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Comparison between different duals in multiobjective fractional programming

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Abstract

The present paper is a continuation of [2] where we deal with the duality for a multiobjective fractional optimization problem. The basic idea in [2] consists in attaching an intermediate multiobjective convex optimization problem to the primal fractional problem, using an approach due to Dinkelbach ([5]), for which we construct then a dual problem expressed in terms of the conjugates of the functions involved. The weak, strong and converse duality statements for the intermediate problems allow us to give dual characterizations for the efficient solutions of the initial fractional problem.

The aim of this paper is to compare the intermediate dual problem with other similar dual problems known from the literature. We completely establish the inclusion relations between the image sets of the duals as well as between the sets of maximal elements of the image sets.

1 Introduction

In this paper we continue the study in [2] on duality assertions for multiobjective fractional optimization problems. In the mentioned paper, considering a primal optimization problem having as objective function a vector function with components that are quotients of a convex and a concave function, we attach to it an intermediate multiobjective convex optimization problem by using an approach due to Dinkelbach ([5]), which we denote by (P_μ) for $\mu \in \mathbb{R}^m$. To this last problem we construct a multiobjective dual (D_μ) expressed in terms

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of the conjugates of the functions involved. For the intermediate primal and dual problems we prove weak, strong and converse duality assertions which we use then to give dual characterizations for the efficient solutions of the initial fractional problem.

The aim we follow in this paper is to make a comparison of the dual (D_μ) with different dual problems to the parametrized multiobjective optimization problem (P_μ) given in the past in literature. On the one hand we consider two multiobjective problems constructed by using the approach described in [3] and on the other hand the multiobjective dual due to Ohlendorf and Tammer ([10]). The approach described by Boţ and Wanka in [3] for constructing multiobjective dual problems by using different scalar dual problems extends the results of Jahn ([7]) for Lagrange duality. Here we consider the multiobjective duals based on some conjugate duality concepts like Fenchel duality and Fenchel - Lagrange duality (for more on this see [13]). For the four dual problems we completely establish inclusion relations between the image sets of their feasible sets through their objective functions. Moreover, we prove that the sets of maximal elements of these image sets are equal for all $\mu \in \mathbb{R}^m_+$.

Similar investigations on the existence of inclusion relations between the image sets and, respectively, between the sets of maximal elements of the image sets of different multiobjective duals have been done by two of the authors in [3] and [4]. A general scheme containing the relations between the multiobjective duals of Jahn ([7]), Nakayama ([9]), Wolfe ([14]), Mond-Weir ([15]) and a conjugate dual introduced by Wanka and Boţ in [12] is presented. Furthermore, conditions under which the dual problems are equivalent are given. In the current paper we extend these investigations to fractional multiobjective optimization problems.

The paper is structured as follows.

In Section 2 we introduce some preliminary notions and we formulate the multiobjective fractional primal problem and the intermediate convex problem $(P_\mu), \mu \in \mathbb{R}^m$, which is equivalent to the original in some sense. Furthermore, we introduce the dual (D_μ) for (P_μ) and recall the weak, strong and converse duality theorems given in [2].

In Section 3 we introduce the other multiobjective duals to (P_μ) and then we give the relations of inclusion between the image sets of these problems. The existence of strict inclusion relations is shown by some examples. Finally, we prove that the sets of maximal elements of the image sets are equal for all $\mu \in \mathbb{R}^m_+$.

2 Preliminaries

In this section we give some notations and preliminary results used later in the paper. The first definition introduces the ordering relation induced on $\mathbb{R}^k$ by the ordering cone $\mathbb{R}^k_+$.

Definition 2.1. For $y, z \in \mathbb{R}^k$ we denote $y \leqq z$ if $z - y \in \mathbb{R}^k_+ = \{u = (u_1, \dots, u_k)^T \in \mathbb{R}^k : u_i \geq 0, i = 1, \dots, k\}$.

The notions we introduce now come from convex analysis.

Definition 2.2. Let be $\mathcal{A} \subseteq \mathbb{R}^n$. The *indicator function of the set* $\mathcal{A}$, $\chi_{\mathcal{A}} : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$\chi_{\mathcal{A}} = \begin{cases} 0, & \text{if } x \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 2.3. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a given function. Then the *conjugate function of* f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by $f^*(p) = \sup\limits_{x \in \mathbb{R}^n} \{p^T x - f(x)\}$. Having a given subset $\mathcal{A} \subseteq \mathbb{R}^n$ we define the *conjugate function of* f *with respect to* $\mathcal{A}$, $f^*_{\mathcal{A}} : \mathbb{R}^n \to \overline{\mathbb{R}}$, as being

$$f^*_{\mathcal{A}}(p) = (f + \chi_{\mathcal{A}})^*(p) = \sup_{x \in \mathcal{A}} \{p^T x - f(x)\}.$$

The primal multiobjective fractional optimization problem considered here is

$$(P) \qquad \begin{aligned} &\text{v-}\min_{x \in \mathcal{A}} \ \Phi(x) \\ &\mathcal{A} = \left\{ x \in \mathbb{R}^n : \ g(x) = \big[g_1(x), \cdots, g_k(x)\big]^T \leqq 0 \right\}, \end{aligned}$$

where $\mathcal{A}$ is assumed to be non-empty, $\forall x \in \mathbb{R}^n$, $\Phi(x) = \big[\Phi_1(x), \cdots, \Phi_m(x)\big]^T = \left[\frac{f_1(x)}{h_1(x)}, \cdots, \frac{f_m(x)}{h_m(x)}\right]^T$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ are convex and proper functions, $(-h_i) : \mathbb{R}^n \to \mathbb{R}$ are convex functions fulfilling $h_i(x) > 0$, $\forall x \in \mathcal{A}$, $i = 1, \dots, m$, $g_j : \mathbb{R}^n \to \mathbb{R}$ are real-valued convex functions, $j = 1, \dots, k$, and $\bigcap\limits_{i=1}^{m} \operatorname{ri}(\operatorname{dom} f_i) \neq \emptyset$.

Note that $\mathcal{A}$ is convex, but nevertheless (P) is in general a non-convex problem.

In order to point out the optimal solutions of the problem (P), let us introduce the following definitions of efficiency and proper efficiency.

Definition 2.4 (Efficiency for problem (P))**.** An element $\bar{x} \in \mathcal{A}$ is said to be *efficient (or minimal) for* (P) if

$$\{\Phi(\bar{x}) - \mathbb{R}^m_+\} \cap \Phi(\mathcal{A}) = \{\Phi(\bar{x})\},$$

or, equivalently, if there is no $x \in \mathcal{A}$ such that

$$\Phi(x) \leqq \Phi(\bar{x})$$

and

$$\Phi(x) \neq \Phi(\bar{x}).$$

Definition 2.5 (Proper efficiency for problem (P))**.** A point $\bar{x} \in \mathcal{A}$ is said to be *properly efficient for* (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \text{int}(\mathbb{R}^m_+)$ such that

$$\sum_{i=1}^{m} \lambda_i \Phi_i(\bar{x}) \leq \sum_{i=1}^{m} \lambda_i \Phi_i(x), \ \forall x \in \mathcal{A}.$$

Let us notice that any properly efficient solution turns out to be an efficient one, too.

