2.4, where we have plotted the errors $\|\underline{S}^{\delta_k} - S^*\|_{C(I)}$ as well as $\|\overline{S}^{\delta_k} - S^*\|_{C(I)}$ in dependence of δ_k. Note that we have used a logarithmic scale on the horizontal axis.

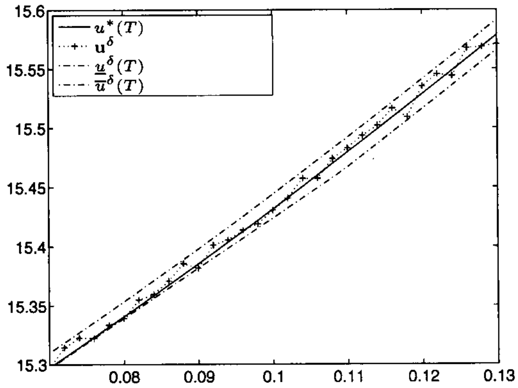

Figure 2.1: Comparison of exact and noisy data

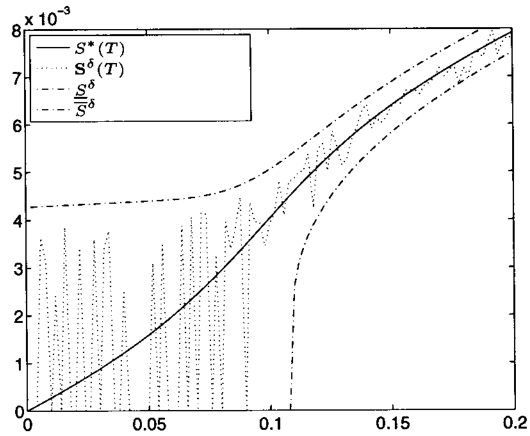

Figure 2.2: Approximate solution $\mathbf{S}^\delta$ obtained from noisy data

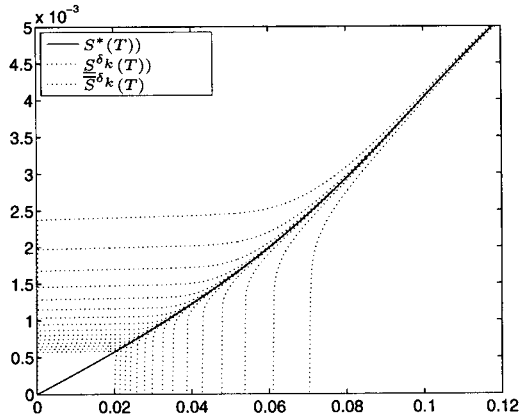

Figure 2.3: Bounds $\underline{S}^{\delta_k}$ and $\overline{S}^{\delta_k}$ restricting S^{δ_k} for $\delta_k = \frac{10^{-3}}{4^k}$ (k=1,...,16)

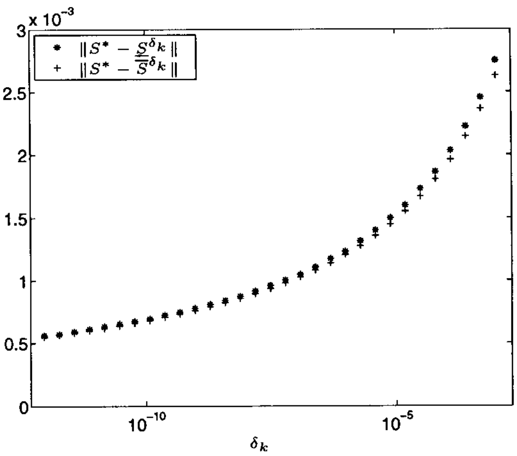

Figure 2.4: Maximal distance of $\underline{S}^{\delta_k}$ and $\overline{S}^{\delta_k}$ from S^*

With respect to the strong oscillations of S^δ which we have seen in Figure 2.1 one could conjecture that a smoothening of the data u^δ should lead to a smoothening of the corresponding solution S^δ. In order to see whether this conjecture holds true, we performed the following experiment.

Remembering that the exact data u^* are strictly monotonically increasing, we removed the (small) oscillations in the data by transforming the vector $\mathbf{u}^\delta$ into a vector $\mathbf{u}^\delta_{mon}$ satisfying

$$\mathbf{u}^\delta_{mon}[i] - \mathbf{u}^\delta_{mon}[i-1] > 0 \qquad \text{for } i = 1, \ldots, N .$$

By this monotonization the error of the data reduced slightly from

$$\max_{0 \le i \le N} \left| \mathbf{u}^\delta[i] - u(T_i) \right| = 0.0126 \quad \text{to} \quad \max_{0 \le i \le N} \left| \mathbf{u}^\delta_{mon}[i] - u(T_i) \right| = 0.0121 .$$

Figure 2.5 compares the monotonized data $\mathbf{u}^\delta_{mon}$ with the exact right hand side u^* and the original noisy data $\mathbf{u}^\delta$. It can be seen that the monotonization has removed the oscillations in the data. However, the corresponding approximate solution $\mathbf{S}^\delta_{umon}$ obtained from the data $\mathbf{u}^\delta_{mon}$ is hardly better than the original solution $\mathbf{S}^\delta$ (cf.Figure 2.6).

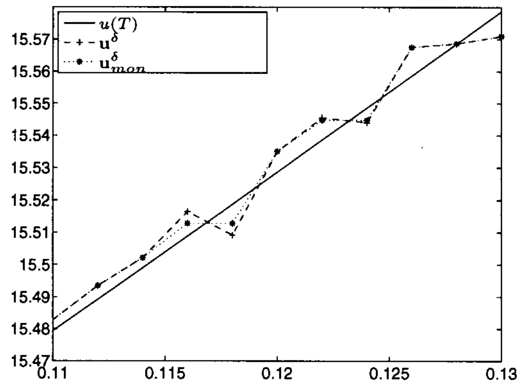

Figure 2.5: Comparison of exact data, noisy data and monotonized noisy data

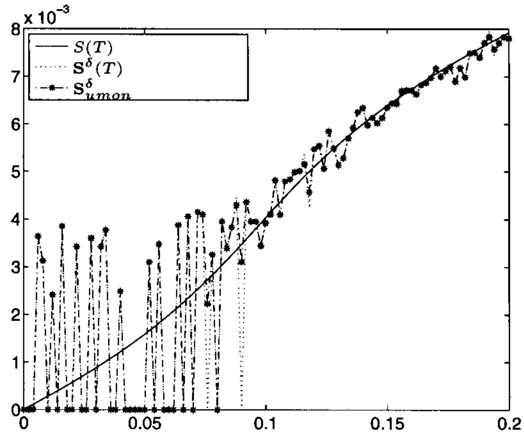

Figure 2.6: Approximate solutions obtained from noisy data and from monotonized noisy data

Next we addressed the question whether a further smoothening of the data by requiring a certain minimal slope of the data u^δ improves the corresponding solution of (1.13). To do this, we remembered (2.4) and introduced the function f_2 as follows

$$f_2(T) := Kre^{-rT}\Phi(d_2(T)) \leq u^{*\prime}(T) \qquad T \in I\,. \tag{2.14}$$

In order to find a minimal slope of u^* we assumed to be aware of certain positive bounds $\underline{c}$ and $\overline{C}$ for $a^*(t)$. Setting $\underline{S}(T) = \underline{c}T$ and $\overline{S}(T) = \overline{C}T$ we got the inequality

$$0 < \underline{S}(T) \leq S^*(T) \leq \overline{S}(T) \quad 0 < T \leq \overline{T}\,. \tag{2.15}$$

Using elementary calculations it was easy to prove the following lemma, which gives a lower bound of f_2 and therefore a minimal slope of u^*.