In order to investigate the duality for (P) we considered in [2] the following parametrized optimization problem by using an idea due to Dinkelbach ([5])

$$(P_\mu) \qquad \text{v-}\min_{x \in \mathcal{A}} \ \Phi^{(\mu)}(x),$$

where

$$\Phi^{(\mu)}(x) = \begin{bmatrix} \Phi_1^{(\mu)}(x) \\ \vdots \\ \Phi_m^{(\mu)}(x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} - \begin{bmatrix} \mu_1 \cdot h_1(x) \\ \vdots \\ \mu_m \cdot h_m(x) \end{bmatrix}$$

and $\mu = (\mu_1, ..., \mu_m)^T \in \mathbb{R}^m$. Note that $\Phi_i^{(\mu)}$ are proper and convex if $\mu_i \geq 0, i = 1, \ldots, m$.

Efficiency and proper efficiency for (P_μ) are defined in an analogous manner as done above for (P).

Kaul and Lyall ([8]) and Bector, Chandra and Singh ([1]) stated the connections between the efficient elements of (P) and (P_μ).

Theorem 2.1 ([1], [8])**.** *A point $\bar{x} \in \mathcal{A}$ is efficient for problem (P) if and only if $\bar{x}$ is efficient for problem $(P_{\bar{\mu}})$, where $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)$ and $\bar{\mu}_i := \frac{f_i(\bar{x})}{h_i(\bar{x})}$, $i = 1, \ldots, m$.*

Another efficiency notion used in [2] is the so-called properly efficiency in the sense of Geoffrion.

Definition 2.6 (Proper efficiency in the sense of Geoffrion [6]). A point $\bar{x} \in \mathcal{A}$ is said to be *properly efficient in the sense of Geoffrion for* (P) if it is efficient and if there is some real number $M > 0$ such that for each $i = 1, \ldots, m$ and each $x \in \mathcal{A}$ satisfying $\Phi_i(x) < \Phi_i(\bar{x})$ there exists at least one $j \in \{1, \ldots, m\}$ such that $\Phi_j(\bar{x}) < \Phi_j(x)$ and

$$\frac{\Phi_i(\bar{x}) - \Phi_i(x)}{\Phi_j(x) - \Phi_j(\bar{x})} \leq M.$$

Proper efficiency in the sense of Geoffrion for problem (P_μ) is defined in an analogous way, with $\Phi^{(\mu)}$ instead of Φ.

Theorem 2.2 ([2]). *Let be $\bar{x} \in \mathcal{A}$ and assume that $\bar{\mu}_i := \frac{f_i(\bar{x})}{h_i(\bar{x})} \geq 0, i = 1, \ldots, m$. The point $\bar{x}$ is properly efficient in the sense of Geoffrion for problem (P) if and only if $\bar{x}$ is properly efficient (in the sense of Definition 2.5) for problem $(P_{\bar{\mu}})$, where $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)^T$.*

The multiobjective dual problem to (P_μ), $\mu \in \mathbb{R}^m$, introduced in [2], based on the duality concept developed by two of the authors in [12], is the following one

$$(D_\mu) \qquad \text{v-}\max_{(u,v,q,\lambda,t)\in\mathcal{B}_\mu} \Psi^{(\mu)}(u, v, q, \lambda, t),$$

where

$$\begin{aligned}
\Psi^{(\mu)}(u, v, q, \lambda, t) &= \left[\Psi_1^{(\mu)}(u, v, q, \lambda, t), \cdots, \Psi_m^{(\mu)}(u, v, q, \lambda, t)\right]^T, \\
\Psi_i^{(\mu)}(u, v, q, \lambda, t) &= - f_i^*(u_i) - (-\mu_i h_i)^*(v_i) \\
&\qquad - (q_i^T g)^* \left(-\frac{1}{m\lambda_i}\sum_{j=1}^m \lambda_j(u_j + v_j)\right) + t_i, \ i = 1, \ldots, m,
\end{aligned}$$

the set of constraints is defined by

$$\mathcal{B}_\mu = \left\{(u, v, q, \lambda, t) : \ \lambda \in \operatorname{int}\mathbb{R}^m_+, \ \sum_{i=1}^m \lambda_i q_i \geqq 0, \ \sum_{i=1}^m \lambda_i t_i = 0\right\}$$

and the dual variables are $u = (u_1, \ldots, u_m), u_i \in \mathbb{R}^n$, $v = (v_1, \ldots, v_m), v_i \in \mathbb{R}^n$, $q = (q_1, \ldots, q_m), q_i \in \mathbb{R}^k, i = 1, \ldots, m$, $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \operatorname{int}\mathbb{R}^m_+$, $t = (t_1, \ldots, t_m)^T \in \mathbb{R}^m$.

The efficient elements of (D_μ) are defined in an analogous manner as for (P).

Definition 2.7 (Efficiency for problem (D_μ))**.** An element $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$ is said to be *efficient (or maximal) for* (D_μ) if

$$\{\Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) + \mathbb{R}^m_+\} \cap \Psi^{(\mu)}(\mathcal{B}_\mu) = \{\Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})\}.$$

We were able to prove the following weak duality result.

Theorem 2.3 (Weak duality [2])**.** *Let be* $\mu \in \mathbb{R}^m$*. There is no* $(u, v, q, \lambda, t) \in \mathcal{B}_\mu$ *and* $x \in \mathcal{A}$ *such that*

$$\Psi^{(\mu)}(u, v, q, \lambda, t) \geqq \Phi^{(\mu)}(x),$$

and

$$\Psi^{(\mu)}(u, v, q, \lambda, t) \neq \Phi^{(\mu)}(x).$$

For the strong duality theorem and the optimality conditions we need a constraint qualification. In order to formulate it let us consider the sets $L = \{j \in \{1, ..., k\} : g_j \text{ is affine}\}$ and $N = \{1, ..., k\} \setminus L$.

Constraint qualification (CQ)
There exists an element $x' \in \bigcap\limits_{i=1}^{m} \operatorname{ri}(\operatorname{dom} f_i)$ such that $g_j(x') < 0$, $j \in N$, and $g_j(x') \leq 0, j \in L$.