Lemma 2.6
Let the functions $\underline{S} : I \to \mathbb{R}$ and $\overline{S} : I \to \mathbb{R}$ be such that (2.15) holds. Defining the functions

$$\underline{d}_2(T) := \begin{cases} \frac{\ln(P/K)+rT-1/2\overline{S}(T)}{\sqrt{\overline{S}(T)}} & \text{if } \ln(P/K)+rT-1/2\overline{S}(T) \geq 0 \\ \frac{\ln(P/K)+rT-1/2\overline{S}(T)}{\sqrt{\underline{S}(T)}} & \text{else} \end{cases} \tag{2.16}$$

$$\overline{d}_2(T) := \begin{cases} \frac{\ln(P/K)+rT-1/2\underline{S}(T)}{\sqrt{\underline{S}(T)}} & \text{if } \ln(P/K)+rT-1/2\underline{S}(T) \geq 0 \\ \frac{\ln(P/K)+rT-1/2\underline{S}(T)}{\sqrt{\overline{S}(T)}} & \text{else} \end{cases} \tag{2.17}$$

the function d_2^* can be bounded by

$$\underline{d}_2(T) \leq d_2^*(T) \leq \overline{d}_2(T) \quad T \in I\,.$$

Furthermore, using

$$\underline{m}(T) := Kre^{-rT}\Phi(\underline{d}_2(T)) \quad \text{and} \quad \overline{m}(T) := Kre^{-rT}\Phi(\overline{d}_2(T)) \tag{2.18}$$

we have

$$\underline{m}(T) \leq f_2(T) \leq \overline{m}(T) \quad T \in I\,.$$

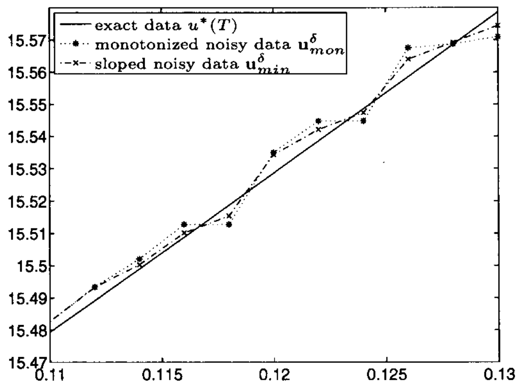

Figure 2.7: Transformation of the monotonized data $\mathbf{u}^\delta_{mon}$ to data $\mathbf{u}^\delta_{min}$ with minimal slope.

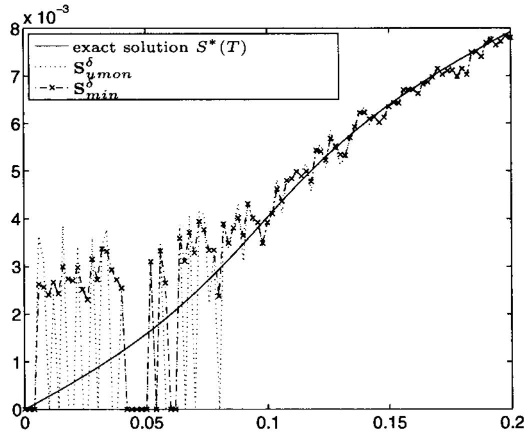

Figure 2.8: Approximate solutions $\mathbf{S}^\delta_{min}$ and $\mathbf{S}^\delta_{mon}$ obtained from data $\mathbf{u}^\delta_{mon}$ and $\mathbf{u}^\delta_{min}$ respectively.

Now we come back to the question whether enforcing a minimal slope $\underline{m}$ in the data u^δ leads to an improvement of the corresponding solution S^δ. We consider again the numerical example introduced above. Using the bounds $\underline{c} = 0.01$ and $\overline{C} = 1$ for a^* we compute $\underline{m}$ via (2.18).

As mentioned before, in Section 3 we will propose an algorithm that transforms a function f into a function f_{min} possessing a predefined minimal slope. Applying a discrete variant of this algorithm to the noisy data $\mathbf{u}^\delta$ we get data $\mathbf{u}^\delta_{min}$ possessing a minimal slope $\underline{m}$.

Figure 2.7 compares the exact data $u^*(T)$, the monotonized data $\mathbf{u}^\delta_{mon}$ and $\mathbf{u}^\delta_{min}$. We see that imposing the minimal slope $\underline{m}$ leads to a further smoothening of the data. In this situation the error of the data in the maximum norm decreased slightly from

$$\|\mathbf{u}^\delta_{mon} - \mathbf{u}^*\| = 0.0121 \quad \text{to} \quad \|\mathbf{u}^\delta_{min} - \mathbf{u}^*\| = 0.0111 .$$

The corresponding solution $\mathbf{S}^\delta_{min}$ is shown in Figure 2.8 in comparison to the exact function $S^*(T)$ and the approximate solution $\mathbf{S}^\delta_{mon}$, which has been computed from the data $\mathbf{u}^\delta_{mon}$. We see that S^δ_{mon} possesses fewer oscillations than S^δ and the maximum norm of the error decreased from

$$\|\mathbf{S}^\delta_{umon} - \mathbf{S}^*\| = 0.00347 \quad \text{to} \quad \|\mathbf{S}^\delta_{min} - \mathbf{S}^*\| = 0.00254 .$$

However, even for close bounds $\underline{c}$ and $\overline{C}$ the computed minimal slope $\underline{m}$ can not guarantee the monotonicity of S^δ. It is therefore desirable to introduce our apriori information about the monotonicity of S^* directly in the solution process. An algorithm which performs this task will be presented in the next section.

3 Regularization by monotonization – Algorithm

In this section we define a numerically efficient algorithm which transforms the input (noisy data u^δ and the corresponding noise level δ) into an approximate solution $\tilde{S}^\delta$ of (1.13). The principal idea of this algorithm can be described as follows. Under all functions $\tilde{S}$ satisfying

$$\|N(\tilde{S}) - u^\delta\|_{C(I)} \leq \delta \tag{3.1}$$

choose a function which matches our apriori information, i. e., which has a predefined minimal slope $\underline{h} > 0$ and is preferably uniformly sloped.

Obviously, (3.1) implies $\|u^* - N(\tilde{S}^\delta)\|_{C(I)} \le 2\delta$. Thus, for a sequence of noisy data u^{δ_k} with given noise levels $\delta_k \to 0$ the approximate solutions $\tilde{S}^{\delta_k}$ obtained by this algorithm converge to S^*.

The first step of the algorithm consists in computing a lower bound $u^{lb,\delta}$ and an upper bound $u^{ub,\delta}$ for u^*, such that it holds

$$u^*(T) - 2\delta \le u^{lb,\delta}(T) \le u^*(T) \le u^{ub,\delta}(T) \le u^*(T) + 2\delta \quad T \in I\,. \tag{3.2}$$

Next, the solutions $S^{lb,\delta}(T)$ and $S^{ub,\delta}(T)$ of

$$u_{BS}(P,K,r,T,S^{lb,\delta}(T)) = u^{lb,\delta}(T) \quad \text{and} \quad u_{BS}(P,K,r,T,S^{ub,\delta}(T)) = u^{ub,\delta}(T)$$

are calculated. From the monotonicity of $u_{BS}(P,K,r,T,s)$ in s it follows that these functions are lower and upper bounds of $S^*(T)$, i. e., it holds

$$0 \le S^{lb,\delta}(T) \le S^*(T) \le S^{ub,\delta}(T) \quad T \in I\,.$$

In general $S^{lb,\delta}$ and $S^{ub,\delta}$ are not yet monotone. Therefore, the third step uses apriori information of the form

$$0 < \underline{h}(t) \le a^*(t) \qquad t \in I \tag{3.3}$$

to transform them into strictly monotonically increasing functions $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ such that it holds

$$S^{lb,\delta}(T) \le S^{lb,\delta}_{mon}(T) \le S^*(T) \le S^{ub,\delta}_{mon}(T) \le S^{ub,\delta}(T) \quad T \in I\,. \tag{3.4}$$

In this context $\underline{h}$ denotes a positive function $\underline{h} : I \to (0,\infty)$, which can be interpreted as minimal slope of S^*, The usage of a time-depending minimal slope $\underline{h}$ instead of a constant is motivated by the fact that it seems realistic that our apriori information of $a^*(t)$ is better for small t than for large values of t.