Theorem 2.4 (Strong duality [2])**.** *Let* $\mu \in \mathbb{R}^m_+$ *and* (CQ) *be fulfilled. If* $\bar{x}$ *is a properly efficient element of* (P_μ)*, then there exists an efficient solution* $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$ *of* (D_μ) *and strong duality holds, i.e.*

$$\Phi^{(\mu)}(\bar{x}) = \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}).$$

Let us introduce now the following condition which will be helpful for the converse duality theorem.

Definition 2.8. Let be $\mu \in \mathbb{R}^m_+$ and $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$. The condition $(C_{\mu,\lambda})$ is fulfilled when from

$$\inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i \Phi_i^{(\mu)}(x) > -\infty$$

it follows that there exists $x_\lambda \in \mathcal{A}$ such that

$$\inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i \Phi_i^{(\mu)}(x) = \sum_{i=1}^{m} \lambda_i \Phi_i^{(\mu)}(x_\lambda).$$

Now the converse duality theorem for (P_μ) can be formulated.

Theorem 2.5 ([2]). *Let be $\mu \in \mathbb{R}^m_+$ given, (CQ) be fulfilled and assume that $(C_{\mu,\lambda})$ holds for all $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$.*

(1) Let $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ be an efficient solution of (D_μ). Then

(a) $\Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \operatorname{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+)$;

(b) there exists a properly efficient solution $\bar{x}_{\bar{\lambda}} \in \mathcal{A}$ of (P_μ) such that

$$\sum_{i=1}^m \bar{\lambda}_i [\Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) - \Psi_i^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})] = 0.$$

(2) If, additionally, $\Phi^{(\mu)}(\mathcal{A})$ is $\mathbb{R}^m_+$-closed ($\Phi^{(\mu)}(\mathcal{A})+\mathbb{R}^m_+$ is closed), then there exists a properly efficient solution $\bar{x} \in \mathcal{A}$ of (P_μ) such that

$$\sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}),$$

and

$$\Phi^{(\mu)}(\bar{x}) = \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}).$$

By using the previous results, one can give dual characterizations for the solutions of the fractional multiobjective optimization problem (P).

Theorem 2.6 ([2]). *Let (CQ) be fulfilled and $\bar{x} \in \mathcal{A}$ be properly efficient in the sense of Geoffrion for problem (P) with $\bar{\mu}_i := \frac{f_i(\bar{x})}{h_i(\bar{x})} \geq 0$, $i = 1, \dots, m$. Let be $\bar{\mu} := (\bar{\mu}_1, \dots, \bar{\mu}_m)^T$. Then $\bar{x}$ is properly efficient for $(P_{\bar{\mu}})$, there exists $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_{\bar{\mu}}$ that is efficient for $(D_{\bar{\mu}})$ and strong duality between $(P_{\bar{\mu}})$ and $(D_{\bar{\mu}})$ holds.*

Theorem 2.7 ([2]). *Let (CQ) be fulfilled and $\bar{\mu} \in \mathbb{R}^m_+$ such that the set $\Phi^{\bar{\mu}}(\mathcal{A})$ is $\mathbb{R}^m_+$-closed. Moreover, assume that $(C_{\bar{\mu},\lambda})$ holds for all $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$. Let $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ be an efficient solution for $(D_{\bar{\mu}})$. Then there exists $\bar{x} \in \mathcal{A}$, a properly efficient solution for $(P_{\bar{\mu}})$, and strong duality between $(P_{\bar{\mu}})$ and $(D_{\bar{\mu}})$ holds. If $\Phi(\bar{x}) = \bar{\mu}$ then $\bar{x}$ is properly efficient in the sense of Geoffrion for (P).*

3 Comparison with other dual problems

In this section we make a comparison between different dual problems to the parametrized multiobjective optimization problem (P_μ) when $\mu \in \mathbb{R}^m_+$. Along the problem (D_μ) introduced in the previous section we consider other two multiobjective problems constructed by using the approach described in [3] as well as the multiobjective dual due to Ohlendorf and Tammer ([10]).

3.1 Formulation of the dual problems

Boţ and Wanka developed in [3] an approach for constructing multiobjective dual problems by using different scalar dual problems. They extended the results of Jahn ([7]) for Lagrange duality to different conjugate duality concepts like Fenchel duality and the so-called Fenchel - Lagrange duality (for more on this see [13]). We also take these two duals into consideration in order to formulate two further multiobjective dual problems to (P_μ)

$$(D_F^\mu) \qquad \text{v-}\max_{(u,v,\lambda,y)\in\mathcal{B}_\mu^F} \Psi^{(F)}(u,v,\lambda,y),$$

where

$$\Psi^{(F)}(u,v,\lambda,y) = \begin{bmatrix} \Psi_1^{(F)}(u,v,\lambda,y_1) \\ \vdots \\ \Psi_m^{(F)}(u,v,\lambda,y_m) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix},$$

$$\mathcal{B}_\mu^F = \Bigg\{ (u,v,\lambda,y):\ \lambda \in \operatorname{int}\mathbb{R}_+^m, \sum_{i=1}^m \lambda_i y_i \leq -\sum_{i=1}^m \lambda_i \left[f_i^*(u_i) + (-\mu_i h_i)^*(v_i)\right] - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^m \lambda_i(u_i+v_i)\right) \Bigg\}$$

and

$$(D_{FL}^\mu) \qquad \text{v-}\max_{(u,v,q,\lambda,y)\in\mathcal{B}_\mu^{FL}} \Psi^{(FL)}(u,v,q,\lambda,y),$$

where

$$\Psi^{(FL)}(u,v,q,\lambda,y) = \begin{bmatrix} \Psi_1^{(FL)}(u,v,q,\lambda,y_1) \\ \vdots \\ \Psi_m^{(FL)}(u,v,q,\lambda,y_m) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix},$$

$$\mathcal{B}_\mu^{FL} = \Bigg\{ (u,v,q,\lambda,y):\ \lambda \in \operatorname{int}\mathbb{R}_+^m,\ q \in \mathbb{R}_+^k, \sum_{i=1}^m \lambda_i y_i \leq -\sum_{i=1}^m \lambda_i \left[f_i^*(u_i) + (-\mu_i h_i)^*(v_i)\right] - (q^T g)^*\left(-\sum_{i=1}^m \lambda_i(u_i+v_i)\right) \Bigg\}.$$

The fourth multiobjective dual problem considered here is the so - called Fenchel - type dual according to Ohlendorf and Tammer [10]

(D^{μ}_{O}) $$\text{v-}\max_{(p,\lambda,y)\in\mathcal{B}^{O}_{\mu}} \Psi^{(O)}(p,\lambda,y),$$

where

$$\Psi^{(O)}(p,\lambda,y) = \begin{bmatrix} \Psi^{(O)}_{1}(p,\lambda,y_1) \\ \vdots \\ \Psi^{(O)}_{m}(p,\lambda,y_m) \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix},$$

$$\mathcal{B}^{O}_{\mu} = \Bigg\{ (p,\lambda,y): \ \lambda \in \operatorname{int}\mathbb{R}^{m}_{+}, p \in \mathbb{R}^{n},$$
$$\sum_{i=1}^{m} \lambda_i y_i = -\left(-\sum_{i=1}^{m} \lambda_i \mu_i h_i\right)^{*}_{\mathcal{A}} (-p) - \left(\sum_{i=1}^{m} \lambda_i f_i\right)^{*}_{\mathcal{A}} (p) \Bigg\}.$$

The weak and strong duality assertions for the presented problems have been proved by Boţ and Wanka in [3] and Ohlendorf and Tammer in [10], respectively.