Finally we choose some monotone function $\tilde{S}^\delta$ between $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$.

The basis of the sketched algorithm are the following two lemmas. The first is concerned with a projector mapping from $\mathcal{C}(I)$ in the set

$$\mathcal{M}_{\underline{h}} := \left\{ f \in \mathcal{C}(I) : f(T_2) \ge f(T_1) + \int_{T_1}^{T_2} \underline{h}(\tau)\, d\tau \quad \forall T_1 \ge T_2 \right\}.$$

containing all continuous functions with minimal slope $\underline{h}$, where $\underline{h} : I \to (0,\infty)$ denotes a predefined integrable function. We remark that $\mathcal{M}_{\underline{h}}$ is contained in the set of strictly monotonically increasing functions.

The second lemma proposes an algorithm that transforms lower and upper bounds f^{lb} and f^{ub} of a function f having a strictly positive derivative $f' \ge \underline{h} > 0$ into lower and upper bounds $f^{lb}_{mon}, f^{ub}_{mon} \in \mathcal{M}_{\underline{h}}$.

Lemma 3.1
Let $\underline{h} : I \to (0,\infty)$ be an integrable function. Then the operators $P^{1/2}_{\underline{h}} : \mathcal{C}(I) \to \mathcal{C}(I)$ defined as

$$[P^{lb}_{\underline{h}} g](T) = \max_{v\in[0,T]} \left(g(v) + \int_v^T \underline{h}(\tau)\, d\tau \right) \tag{3.5a}$$

$$[P^{ub}_{\underline{h}} g](T) = \min_{v\in[T,\overline{T}]} \left(g(v) - \int_T^v \underline{h}(\tau)\, d\tau \right) \tag{3.5b}$$

are well-defined projectors onto the set $\mathcal{M}_{\underline{h}}$. Furthermore, it holds

$$[P_{\underline{h}}^{lb} g](T) \geq g(T) \quad \text{and} \quad [P_{\underline{h}}^{ub} g](T) \leq g(T) \quad \forall T \in I\,. \tag{3.5c}$$

Proof: In this proof we use the notation $P_{\underline{h}}^{\#}$ if we mean that the corresponding formula holds true for $P_{\underline{h}}^{lb}$ as well as for $P_{\underline{h}}^{ub}$.

As the integrals

$$\int_T^v \underline{h}(\tau)\,d\tau \quad \text{and} \quad \int_v^T \underline{h}(\tau)\,d\tau$$

depend continuously on the limit of integration v and the function g is continuous, there exist the maxima and minima in (3.5) and the function $[P_{\underline{h}}^{\#} g](T)$ is continuous. In other words, the operators $P_{\underline{h}}^{\#} : \mathcal{C}(I) \to \mathcal{C}(I)$ are well defined. Furthermore, the relation (3.5c) follows immediately from the construction.

It remains to show

$$P_{\underline{h}}^{\#} g \in \mathcal{M}_{\underline{h}} \quad \forall\, g \in \mathcal{C}(I) \qquad \text{and} \qquad P_{\underline{h}}^{\#} \tilde{g} = \tilde{g} \quad \forall\, \tilde{g} \in \mathcal{M}_{\underline{h}}$$

for the operators $P_{\underline{h}}^{lb}$ and $P_{\underline{h}}^{ub}$. We show these assertions only for the operator $P_{\underline{h}}^{lb}$, the proof for $P_{\underline{h}}^{ub}$ is analogous. Let $0 \leq T_1 < T_2 \leq \overline{T}$. Obviously we have

$$\max_{v \in [0,T_1]} \left(g(v) + \int_v^{T_2} \underline{h}(\tau)\,d\tau \right) \leq \max_{v \in [0,T_2]} \left(g(v) + \int_v^{T_2} \underline{h}(\tau)\,d\tau \right).$$

Thus it follows

$$\begin{aligned}
[P_{\underline{h}}^{lb} g](T_1) + \int_{T_1}^{T_2} \underline{h}(\tau)\,d\tau &= \max_{v \in [0,T_1]} \left(g(v) + \int_v^{T_1} \underline{h}(\tau)\,d\tau \right) + \int_{T_1}^{T_2} \underline{h}(\tau)\,d\tau \\
&= \max_{v \in [0,T_1]} \left(g(v) + \int_v^{T_2} \underline{h}(\tau)\,d\tau \right) \\
&\leq \max_{v \in [0,T_2]} \left(g(v) + \int_v^{T_2} \underline{h}(\tau)\,d\tau \right) = [P_{\underline{h}}^{ub} g](T_2)\,,
\end{aligned}$$

which gives $P_{\underline{h}}^{lb} g \in \mathcal{M}_{\underline{h}}$.

Let now $\tilde{g} \in \mathcal{M}_{\underline{h}}$, i. e., it holds

$$\tilde{g}(T) \geq \tilde{g}(v) + \int_v^T \underline{h}(\tau)\,d\tau \quad \forall\, v \in [0,T]$$

which implies

$$[P_{\underline{h}}^{lb} \tilde{g}](T) = \max_{v \in [0,T]} \left(\tilde{g}(v) + \int_v^T \underline{h}(\tau)\,d\tau \right) = \tilde{g}(T)$$

and therefore $P_{\underline{h}}^{lb} \tilde{g} = \tilde{g}$. ∎

Lemma 3.2

Let $f : I \to \mathbb{R}$ be a strictly monotonically increasing function, which is continuously differentiable in $(0,\overline{T}]$. Furthermore, let $\underline{h} : I \to (0,\infty)$ be an integrable function such that

$$0 < \underline{h}(t) \leq f'(t) \qquad \forall\, t \in (0,\overline{T}] \tag{3.6}$$

holds. Let furthermore $f^{lb}: I \to \mathbb{R}$ and $f^{ub}: I \to \mathbb{R}$ be continuous lower and upper bounds of f, i.e., it holds

$$f^{lb}(T) \le f(T) \le f^{ub}(T) \qquad \forall T \in I\,.$$

Then the functions

$$f^{lb}_{mon} := P^{lb}_{\underline{h}} f^{lb} \quad \text{and} \quad f^{ub}_{mon} := P^{ub}_{\underline{h}} f^{ub} \tag{3.7}$$

are continuous, strictly monotonically increasing and satisfy the inequality

$$f^{lb}(T) \le f^{lb}_{mon}(T) \le f(T) \le f^{ub}_{mon}(T) \le f^{ub}(T) \quad T \in I\,. \tag{3.8}$$

Proof: In view of Lemma 3.1 it remains to show

$$f^{lb}_{mon}(T) \le f(T) \le f^{ub}_{mon}(T) \quad T \in I. \tag{3.9}$$

We use the inequality

$$f^{lb}(v) \le f(v) \quad \text{for all } v \in I$$

and the relation

$$\int_v^T \underline{h}(\tau)\, d\tau \le \int_v^T f'(\tau)\, d\tau = f(T) - f(v) \quad \text{for } 0 \le v \le T \le \overline{T}.$$

Adding these two inequalities and computing the maximum over $v \in [0, T]$ gives the left inequality of (3.9). The right inequality of (3.9) can be proven analogously. ∎

Remark 3.3 Let f be as in Lemma 3.2. Assume that we are not aware of bounds f^{lb} and f^{ub} of f but we have some approximation $f^\delta \in C(I)$ with noiselevel δ, i.e., it holds $\|f - f^\delta\|_{C(I)} \le \delta$. Then one can show by similar considerations as in the proofs of Lemma 3.1 and 3.2 that the functions f^δ_{lb} and f^δ_{ub} defined by

$$f^\delta_{lb}(T) := \min_{v \in [T,\overline{T}]} f^\delta(v) \quad \text{and} \quad f^\delta_{ub}(T) := \max_{v \in [0,T]} f^\delta(v) \qquad \forall T \in I$$

are continuous and it holds

$$f(T) - \delta \le f^\delta_{lb}(T) \le f^\delta(T) \le f^\delta_{ub}(T) \le f(T) + \delta\,.$$

However, it can neither be guaranteed that

$$f^\delta_{lb}(T) \le f(T) \le f^\delta_{ub}(T)$$

holds nor that there exists a function f^δ_{min} between f^δ_{lb} and f^δ_{ub} possessing a minimal slope $\underline{h}$.

Lemma 3.2 indicated how we can compute monotone bounds $S^{lb,\delta}_{mon}(T)$ and $S^{ub,\delta}_{mon}(T)$ from $S^{lb,\delta}$ and $S^{ub,\delta}$. The question which has been left open is how to choose a monotone function $\tilde{S}^\delta$ between these bounds. A first idea was to compute $\tilde{S}^\delta(T)$ pointwise as arithmetic mean, i.e., to choose $\tilde{S}^\delta(T)$ as

$$S^\delta_{conv}(T) = \frac{1}{2} S^{lb,\delta}_{mon}(T) + \frac{1}{2} S^{ub,\delta}_{mon}(T) \quad T \in I\,. \tag{3.10}$$

This approach works quite well in the regions where the difference between $S^{ub,\delta}_{mon}(T)$ and $S^{lb,\delta}_{mon}(T)$ is relatively small. Unfortunately, in the first part of the interval I this difference is in general rather large. This is due to the fact that in general $S^{lb,\delta}_{mon}(T)$ and $S^{ub,\delta}_{mon}(T)$ have a similar behaviour as the functions $\underline{S}^\delta$ and $\overline{S}^\delta$ defined by (2.13). That is, for small T the function $S^{lb,\delta}_{mon}(T)$ is (nearly) equal to $\underline{c}T$ and the function $S^{ub,\delta}_{mon}(T)$ is close to a nearly horizontal line. Then at some timepoint the function $S^{lb,\delta}_{mon}(T)$ grows very quickly. Consequently, the difference between $S^{ub,\delta}_{mon}(T)$ and $S^{lb,\delta}_{mon}(T)$ decreases rapidly and after some timepoint T^* this difference is sufficiently small.

From these considerations we can now conclude that S^δ_{conv} is in general not the best approximation of S^* in the first part of the interval. Indeed, the large slope of $S^{lb,\delta}_{mon}(T)$ near T^* leads also to a quite large slope in the convex linear combination (3.10). This would correspond to a volatility function $\tilde{\sigma}^\delta(T)$ with a very sharp and high peak. Clearly, in absence of any other information we would rather prefer a smooth or even nearly constant volatility. This would lead to a choice of $\tilde{S}^\delta$ as (3.10) in the region $T > T^*$ and as linear interpolation in the interval $[0, T^*]$.

Finally we have to be aware of the fact that it might of course happen that the underlying volatility σ^* possesses really a sharp peak. Then the data u^δ can be such that the linear interpolation intersects the bounds $S^{ub,\delta}_{mon}(T)$ and $S^{lb,\delta}_{mon}(T)$. In this situation we should therefore choose $\tilde{S}^\delta$ on the entire interval I as S^δ_{conv}.

Combining all these ideas we suggest the following pragmatic approach. First we compute the mean distance

$$md := \frac{1}{0.2\overline{T}} \int_{0.8\overline{T}}^{\overline{T}} \left| S^{ub,\delta}_{mon}(T) - S^{lb,\delta}_{mon}(T) \right| dT \tag{3.11a}$$

over the last part of the interval I and define

$$T^* := \inf\left\{ T \in I : \left| S^{ub,\delta}_{mon}(T) - S^{lb,\delta}_{mon}(T) \right| \le 3\, md \right\} . \tag{3.11b}$$

Now, if the function

$$S^\delta_{lin}(T) := \frac{T}{T^*} S^\delta_{conv}(T^*) \tag{3.11c}$$

with S^δ_{conv} defined by (3.10) satisfies

$$S^{lb,\delta}_{mon}(T) \le S^\delta_{lin}(T) \le S^{ub,\delta}_{mon}(T) \quad T \le T^* \tag{3.11d}$$

then set $\tilde{S}^\delta(T) = S^\delta_{lin}(T) \quad T \le T^*$, otherwise

$$\tilde{S}^\delta(T) = \frac{T}{2T^*} S^{ub,\delta}_{mon}(T) + \left(1 - \frac{T}{2T^*}\right) S^{lb,\delta}_{mon}(T) , \tag{3.11e}$$

which is a pointwise linear combination of $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$. In both cases it holds

$$\tilde{S}^\delta(0) = 0 \quad \text{and} \quad \tilde{S}^\delta(T^*) = S^\delta_{conv}(T^*).$$

Finally we set

$$\tilde{S}^\delta(T) := S^\delta_{conv}(T) \quad T > T^*\,. \tag{3.11f}$$

At the end of this section we combine all our considerations to formulate an algorithm that realizes the choice of a function $\tilde{S}^\delta$ possessing a minimal slope $\underline{h}$ and satisfying (3.1).

Algorithm 3.4
Input:

- a noise level δ and a continuous function $u^\delta : I \to \mathbb{R}$ such that $\left\|u^\delta - u^*\right\|_{C(I)} \leq \delta$ and $u^\delta(T) < P - \delta$ holds
- an integrable function $\underline{h}$ such that $0 < \underline{h}(t) \leq a^*(t)$ holds for all $t \in I$

Output: approximate solution $\tilde{S}^\delta(T)$ such that

- $\tilde{S}^\delta(0) = 0$, $\tilde{S}^\delta \in \mathcal{M}_{\underline{h}}$
- $\|N(\tilde{S}^\delta) - u^\delta\|_{C(I)} \leq \delta$.

Algorithm:

1. For $T \in I$ compute lower and upper bounds of $u(T)$ as follows

$$\begin{aligned} u^{lb,\delta}(T) &:= \max\left(u^\delta(T) - \delta, u_{BS}\left(P, K, r, T, \int_0^T \underline{h}(\tau)\,d\tau\right)\right) \\ u^{ub,\delta}(T) &:= u^\delta(T) + \delta\,. \end{aligned}$$

2. Compute pointwise lower bounds $S^{lb,\delta}(t)$ and upper bounds $S^{ub,\delta}(T)$ for $S^*(T)$ as solution of

$$\begin{aligned} u_{BS}\left(P, K, r, T, S^{lb,\delta}(T)\right) &= u^{lb,\delta}(T) \\ u_{BS}\left(P, K, r, T, S^{ub,\delta}(T)\right) &= u^{ub,\delta}(T) \end{aligned} \qquad T \in I\,.$$

3. Monotonize the bounds: Set $S^{lb,\delta}_{mon} := P^{lb}_{\underline{h}} S^{lb,\delta}$ and $S^{ub,\delta}_{mon} := P^{ub}_{\underline{h}} S^{ub,\delta}$.