3.2 Inclusions between the image sets

For $\mu = (\mu_1, ..., \mu_m)^T \in \mathbb{R}^m_+$ we denote the image sets of the feasible sets of the four multiobjective duals through their objective functions by $\mathcal{D}_\mu := \Psi^{(\mu)}(\mathcal{B}_\mu)$, $\mathcal{D}^{(\mu)}_{FL} := \Psi^{(FL)}(\mathcal{B}^{FL}_{\mu})$, $\mathcal{D}^{(\mu)}_{F} := \Psi^{(F)}(\mathcal{B}^{F}_{\mu})$ and $\mathcal{D}^{(\mu)}_{O} := \Psi^{(O)}(\mathcal{B}^{O}_{\mu})$. Next we study the inclusion relations which exist between them.

We omit proving the theorem below as this result can be derived from Proposition 5.2 in [3] and Proposition 2.1 in [4].

Theorem 3.1. *It holds* $\mathcal{D}_\mu \cap \mathbb{R}^m \subseteq \mathcal{D}^{(\mu)}_{FL} \subseteq \mathcal{D}^{(\mu)}_{F}, \ \forall \mu \in \mathbb{R}^m_+$.

Example 5.2 in [3] and Example 2.1 in [4] show that the relations of inclusion in Theorem 3.1 can be also strict.

Assuming the constraint qualification (CQ) is fulfilled, Proposition 3.1 in [4] offers a refinement of the relation above.

Theorem 3.2 (Proposition 3.1, [4]). *Let* (CQ) *be fulfilled. Then it holds*

$$\mathcal{D}^{(\mu)}_{FL} = \mathcal{D}^{(\mu)}_{F}, \ \forall \mu \in \mathbb{R}^m_+.$$

This means that if (CQ) is fulfilled, then we have for all $\mu \in \mathbb{R}^m_+$

$$\mathcal{D}_\mu \cap \mathbb{R}^m \subseteq \mathcal{D}^{(\mu)}_{FL} = \mathcal{D}^{(\mu)}_{F}.$$

In the next example we show that the first inclusion in the relation above can be strict.

Example 3.1. *Let* $n = 1, m = 2, k = 1$, $f_1(x) = x$, $f_2(x) = 0, x \in \mathbb{R}$, $g(x) = -x, x \in \mathbb{R}$, $h_1(x) = h_2(x) = 1, x \in \mathbb{R}$, *and* $\mu = (1,1)^T$. *Thus the feasible set* $\mathcal{A}$ *looks like* $\mathcal{A} = \{x \in \mathbb{R} : x \geq 0\}$ *and it is obvious that the constraint qualification* (CQ) *is fulfilled.*

The conjugate functions turn out to be

$$f_1^*(u_1) = \begin{cases} 0, & \text{if } u_1 = 1, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$f_2^*(u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

and, respectively, for $i = 1, 2$,

$$(-h_i)^*(v_i) = \begin{cases} 1, & \text{if } v_i = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

For $u = (1,0), v = (0,0), \lambda = (1,1)^T$ *and* $d = (-2,-2)^T$ *we have that* $\lambda_1 d_1 + \lambda_2 d_2 = -4$ *and, on the other hand,*

$$-\lambda_1[f_1^*(u_1) + (-h_1)^*(v_1)] - \lambda_2[f_2^*(u_2) + (-h_2)^*(v_2)]$$
$$-\chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{2}\lambda_i(u_i + v_i)\right) = -2 + \inf_{x \geq 0} x = -2.$$

This means that $(u, v, \lambda, d) \in \mathcal{B}_\mu^F$, *which is nothing else than* $d \in \mathcal{D}_F^{(\mu)}$.

Let us show now that $d \notin \mathcal{D}_\mu$. *If this were not true, then there would exist an element* $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$ *such that*

$$\begin{pmatrix} -2 \\ \\ -2 \end{pmatrix} = \begin{pmatrix} -f_1^*(\bar{u}_1) - (-h_1)^*(\bar{v}_1) - (\bar{q}_1 g)^*\left(-\frac{1}{2\lambda_1}\sum_{j=1}^{2}\bar{\lambda}_j(\bar{u}_j + \bar{v}_j)\right) + \bar{t}_1 \\ \\ -f_2^*(\bar{u}_2) - (-h_2)^*(\bar{v}_2) - (\bar{q}_2 g)^*\left(-\frac{1}{2\lambda_2}\sum_{j=1}^{2}\bar{\lambda}_j(\bar{u}_j + \bar{v}_j)\right) + \bar{t}_2 \end{pmatrix}.$$

In order to happen this we must have $\bar{u}_1 = 1, \bar{u}_2 = 0, \bar{v}_1 = 0, \bar{v}_2 = 0$ *and so*

$$\begin{pmatrix} -2 \\ \\ -2 \end{pmatrix} = \begin{pmatrix} -1 + \inf_{x \in \mathbb{R}}\left[\left(\frac{1}{2} - \bar{q}_1\right)x\right] + \bar{t}_1 \\ \\ -1 + \inf_{x \in \mathbb{R}}\left[\left(\frac{\bar{\lambda}_1}{2\bar{\lambda}_2} - \bar{q}_2\right)x\right] + \bar{t}_2 \end{pmatrix}.$$

This relation can be true just if $\bar{q}_1 = \frac{1}{2}$, $\bar{q}_2 = \frac{\bar{\lambda}_1}{2\bar{\lambda}_2}$ *and* $\bar{t}_1 = \bar{t}_2 = -1$. *As* $\bar{\lambda}_1\bar{t}_1 + \bar{\lambda}_2\bar{t}_2 < 0$, *this leads to a contradiction to* $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$.