4. Choose $\tilde{S}^\delta(T)$ between $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ by the algorithm (3.11).

Proof: We show that the algorithm terminates without error and the constructed function $\tilde{S}^\delta$ possesses the asserted properties.

1. Obviously the functions $u^{lb,\delta}$ and $u^{ub,\delta}$ satisfy

$$u^\delta(T) - \delta \leq u^{lb,\delta}(T) \leq u^*(T) \leq u^{ub,\delta}(T) \leq u^\delta(T) + \delta\,.$$

   Furthermore, using the properties of the function $u_{BS}(P, K, r, T, s)$ together with the relation $\int_0^T \underline{h}(\tau)\,d\tau \leq S^*(T)$ one gets the chain of inequalities

$$\begin{aligned} \max(P - Ke^{-rT}, 0) < u_{BS}&\left(P, K, r, T, \int_0^T \underline{h}(\tau)\,d\tau\right) \\ &\leq u^{lb,\delta}(T) \leq u^*(T) \leq u^{ub,\delta}(T) < P \quad \forall T \in (0, \overline{T}]\,. \end{aligned}$$

and the relation $u^{lb,\delta}(0) = u_{BS}(P,K,r,0,0) = \max(P-K,0)$. In addition, the continuity of u^δ and the continuity of $u_{BS}(P,K,r,T,s)$ with respect to s implies the continuity of $u^{lb,\delta}$ and $u^{ub,\delta}$.

2. As the function $u^{lb,\delta}$ satisfies Assumption 2.3 the existence and uniqueness of the pointwise defined function $S^{lb,\delta}$ is ensured. Furthermore, this function is an element of $\mathcal{D}_{0,+}$. The existence and uniqueness of the pointwise defined, continuous function $S^{ub,\delta}$ can be shown by analogous considerations. In fact, the only difference is that $u^{ub,\delta}(0) > \max(P-K,0)$ which implies $S^{ub,\delta}(0) > 0$. The strict monotonicity of $u_{BS}(P,K,r,T,s)$ with respect to s implies

$$S^{lb,\delta}(T) \le S^*(T) \le S^{ub,\delta}(T) \qquad \forall T \in I\,.$$

3. Now, Lemma 3.2 guarantees that the functions $S^{lb,\delta}_{mon}(T)$ and $S^{ub,\delta}_{mon}(T)$ are continuous, possess a minimal slope $\underline{c} > 0$ and satisfy the chain of inequalities

$$S^{lb,\delta}(T) \le S^{lb,\delta}_{mon}(T) \le S^*(T) \le S^{ub,\delta}_{mon}(T) \le S^{ub,\delta}(T) \qquad \forall T \in I\,. \tag{3.12}$$

4. The continuity and the minimal slope of $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ imply the continuity and the minimal slope of S^δ_{conv}. Besides, the chain of inequalities $\underline{c}T^* \le S^{lb,\delta}_{mon}(T^*) \le S^\delta_{conv}(T^*)$ implies the minimal slope of S^δ_{lin}. The continuity of S^δ_{lin} is obvious. Furthermore, the weight function $\frac{T}{2T^*}$ is monotonically increasing. Together with the inequality $S^{lb,\delta}_{mon} \le S^{ub,\delta}_{mon}$ and the continuity and minimal slope of $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ this shows the continuity and minimal slope of $\tilde{S}^\delta(T)$ $(T \le T^*)$ if it is defined by (3.11).

   Finally, the property $S^{lb,\delta} \le \tilde{S}^\delta \le S^{ub,\delta}$ together with the strict monotonicity of $u_{BS}(P,K,r,T,s)$ with respect to s implies $u^{lb,\delta} \le \tilde{N}(\tilde{S}^\delta) \le u^{ub,\delta}$ and thus (3.1). ■

Remark 3.5 *1. The functions $\tilde{S}^\delta$ and $\tilde{u}^\delta$ are continuous. However, in general they are not continuously differentiable.*

2. For the algorithm the knowledge of the noise level δ is crucial. If one is not aware of this level, one could compute S^δ from $N(S^\delta) = u^\delta$ and use the ideas of Remark 3.3 and (3.11) to monotonize this function. However, this approach guarantees only monotonicity, no strict monotonicity or even a minimal slope.

3. Instead of the noisy data u^δ and the noiselevel δ one could also use bounds $u^{lb,\delta}$ and $u^{ub,\delta}$ for u^* as input for the algorithm. This would be motivated by the assumption that the noise in the data is solely due to the bid-ask spread.

4 Discrete Variant

Instead of observing the data $u^\delta : I \to \mathbb{R}$ on the entire interval we can rather expect to observe only discrete values $u^\delta(T_i)$ for several maturities T_i $(i = 0,\dots,N)$. In order to keep notation simple, we assume that the T_i form a uniform grid on I, i. e., we assume $T_i = i\Delta$ $(i = 0,\dots,N)$ with a positive increment Δ.

Combining the observed prices into the vector $\mathbf{u}^\delta := \big(u^\delta(T_0),\dots,u^\delta(T_N)\big)$, we can formulate the following modification of Algorithm 3.4. For the seek of simple notation we formulate the algorithm only for the case where the function $\underline{h}$ is constant.

Algorithm 4.1
Input:

- step width Δ
- number of observations $N+1$ and observation points $T_i = i\Delta\,(i = 0, \ldots, N)$.
- vector $\mathbf{u}^\delta \in \mathbb{R}^{N+1}$ such that $\max_{i=0,\ldots,N} \left|\mathbf{u}^\delta[i] - u^*(T_i)\right| \le \delta$ and $\max_{i=0,\ldots,N} \mathbf{u}^\delta[i] < P$ is satisfied
- a constant $\underline{h}$ such that $0 < \underline{h} \le a^*(t)$ holds for all $t \in I$

Output: an approximate solution $\tilde{S}^\delta : I \to \mathbb{R}$ such that

- $\tilde{S}^\delta(0) = 0$
- $\tilde{S}^\delta \in \mathcal{M}_{\underline{h}}$
- the corresponding function $\tilde{u}^\delta = N(\tilde{S}^\delta)$ satisfies $\max_{0\le i\le N} \left|\tilde{u}^\delta(T_i) - u^*(T_i)\right| \le 2\delta$.

Algorithm:

0. Initialization

$\mathbf{T} := [0 : \overline{T}/N : \overline{T}]$
`for i from 0 step 1 to N`
  $\underline{\mathbf{u}}[i] := u_{BS}(P, K, r, \mathbf{T}[i], \underline{h}\,\mathbf{T}[i])$
`end %for`
$\mathbf{u}^{lb,\delta} := \mathtt{max}(\mathbf{u}^\delta - \delta, \underline{\mathbf{u}})$
$\mathbf{u}^{ub,\delta} := \mathbf{u}^\delta + \delta$

1. Using a simple bisection algorithm compute $\mathbf{S}^{lb,\delta}$ and $\mathbf{S}^{ub,\delta}$ from

$$\begin{aligned} u_{BS}(P, K, r, \mathbf{T}[i], \mathbf{S}^{lb,\delta}[i]) &= \mathbf{u}^{lb,\delta}[i] \\ u_{BS}(P, K, r, \mathbf{T}[i], \mathbf{S}^{ub,\delta}[i]) &= \mathbf{u}^{ub,\delta}[i] \end{aligned} \qquad (i = 0, \ldots, N)$$

2. Compute the strictly monotonically increasing vectors $\mathbf{S}^{lb,\delta}_{mon}$ and $\mathbf{S}^{ub,\delta}_{mon}$ by

$\mathbf{S}^{lb,\delta}_{mon}[0] := 0;$
$\mathbf{S}^{ub,\delta}_{mon}[N] := \mathbf{S}^{ub,\delta}[N]$
`for i from 1 step 1 to N`
  $\mathbf{S}^{lb,\delta}_{mon}[i] := \max(\mathbf{S}^{lb,\delta}_{mon}[i-1] + \Delta\underline{h}, \mathbf{S}^{lb,\delta}[i]);$
`end %for`
`for i from N-1 step -1 to 0`
  $\mathbf{S}^{ub,\delta}_{mon}[i] := \min(\mathbf{S}^{ub,\delta}_{mon}[i+1] - \Delta\underline{h}, \mathbf{S}^{ub,\delta}[i]);$
`end %for`

3. Compute $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ by linear interpolation such that

$$\begin{aligned} S^{lb,\delta}_{mon}(T_i) &= \mathbf{S}^{lb,\delta}_{mon}[i] \\ S^{ub,\delta}_{mon}(T_i) &= \mathbf{S}^{ub,\delta}_{mon}[i] \end{aligned} \qquad (i = 0, \ldots, N)$$

holds. Choose $\tilde{S}^\delta$ between $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ by the algorithm (3.11).

Remark 4.2

1. Note that for computing $\mathbf{S}^{lb,\delta}_{mon}[i]$ it is not necessary to compute

$$\max_{j\le i}\left(\mathbf{S}^{lb,\delta}[j]+(i-j)\Delta\underline{h}\right),$$

which would mean to compute the maximum of i values. It suffices to compute the maximum

$$\max\left(\mathbf{S}^{lb,\delta}_{mon}[i-1]+\Delta\underline{h},\mathbf{S}^{lb,\delta}[i]\right)$$

of two values.

2. Concerning the residuum the algorithm guarantees the estimate $\left|\tilde{u}^\delta(T_i)-u^*(T_i)\right|\le 2\delta$ at the points T_i $(i=0,\dots,N)$. As the functions $\tilde{u}^\delta$ and u^* are strictly monotonically increasing this implies for $T\in[T_{i-1},T_i]$ the estimate

$$u^*(T_{i-1})-2\delta\le\tilde{u}^\delta(T_{i-1})\le\tilde{u}^\delta(T)\le\tilde{u}^\delta(T_i)\le u^*(T_i)+2\delta$$

and therefore

$$u^*(T)-2\delta-(u^*(T_i)-u^*(T_{i-1}))\le\tilde{u}^\delta(T)\le u^*(T)+2\delta+u^*(T_i)-u^*(T_{i-1}).$$

Thus, defining the constant $C:=\max_{T\in I}|u^{*\prime}(T)|$ it holds

$$\left|\tilde{u}^\delta(T)-u^*(T)\right|\le 2\delta+\Delta C.$$

We will now illustrate the performance of Algorithm 4.1. We use again the noisy data $\mathbf{u}^\delta$ which had been defined in Section 2 and illustrated in Figure 2.1. Furthermore, we use the constant $\underline{h}=0.01$ as input of the algorithm. Figure 4.1 compares the exact data u^* with the lower and upper bounds $\mathbf{u}^{lb,\delta}$ and $\mathbf{u}^{ub,\delta}$ which have been computed in Step 1 of the algorithm. As in Section 2 we zoomed into the picture and plotted the function only in the interval $[0.07,0.13]$. The corresponding lower and upper bounds $\mathbf{S}^{lb,\delta}$ and $\mathbf{S}^{ub,\delta}$, which have been constructed in Step 3 are illustrated in Figure 4.2. As expected, the lower bound is in the first part (nearly) equal to $\underline{h}T$. Furthermore, the noisy data were such that at $T_1=0.036$ and $T_2=0.06$ the function $\mathbf{S}^{ub,\delta}$ is quite close to S^*, whereas the other points in this region are further away. Consequently, the monotonized bound $\mathbf{S}^{ub,\delta}_{mon}$, which is illustrated together with $\mathbf{S}^{lb,\delta}_{mon}$ and S^* in Figure 4.3, possesses at these points jumps. Finally, Figure 4.4 shows $\tilde{S}^\delta$ which has been computed in Step 4. We see that $T^*=0.0104$ and $\tilde{S}^\delta(T)$ $(T\le T^*)$ is chosen as S^δ_{lin}.

Let us now consider a sequence of stepwidths Δ_k and corresponding grids having $N_k:=\frac{\overline{T}}{\Delta_k}$ points. Let us furthermore assume that we can for each $k\in\mathbb{N}$ observe noisy data $\left(u^{\delta_k}(j\Delta_k)\right)_{j=1}^{N_k}$ with noise level δ_k, i. e., satisfying

$$\left|u^*(j\Delta_k)-u^{\delta_k}(j\Delta_k)\right|\le\delta_k.$$

Then we can for each $k\in\mathbb{N}$ construct an approximate solution $\tilde{S}^{\delta_k}\in\mathcal{C}(I)$. Furthermore, if the noiselevels δ_k and the stepwidths Δ_k converge to 0 then $\tilde{S}^{\delta_k}$ converges to S^* in $\mathcal{C}(I)$.

Before we investigate the situation of discrete data with random noise we want to remark that the algorithm might not terminate correctly if the noiselevel δ in Algorithm 4.1 is misspecified, i. e., as input we use a level δ_1 und noisy data $\mathbf{u}^{\delta_2}$ such that

$$\max_{i=0,\dots,N}|\mathbf{u}^{\delta_2}[i]-u^*(T_i)|=\delta_2>\delta_1$$

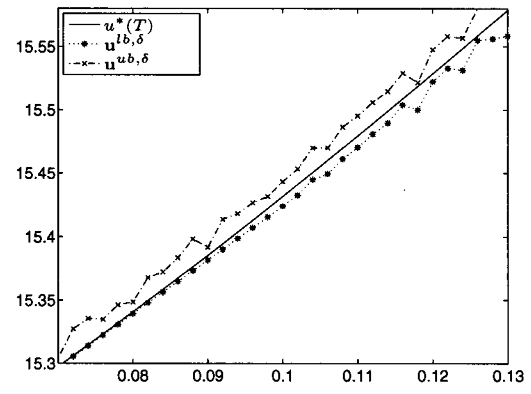

Figure 4.1: Result of Step 1: Lower and upper bounds $\mathbf{u}^{lb,\delta}$ $\mathbf{u}^{ub,\delta}$ in comparison with the exact data $u^*(T)$

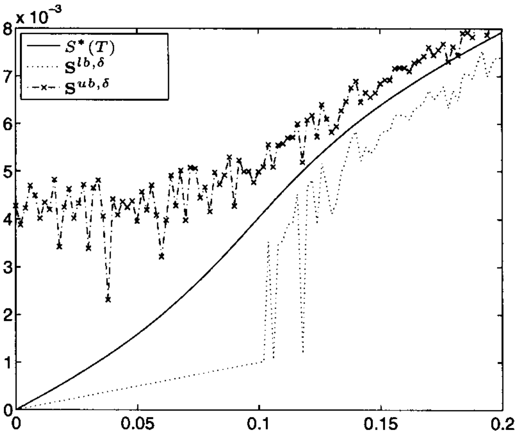

Figure 4.2: Result of Step 2: Corresponding lower and upper bounds $\mathbf{S}^{lb,\delta}$ and $\mathbf{S}^{ub,\delta}$