Next we study the existence of an inclusion between $\mathcal{D}_O^{(\mu)}$ and $\mathcal{D}_F^{(\mu)}$, assuming that the constraint qualification (CQ) is fulfilled. To this end we formulate and study the Fenchel dual to the following scalarized primal

$$(P_{\mu,\lambda}) \qquad \inf_{x\in\mathcal{A}} \Phi^{(\mu,\lambda)}(x),$$

where

$$\Phi^{(\mu,\lambda)}(x) = \sum_{i=1}^{m} \lambda_i \cdot \Phi_i^{(\mu)}(x) = \sum_{i=1}^{m} \lambda_i \cdot \left(f_i(x) - \mu_i \cdot h_i(x)\right), x \in \mathbb{R}^n$$

and $\lambda_i > 0, i = 1, \dots, m$.
The Fenchel dual to $(P_{\mu,\lambda})$ is (see, for example, [11])

$$(D_{\mu,\lambda}^{(F)}) \qquad \sup_{p\in\mathbb{R}^n} \left\{-(\Phi^{(\mu,\lambda)})^*(p) - \chi_{\mathcal{A}}^*(-p)\right\}.$$

As we will see in the proof of the next lemma, the Fenchel dual to $(P_{\mu,\lambda})$ turns out to be

$$(D_{\mu,\lambda}^{(F)}) \quad \sup_{\substack{u_i,v_i\in\mathbb{R}^n,\\ i=1\dots m}} \left\{-\sum_{i=1}^{m} \lambda_i[f_i^*(u_i) + (-\mu_i h_i)^*(v_i)] - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m} \lambda_i(u_i+v_i)\right)\right\}.$$

Lemma 3.1. *Assume that (CQ) is fulfilled and* $\inf(P_{\mu,\lambda})$ *is finite. Then there is*

$$\inf(P_{\mu,\lambda}) = \max(D_{\mu,\lambda}^{(F)}),$$

and the dual problem $(D_{\mu,\lambda}^{(F)})$ has an optimal solution.

Proof. The constraint qualification (CQ) being fulfilled, according to Theorem 31.1 in [11], it follows that between $(P_{\mu,\lambda})$ and $(D_{\mu,\lambda}^{(F)})$ strong duality holds, namely

$$\inf(P_{\mu,\lambda}) = \max(D_{\mu,\lambda}^{(F)}),$$

and $(D_{\mu,\lambda}^{(F)})$ has an optimal solution. Thus there exists $\bar{p} \in \mathbb{R}^n$ such that

$$\inf(P_{\mu,\lambda}) = -(\Phi^{(\mu,\lambda)})^*(\bar{p}) - \chi_{\mathcal{A}}^*(-\bar{p}).$$

On the other hand, as $\bigcap_{i=1}^{m} (\operatorname{ri}(\operatorname{dom} f_i)) \neq \emptyset$, the conjugate function of $\Phi^{(\mu,\lambda)}$ turns out to be $\forall p \in \mathbb{R}^n$ (cf. Theorem 16.4 in [11])

$$\begin{aligned}(\Phi^{(\mu,\lambda)})^*(p) &= \min\left\{\sum_{i=1}^{m}(\lambda_i f_i)^*(r_i) + \sum_{i=1}^{m}(-\lambda_i\mu_i h_i)^*(s_i) : \sum_{i=1}^{m}(r_i+s_i) = p\right\}\\ &= \min\left\{\sum_{i=1}^{m}\lambda_i f_i^*(u_i) + \sum_{i=1}^{m}\lambda_i(-\mu_i h_i)^*(v_i) : \sum_{i=1}^{m}\lambda_i(u_i+v_i) = p\right\}.\end{aligned}$$

One can see that indeed

$$(D^{(F)}_{\mu,\lambda}) \quad \sup_{\substack{u_i,v_i\in\mathbb{R}^n,\\ i=1\dots m}} \left\{ -\sum_{i=1}^m \lambda_i[f_i^*(u_i)+(-\mu_i h_i)^*(v_i)] - \chi^*_{\mathcal{A}}\left(-\sum_{i=1}^m \lambda_i(u_i+v_i)\right)\right\}$$

is the Fenchel dual of $(P_{\mu,\lambda})$ and that strong duality holds. □

Theorem 3.3. *Let (CQ) be fulfilled. Then it holds $\mathcal{D}_O^{(\mu)} \subseteq \mathcal{D}_F^{(\mu)}$.*

Proof. Let be $\bar{d} \in \mathcal{D}_O^{(\mu)}$ with corresponding $(\bar{\lambda},\bar{p}) \in \operatorname{int}\mathbb{R}^m_+ \times \mathbb{R}^n$. Then we get the following relations

$$\begin{aligned}
\sum_{i=1}^m \bar{\lambda}_i\bar{d}_i &= -\left(-\sum_{i=1}^m \bar{\lambda}_i\mu_i h_i\right)^*_{\mathcal{A}}(-\bar{p}) - \left(\sum_{i=1}^m \bar{\lambda}_i f_i\right)^*_{\mathcal{A}}(\bar{p}) \\
&= \inf_{x\in\mathcal{A}}\left\{\bar{p}^T x - \sum_{i=1}^m \bar{\lambda}_i\mu_i h_i(x)\right\} + \inf_{x\in\mathcal{A}}\left\{-\bar{p}^T x + \sum_{i=1}^m \bar{\lambda}_i f_i(x)\right\} \\
&\le \inf_{x\in\mathcal{A}}\left\{\bar{p}^T x - \sum_{i=1}^m \bar{\lambda}_i\mu_i h_i(x) - \bar{p}^T x + \sum_{i=1}^m \bar{\lambda}_i f_i(x)\right\} \\
&= \inf_{x\in\mathcal{A}}\left\{\sum_{i=1}^m \bar{\lambda}_i(f_i(x)-\mu_i h_i(x))\right\}.
\end{aligned}$$

The right-hand side is nothing else but the scalarization of the parameterized primal problem (P_μ). According to Lemma 3.1 there exists an optimal solution to $(D^{(F)}_{\mu,\bar{\lambda}})$, say $(\bar{u},\bar{v}) = (\bar{u}_1,\dots,\bar{u}_m,\bar{v}_1,\dots,\bar{v}_m)$, $\bar{u}_i,\bar{v}_i \in \mathbb{R}^n, i=1,\dots,m$, such that strong duality holds.Thus,

$$\begin{aligned}
\sum_{i=1}^m \bar{\lambda}_i\bar{d}_i &\le \max(D^{(F)}_{\mu,\bar{\lambda}}) \\
&= -\sum_{i=1}^m \bar{\lambda}_i f_i^*(\bar{u}_i) - \sum_{i=1}^m \bar{\lambda}_i(-\mu_i h_i)^*(\bar{v}_i) - \chi^*_{\mathcal{A}}\left(-\sum_{i=1}^m \bar{\lambda}_i(\bar{u}_i+\bar{v}_i)\right).
\end{aligned}$$

This means that $(\bar{u},\bar{v},\bar{\lambda},\bar{d}) \in \mathcal{B}^F_\mu$ and so $\bar{d} \in \Psi^F(\mathcal{B}^F_\mu) = \mathcal{D}_F^{(\mu)}$. □

Assuming the constraint qualification (CQ) is fulfilled, by Theorem 3.2 and Theorem 3.3, we have for all $\mu \in \mathbb{R}^m_+$

$$\mathcal{D}_O^{(\mu)} \subseteq \mathcal{D}_{FL}^{(\mu)} = \mathcal{D}_F^{(\mu)}.$$

Below we introduce a further example which shows that, in general, one can find an element $\mu \in \mathbb{R}^m_+$ such that the inclusion $\mathcal{D}_\mu \cap \mathbb{R}^m \subseteq \mathcal{D}_O^{(\mu)}$ fails. This implies that the inclusion in the relation above can indeed be strict.