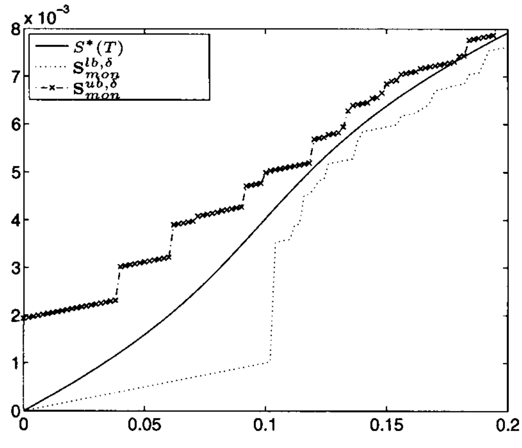

Figure 4.3: Result of Step 3: Monotone lower and upper bounds $\mathbf{S}^{lb,\delta}_{mon}$ and $\mathbf{S}^{ub,\delta}_{mon}$ in comparison with the exact function $S^*(T)$

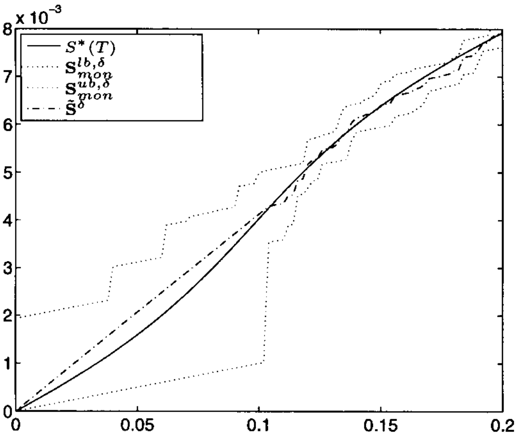

Figure 4.4: Result of Step 4: Approximate solution $\tilde{\mathbf{S}}^\delta$ between $\mathbf{S}^{lb,\delta}$ and $\mathbf{S}^{ub,\delta}$

holds. Indeed, in this situation the computed vectors $\mathbf{u}^{lb,\delta}$ and $\mathbf{u}^{ub,\delta}$ do not satisfy

$$\mathbf{u}^{lb,\delta}[i] \leq u^*(T_i) \leq \mathbf{u}^{ub,\delta}[i] \quad (i = 0, \ldots, N)$$

and hence also the corresponding vectors $\mathbf{S}^{lb,\delta}$ and $\mathbf{S}^{ub,\delta}$ do not satisfy

$$\mathbf{S}^{lb,\delta}[i] \leq S^*(T_i) \leq \mathbf{S}^{ub,\delta}[i] \quad (i = 0, \ldots, N)\,.$$

Therefore, it might happen that the constructed functions $\mathbf{S}^{lb,\delta}_{mon}$ and $\mathbf{S}^{ub,\delta}_{mon}$ intersect, i. e., that there exists some i_1 such that $\mathbf{S}^{lb,\delta}_{mon}[i_1] > \mathbf{S}^{ub,\delta}_{mon}[i_1]$. Clearly, in this situation it is not possible to choose $\tilde{S}^\delta$ such that

$$\mathbf{S}^{lb,\delta}_{mon}[i] \leq \tilde{S}^\delta(T_i) \leq \mathbf{S}^{ub,\delta}_{mon}[i] \quad i = 0, \ldots, N$$

holds.

Now we are ready for addressing the situation of discrete and independent (normally distributed) noise, i. e., we assume that we are able to observe for several discrete maturities $T_i = i\Delta$ $(i = 0, \ldots, N)$ option prices $\mathbf{u}^\delta[i]$ such that the errors $X_i := \mathbf{u}^\delta[i] - u^*(T_i)$ are i.i.d random variables with normal distribution, mean zero and variance δ^2, i. e., $X_i \sim \mathcal{N}(0, \delta^2)$. Furthermore, we will assume to be aware of some upper bound $\overline{S}$ such that it holds

$$S^*(\overline{T}) \leq \overline{S}\,. \tag{4.1}$$

In order to apply the ideas of Algorithm 4.1 we would need an upper bound of $\max_{i=1,\dots,N} |X_i|$. Clearly, we can not expect that the maximum of normally distributed random variables to be bounded almost surely. However, we can use the following result.

Lemma 4.3
Let Y_i be i. i. d. random variables with $Y_i \sim \mathcal{N}(0,1)$. Then it holds

$$\mathbb{P}\left(\max_{i=1,\dots,n} |Y_i| > 2\sqrt{\ln(n)}\right) \to 0 \quad \text{for } n \to \infty$$

Proof: Noting $\max\limits_{i=1,\dots,n} |Y_i| = \max\left(\max_{i=1,\dots,n} Y_i, \max_{i=1,\dots,n}(-Y_i)\right)$ we see

$$\begin{aligned}\mathbb{P}\left(\max_{i=1,\dots,n} |Y_i| > 2\sqrt{\ln(n)}\right) &= \mathbb{P}\left(\left\{\max_{i=1,\dots,n} Y_i > 2\sqrt{\ln(n)}\right\} \cup \left\{\max_{i=1,\dots,n} (-Y_i) > 2\sqrt{\ln(n)}\right\}\right) \\ &\leq \mathbb{P}\left(\left\{\max_{i=1,\dots,n} Y_i > 2\sqrt{\ln(n)}\right\}\right) + \mathbb{P}\left(\left\{\max_{i=1,\dots,n} (-Y_i) > 2\sqrt{\ln(n)}\right\}\right).\end{aligned}$$

As because of the symmetry of the standard normal distribution

$$\mathbb{P}\left(\left\{\max_{i=1,\dots,n} Y_i > 2\sqrt{\ln(n)}\right\}\right) = \mathbb{P}\left(\left\{\max_{i=1,\dots,n} (-Y_i) > 2\sqrt{\ln(n)}\right\}\right)$$

holds, it remains to show $\mathbb{P}\left(\left\{\max\limits_{i=1,\dots,n} Y_i > 2\sqrt{\ln(n)}\right\}\right) \to 0$ for n tending to infinity.

In view of Theorem 1.5.1 of [7] it is sufficient to prove

$$\lim_{n\to\infty} n\left(1 - \Phi\left(2\sqrt{\ln(n)}\right)\right) \to 0\,,$$

which can be done using L'Hospital's rule. ∎

Defining the constant $\hat{\delta} := 2\sqrt{\ln(N)}\delta$, which can be interpreted as estimated noiselevel, Lemma 4.3 implies that the probability of the event

$$\Omega_+ := \left\{\max_{i=0,\dots,N} \left|\mathbf{u}^\delta[i] - u^*(T_i)\right| \leq \hat{\delta}\right\}$$

is roughly speaking *high*. Furthermore, in this situation the Algorithm 4.1 with input $\hat{\delta}$, $\mathbf{u}^\delta$ terminates without error and we find $\tilde{S}^\delta$ such that the residuum $N(\tilde{S}^\delta) - u^*$ is bounded by $2\hat{\delta} + \Delta C$ in the $C(I)$ norm.

In the situation

$$\Omega_- := \left\{\max_{i=0,\dots,N} \left|\mathbf{u}^\delta[i] - u^*(T_i)\right| > \hat{\delta}\right\}$$

three cases might occur:

A The algorithm might not terminate without error.