Example 3.2. *Let $n = 1, m = 2, k = 1,\ f_1(x) = x + 2, f_2(x) = -x + 2,$ $h_1(x) = h_2(x) = 1, x \in \mathbb{R}$ and $\mu = (1,1)^T$. For*

$$g(x) = \begin{cases} (x-1)^2 - 1, & \text{if } x \leq 1, \\ -1, & \text{otherwise,} \end{cases}$$

the feasible set is defined as $\mathcal{A} = \{x \in \mathbb{R} : g(x) \leq 0\} = [0, +\infty)$. The constraint qualification (CQ) is again fulfilled. The conjugate functions become

$$f_1^*(u_1) = \begin{cases} -2, & \text{if } u_1 = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f_2^*(u_2) = \begin{cases} -2, & \text{if } u_2 = -1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and for $i = 1, 2$

$$(-h_i)^*(v_i) = \begin{cases} 1, & \text{if } v_i = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Choosing $u = (1,-1), v = (0,0), q = (1,1), \lambda = (1,1)^T, t = (0,0)^T$, we get that $(u, v, q, \lambda, t) \in \mathcal{B}_\mu$ because $\lambda \in \operatorname{int} \mathbb{R}^2_+$, $\sum\limits_{i=1}^{2} \lambda_i q_i = 1 + 1 = 2 \geq 0$ and $\sum\limits_{i=1}^{2} \lambda_i t_i = 0$. Furthermore

$$\begin{aligned} \Psi_i^{(\mu)}(u, v, q, \lambda, t) &= 2 - 1 - (1 \cdot g)^* \left(-\frac{1}{2} \sum_{i=1}^{2} (u_i + v_i) \right) + 0 \\ &= 1 + \inf_{x \in \mathbb{R}} g(x) = 0, i = 1, 2. \end{aligned}$$

This means that the element $d = (0,0)^T \in \mathcal{D}_\mu \cap \mathbb{R}^2$.

But $d \notin \mathcal{D}_O^{(\mu)}$, because in the opposite situation there would exist $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in \operatorname{int} \mathbb{R}^2_+$ and $\bar{p} \in \mathbb{R}$ such that

$$\begin{aligned} \bar{\lambda}_1 d_1 + \bar{\lambda}_2 d_2 = 0 &= -\left(-\sum_{i=1}^{2} \bar{\lambda}_i h_i \right)^*_{\mathcal{A}} (-\bar{p}) - \left(\sum_{i=1}^{2} \bar{\lambda}_i f_i \right)^*_{\mathcal{A}} (\bar{p}) \\ &= \inf_{x \in \mathcal{A}} [\bar{p}x] + \bar{\lambda}_1 + \bar{\lambda}_2 + \inf_{x \in \mathcal{A}} [(-\bar{p} + \bar{\lambda}_1 - \bar{\lambda}_2)x]. \end{aligned}$$

This can be the case just if $\bar{p} \geq 0$, $-\bar{p} + \bar{\lambda}_1 - \bar{\lambda}_2 \geq 0$ and $\bar{\lambda}_1 + \bar{\lambda}_2 = 0$. As this can never be the case, the assertion is proved.

The last example of this section shows that, in general, one can find an element $\mu \in \mathbb{R}^m_+$ such that the inclusion $\mathcal{D}_O^{(\mu)} \subseteq \mathcal{D}_\mu \cap \mathbb{R}^m$ also fails. This means that, even if the constraint qualification (CQ) is fulfilled, between the image sets $\mathcal{D}_O^{(\mu)}$ and $\mathcal{D}_\mu \cap \mathbb{R}^m$ there exists no relation of inclusion which holds for all $\mu \in \mathbb{R}^m_+$.

Example 3.3. *Let $n = 2, m = 2, k = 1$, $f_1(x_1, x_2) = x_2$, $f_2(x_1, x_2) = 0, (x_1, x_2)^T \in \mathbb{R}^2$, $h_1(x_1, x_2) = h_2(x_1, x_2) = 1, (x_1, x_2)^T \in \mathbb{R}^2$, and $\mu = (1, 1)^T$. For $g(x_1, x_2) = x_1^2 - x_2, (x_1, x_2)^T \in \mathbb{R}^2$, the feasible set looks like $\mathcal{A} = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1^2 \leq x_2\}$ and it obvious that the constraint qualification (CQ) is fulfilled.*

The conjugate functions turn out to be

$$f_1^*(u^1) = \begin{cases} 0, & \text{if } u^1 = (0,1)^T, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$f_2^*(u^2) = \begin{cases} 0, & \text{if } u^2 = (0,0)^T, \\ +\infty, & \text{otherwise}, \end{cases}$$

and, respectively, for $i = 1, 2$,

$$(-h_i)^*(v^i) = \begin{cases} 1, & \text{if } v^i = (0,0)^T \\ +\infty, & \text{otherwise}. \end{cases}$$

For $p = \left(-1, \frac{1}{2}\right)$, $\lambda = (1,1)^T$ and $d = (-2,-1)^T$ we have that $\lambda_1 d_1 + \lambda_2 d_2 = -3$ and, on the other hand,

$$-(-\lambda_1 h_1 - \lambda_2 h_2)^*_{\mathcal{A}}(-p) - (\lambda_1 f_1 + \lambda_2 f_2)^*_{\mathcal{A}}(p) =$$

$$\inf_{x_1^2 \leq x_2} \left[-x_1 + \frac{1}{2}x_2 - 2\right] + \inf_{x_1^2 \leq x_2} \left[x_1 + \frac{1}{2}x_2\right] = -\frac{1}{2} - 2 - \frac{1}{2} = -3.$$

This means that $(p, \lambda, d) \in \mathcal{B}^O_\mu$, which is nothing else than $d \in \mathcal{D}_O^{(\mu)}$.