B The algorithm terminates without error and computes the strictly monotone function $\tilde{S}^\delta$ but it holds $\tilde{S}^\delta(\overline{T}) > \overline{S}$.

C The algorithm terminates without error and the strictly monotone function $\tilde{S}^\delta$ satisfies $\tilde{S}^\delta(\overline{T}) \le \overline{S}$ and therefore $\|\tilde{S}^\delta - S^*\|_{C(I)} \le \overline{S}$.

In order to bound the error we modify the Algorithm 4.1 inasmuch as we set $\tilde{S}^\delta(T) = \frac{T}{T^*}\overline{S}(T)$ if the cases A or B occur. Thus we get the following algorithm.

Algorithm 4.4 Input:

- step width Δ, number of observations $N+1$ and observation points $T_i = i\Delta$ $(i = 0, \ldots, N)$
- variance δ and a random vector $\mathbf{u}^\delta \in \mathbb{R}^N$ such that the components $\mathbf{u}^\delta[i]$ $(i = 1, \ldots, N)$ are i. i. d. normal random variables with expectation $u^*(T_i)$ and variance δ^2. The first component is deterministic $\mathbf{u}^\delta[0] = u^*(0) = \max(P - K, 0)$.
- a constant $\underline{h}$ such that $0 < \underline{h} \le a^*(t)$ holds for all $t \in I$
- an upper bound $\overline{S}$ for S^* such that (4.1) is satisfied

Output: an approximate solution $\tilde{S}^\delta : I \to \mathbb{R}$ such that

- $\tilde{S}^\delta(0) = 0$
- $\tilde{S}^\delta \in \mathcal{M}_{\underline{h}}$
-
  - if the event $\Omega_+ := \left\{\max_{i=1,\ldots,N} \left|\mathbf{u}^\delta[i] - u^*(T_i)\right| \le \hat{\delta}\right\}$ with $\hat{\delta} := 2\delta\sqrt{\ln(N)}$ occurs, then the computed function $\tilde{S}^\delta$ is such that the corresponding function $\tilde{u}^\delta = N(\tilde{S}^\delta)$ satisfies $\left\|\tilde{u}^\delta(T_i) - u^*(T_i)\right\|_{C(I)} \le 2\hat{\delta} + \Delta C$.
  - if the event $\Omega_- := \left\{\max_{i=1,\ldots,N} \left|\mathbf{u}^\delta[i] - u^*(T_i)\right| > \hat{\delta}\right\}$ occurs, the computed function $\tilde{S}^\delta$ is such that $\|\tilde{S}^\delta - S^*\|_{C(I)} \le \overline{S}$ holds.

Algorithm:

1. Set $\hat{\delta} := 2\sqrt{\ln(N)}\delta$ and perform the Steps 0.-3. of the Algorithm 4.1 with δ replaced by $\hat{\delta}$
2. If $\mathbf{S}^{lb,\delta}_{mon}[i] \le \mathbf{S}^{ub,\delta}_{mon}[i] \le \overline{S}$ holds for all $i = 0, \ldots, N$ then choose $\tilde{S}^\delta$ between $S^{lb,\delta}_{mon}$ and $S^{ub,\delta}_{mon}$ by the algorithm (3.11). Otherwise set $\tilde{S}^\delta(T) := \frac{T}{\overline{T}}\overline{S}$ $\quad (T \in I)$

Now we are able to formulate and prove a convergence result for the situation of discrete and noisy errors.

Theorem 4.5
Let $\{N_k\}_{k=1}^\infty$, $N_k \in \mathbb{N}$ and $\{\delta_k\}_{k=1}^\infty \subset (0, \infty)$ be such that

$$\Delta_k := \frac{\overline{T}}{N_k} \to 0 \quad \text{and} \quad \delta_k\sqrt{\ln(N_k)} \to 0 \qquad \text{for } k \to \infty \tag{4.2}$$

holds. Let furthermore exist a sequence $\{\mathbf{u}^{\delta_k}\}_{k=1}^{\infty}$ *such that for every* k

$$\mathbf{u}^{\delta_k}[i] - u^*(i\Delta_k) \quad i = 1, \dots, N$$

are i. i. d. normal variables with expectation zero and variance δ_k^2. *Given constants* $0 < \underline{h} \leq a^*(t)$ $(t \in I)$ *and* $\overline{S}$ *such that* (4.1) *is satisfied, for every* k *the Algorithm 4.4 can be applied and computes a function* $\tilde{S}^{\delta_k}$. *Furthermore we have*

$$\mathbb{E} \left\| \tilde{S}^{\delta_k} - S^* \right\|_{C(I)} \to 0 \quad \textit{for } k \to \infty\,.$$

Proof: Let $\varepsilon > 0$. Lemma 4.3 yields

$$\begin{aligned}\mathbb{P}(\Omega_-(k)) &= \mathbb{P}\left(\left\{\max_{i=1,\dots,N_k} \left|\mathbf{u}^{\delta_k}[i] - u^*(T_i)\right| > 2\sqrt{\ln(N_k)}\delta_k\right\}\right)\\ &= \mathbb{P}\left(\left\{\max_{i=1,\dots,N_k} \frac{\left|\mathbf{u}^{\delta_k}[i] - u^*(T_i)\right|}{\delta_k} > 2\sqrt{\ln(N_k)}\right\}\right) \to 0 \quad \text{for } N_k \to \infty.\end{aligned}$$

Thus, there exists a value k_1 such that

$$\mathbb{P}(\Omega_-(k)) \leq \frac{\varepsilon}{2\overline{S}} \quad \forall\, k \geq k_1\,.$$

Furthermore, on

$$\Omega_+(k) := \left\{\max_{i=1,\dots,k} \left|\mathbf{u}^{\delta_k}[i] - u^*(T_i)\right| \leq \hat{\delta}_k\right\}$$

the residuum $\|N(\tilde{S}^{\delta_k}) - N(S^*)\|_{C(I)}$ is bounded by $2\hat{\delta}_k + C\Delta_k$, which tends to zero for $k \to \infty$. Together with the continuity of the inverse operator this implies that there exists some value k_2 such that on the sets $\Omega_+(k)$

$$\|\tilde{S}^{\delta_k} - S^*\|_{C(I)} \leq \frac{\varepsilon}{2} \quad \forall\, k \geq k_2$$

holds.

Combining these results we obtain for $k \geq \max(k_1, k_2)$

$$\begin{aligned}\mathbb{E} \left\| \tilde{S}^{\delta_k} - S^* \right\|_{C(I)} &\leq \mathbb{P}\left(\left\{\max_{i=1,\dots,k} \left|\mathbf{u}^{\delta_k}[i] - u^*(T_i)\right| > 2\sqrt{\ln(N_k)}\delta_k\right\}\right)\overline{S}\\ &\quad + \mathbb{P}\left(\left\{\max_{i=1,\dots,k} \left|\mathbf{u}^{\delta_k}[i] - u^*(T_i)\right| \leq 2\sqrt{\ln(N_k)}\delta_k\right\}\right)\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\,.\end{aligned}$$

∎

5 Conclusion

The aim of the present paper was to examine the instability occurring in a specific inverse problem in option pricing and to study how apriori information about the monotonicity of the searched function and the data can be used for regularization. After showing that the instability is due to ill-conditioning effects (and not to ill-posedness) we have illustrated in a numerical case study that monotonization of the data leads in general only to a slight improvement of the reconstructed solution. It was therefore necessary to find a numerically efficient algorithm that introduces the monotonicity of the searched function in the solution process. Using the specific structure of the forward operator we have suggested such an algorithm in the continuous setting and discussed its adaptation to the situation of discrete data with deterministic and stochastic noise.

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