Let us show now that $d \notin \mathcal{D}_\mu$. If this were not true, then there would exist an element $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$, $\bar{u} = (\bar{u}^1, \bar{u}^2) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\bar{v} = (\bar{v}^1, \bar{v}^2) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\bar{q} = (\bar{q}_1, \bar{q}_2) \in \mathbb{R}^2_+$, $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in \operatorname{int} \mathbb{R}^2_+$, $\bar{t} = (\bar{t}_1, \bar{t}_2)^T \in \mathbb{R}^2$, such that

$$\begin{pmatrix} -2 \\ \\ -1 \end{pmatrix} = \begin{pmatrix} -f_1^*(\bar{u}^1) - (-h_1)^*(\bar{v}^1) - (\bar{q}_1 g)^* \left(-\frac{1}{2\bar{\lambda}_1} \sum\limits_{j=1}^{2} \bar{\lambda}_j (\bar{u}^j + \bar{v}^j)\right) + \bar{t}_1 \\ \\ -f_2^*(\bar{u}_2) - (-h_2)^*(\bar{v}^2) - (\bar{q}_2 g)^* \left(-\frac{1}{2\bar{\lambda}_2} \sum\limits_{j=1}^{2} \bar{\lambda}_j (\bar{u}^j + \bar{v}^j)\right) + \bar{t}_2 \end{pmatrix}.$$

In order to happen this we must have $\bar{u}^1 = (0,1), \bar{u}^2 = (0,0), \bar{v}^1 = (0,0), \bar{v}^2 = (0,0)$ *and, so,*

$$\begin{pmatrix} -2 \\ \\ -1 \end{pmatrix} = \begin{pmatrix} -1 + \inf\limits_{(x_1,x_2)^T \in \mathbb{R}^2} \left[\bar{q}_1 x_1^2 + \left(\frac{1}{2} - \bar{q}_1\right) x_2\right] + \bar{t}_1 \\ \\ -1 + \inf\limits_{(x_1,x_2)^T \in \mathbb{R}^2} \left[\bar{q}_2 x_1^2 + \left(\frac{\bar{\lambda}_1}{2\bar{\lambda}_2} - \bar{q}_2\right) x_2\right] + \bar{t}_2 \end{pmatrix}.$$

This relation can be true just if $\bar{q}_1 = \frac{1}{2}$, $\bar{q}_2 = \frac{\bar{\lambda}_1}{2\bar{\lambda}_2}$, $\bar{t}_1 = -1$ *and* $\bar{t}_2 = 0$. *As* $\bar{\lambda}_1 \bar{t}_1 + \bar{\lambda}_2 \bar{t}_2 < 0$, *this leads to a contradiction.*

What we succeeded to prove is that, assuming that (CQ) holds, for all $\mu \in \mathbb{R}^m_+$,

$$\mathcal{D}_\mu \cap \mathbb{R}^m \subseteq \mathcal{D}_{FL}^{(\mu)} = \mathcal{D}_F^{(\mu)}$$

and

$$\mathcal{D}_O^{(\mu)} \subseteq \mathcal{D}_{FL}^{(\mu)} = \mathcal{D}_F^{(\mu)}.$$

In general, both inclusions in the relations above can be strict. Moreover, between the image sets $\mathcal{D}_O^{(\mu)}$ and $\mathcal{D}_\mu \cap \mathbb{R}^m$ there exists no relation of inclusion which holds for all $\mu \in \mathbb{R}^m_+$.

3.3 Inclusion between the efficiency sets

In this section we extend our study to the comparison of the sets of maximal elements of the image sets we dealt with in the previous subsection. Having a given subset $\mathcal{D} \subseteq \mathbb{R}^m$, an element $d \in \mathcal{D}$ is said to be *maximal* if there exists no $\bar{d} \in \mathcal{D}$ such that $\bar{d} - d \in \mathbb{R}^m_+$ and $\bar{d} \neq d$. The set of maximal elements of $\mathcal{D}$ will be denoted by $\text{vmax}(\mathcal{D})$.

In the following we assume that the constraint qualification (CQ) is fulfilled. Under this assumption we can derive the first theorem from Theorem 3.2 and, respectively, Theorem 5.4 in [3].

Theorem 3.4. *It holds*

$$\text{vmax}(\mathcal{D}_\mu) = \text{vmax}(\mathcal{D}_F^{(\mu)}) = \text{vmax}(\mathcal{D}_{FL}^{(\mu)}), \ \forall \mu \in \mathbb{R}^m_+.$$

The sets of maximal elements are nothing else than the image sets of the efficiency sets of the corresponding dual problems.

Now it remains to investigate if there are some connections between the set of maximal elements of $\text{vmax}(\mathcal{D}_O^{(\mu)})$ and the sets in the relation above. We prove first the following theorem.

Theorem 3.5. *It holds*

$$\text{vmax}(\mathcal{D}_O^{(\mu)}) \subseteq \text{vmax}(\mathcal{D}_F^{(\mu)}),\ \forall \mu \in \mathbb{R}_+^m.$$

Proof. Let be $\mu \in \mathbb{R}_+^m$ fixed and $d \in \text{vmax}(\mathcal{D}_O^{(\mu)})$. This implies that $d \in \mathcal{D}_O^{(\mu)}$ and, according to Theorem 3.3, we have $d \in \mathcal{D}_F^{(\mu)}$.

Now assume that $d \notin \text{vmax}(\mathcal{D}_F^{(\mu)})$. This means that there exists $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{d}) \in \mathcal{B}_\mu^F$ such that $d \in \bar{d} - \{\mathbb{R}_+^m \backslash \{0\}\}$. Furthermore it holds

$$\begin{aligned}
\sum_{i=1}^m \bar{\lambda}_i d_i < \sum_{i=1}^m \bar{\lambda}_i \bar{d}_i & \\
&\leq -\sum_{i=1}^m \bar{\lambda}_i \left[f_i^*(\bar{u}_i) + (-\mu_i h_i)^*(\bar{v}_i)\right] - \chi_\mathcal{A}^*\left(-\sum_{i=1}^m \bar{\lambda}_i(\bar{u}_i + \bar{v}_i)\right) \\
&= -\sum_{i=1}^m \bar{\lambda}_i \sup_{x\in\mathbb{R}^n}\{\bar{u}_i^T x - f_i(x)\} - \sum_{i=1}^m \bar{\lambda}_i \sup_{x\in\mathbb{R}^n}\{\bar{v}_i^T x - (-\mu_i h_i)(x)\} \\
&\quad - \sup_{x\in\mathbb{R}^n}\left\{\left(-\sum_{i=1}^m \bar{\lambda}_i(\bar{u}_i + \bar{v}_i)\right)^T x - \chi_\mathcal{A}(x)\right\}.
\end{aligned}$$

The supremum of a function over the whole space is always greater than or equal to the supremum over a subset of this space. Thus it follows

$$\begin{aligned}
\sum_{i=1}^m \bar{\lambda}_i d_i < &-\sum_{i=1}^m \bar{\lambda}_i \sup_{x\in\mathcal{A}}\{\bar{u}_i^T x - f_i(x)\} - \sum_{i=1}^m \bar{\lambda}_i \sup_{x\in\mathcal{A}}\{\bar{v}_i^T x + \mu_i h_i(x)\} \\
&- \sup_{x\in\mathcal{A}}\left\{\left(-\sum_{i=1}^m \bar{\lambda}_i(\bar{u}_i + \bar{v}_i)\right)^T x\right\},
\end{aligned}$$

and from here there is

$$\begin{aligned}
\sum_{i=1}^m \bar{\lambda}_i d_i < &-\sup_{x\in\mathcal{A}}\left\{\sum_{i=1}^m \bar{\lambda}_i[\bar{u}_i^T x - f_i(x)]\right\} - \sup_{x\in\mathcal{A}}\left\{\sum_{i=1}^m \bar{\lambda}_i[\bar{v}_i^T x + \mu_i h_i(x)]\right\} \\
&- \sup_{x\in\mathcal{A}}\left\{\left(-\sum_{i=1}^m \bar{\lambda}_i(\bar{u}_i + \bar{v}_i)\right)^T x\right\} \\
&\leq -\sup_{x\in\mathcal{A}}\left\{-\sum_{i=1}^m \bar{\lambda}_i[\bar{u}_i^T x - \mu_i h_i(x)]\right\} - \sup_{x\in\mathcal{A}}\left\{\sum_{i=1}^m \bar{\lambda}_i[\bar{u}_i^T x - f_i(x)]\right\}
\end{aligned}$$

$$= \inf_{x\in\mathcal{A}} \left\{ \left(\sum_{i=1}^{m} \bar{\lambda}_i \bar{u}_i \right)^T x - \sum_{i=1}^{m} \bar{\lambda}_i \mu_i h_i(x) \right\}$$
$$- \sup_{x\in\mathcal{A}} \left\{ \left(\sum_{i=1}^{m} \bar{\lambda}_i \bar{u}_i \right)^T x - \sum_{i=1}^{m} \bar{\lambda}_i f_i(x) \right\}$$
$$= - \left(- \sum_{i=1}^{m} \bar{\lambda}_i \mu_i h_i \right)^*_{\mathcal{A}} \left(- \sum_{i=1}^{m} \bar{\lambda}_i \bar{u}_i \right) - \left(\sum_{i=1}^{m} \bar{\lambda}_i f_i \right)^*_{\mathcal{A}} \left(\sum_{i=1}^{m} \bar{\lambda}_i \bar{u}_i \right).$$

Choose now $\tilde{d} \in \bar{d} + \mathbb{R}^m_+$ such that

$$\sum_{i=1}^{m} \bar{\lambda}_i \tilde{d}_i = - \left(- \sum_{i=1}^{m} \bar{\lambda}_i \mu_i h_i \right)^*_{\mathcal{A}} \left(- \sum_{i=1}^{m} \bar{\lambda}_i \bar{u}_i \right) - \left(\sum_{i=1}^{m} \bar{\lambda}_i f_i \right)^*_{\mathcal{A}} \left(\sum_{i=1}^{m} \bar{\lambda}_i \bar{u}_i \right).$$

As for $\bar{p} := \sum\limits_{i=1}^{m} \bar{\lambda}_i \bar{u}_i$, $(\bar{p}, \lambda, \tilde{d}) \in \mathcal{B}^O_\mu$, it follows $\tilde{d} \in \mathcal{D}^{(\mu)}_O$. But $\tilde{d} \in d + \{\mathbb{R}^m_+ \backslash \{0\}\}$ and this contradicts the maximality of d in $\mathcal{D}^{(\mu)}_O$. □

The next theorem shows that the reverse inclusion also holds.

Theorem 3.6. *It holds*

$$\text{vmax}(\mathcal{D}^{(\mu)}_F) \subseteq \text{vmax}(\mathcal{D}^{(\mu)}_O), \; \forall \mu \in \mathbb{R}^m_+.$$

Proof. Let be $\mu \in \mathbb{R}^m_+$ fixed and $d \in \text{vmax}(\mathcal{D}^{(\mu)}_F)$. As $d \in \mathcal{D}^{(\mu)}_F$, there exists $\bar{u} = (\bar{u}_1, ..., \bar{u}_m)$, $\bar{v} = (\bar{v}_1, ..., \bar{v}_m)$, $\bar{u}_i, \bar{v}_i \in \mathbb{R}^n, i = 1, ..., m$, and $\bar{\lambda} \in \text{int}\, \mathbb{R}^m_+$, such that $(\bar{u}, \bar{v}, \bar{\lambda}, d) \in \mathcal{B}^F_\mu$. Thus

$$\sum_{i=1}^{m} \bar{\lambda}_i d_i \le - \sum_{i=1}^{m} \bar{\lambda}_i \left[f_i^*(\bar{u}_i) + (-\mu_i h_i)^*(\bar{v}_i) \right] - \chi^*_{\mathcal{A}} \left(- \sum_{i=1}^{m} \bar{\lambda}_i (\bar{u}_i + \bar{v}_i) \right)$$
$$\le \sup_{\substack{u_i, v_i \in \mathbb{R}^n, \\ i=1,...,m}} \left\{ - \sum_{i=1}^{m} \bar{\lambda}_i \left[f_i^*(u_i) + (-\mu_i h_i)^*(v_i) \right] - \chi^*_{\mathcal{A}} \left(- \sum_{i=1}^{m} \bar{\lambda}_i (u_i + v_i) \right) \right\}$$
$$= \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i (f_i(x) - \mu_i h_i(x)) < +\infty,$$

because of Lemma 3.1. The supremum in the relation above must be finite and, from the maximality of d in $\mathcal{D}^{(\mu)}_F$, one has the following equality

$$\sum_{i=1}^{m} \bar{\lambda}_i d_i = \sup_{\substack{u_i, v_i \in \mathbb{R}^n, \\ i=1,...,m}} \left\{ - \sum_{i=1}^{m} \bar{\lambda}_i \left[f_i^*(u_i) + (-\mu_i h_i)^*(v_i) \right] - \chi^*_{\mathcal{A}} \left(- \sum_{i=1}^{m} \bar{\lambda}_i (u_i + v_i) \right) \right\}.$$

This means that

$$\sum_{i=1}^{m}\bar{\lambda}_i d_i = \inf_{x\in\mathcal{A}}\sum_{i=1}^{m}\bar{\lambda}_i(f_i(x)-\mu_i h_i(x)).$$

On the other hand, the infimum above can be written, equivalently, in the following way

$$\inf_{x\in\mathcal{A}}\sum_{i=1}^{m}\bar{\lambda}_i(f_i(x)-\mu_i h_i(x)) = \inf_{x\in\mathbb{R}^n}\left(\sum_{i=1}^{m}\bar{\lambda}_i(f_i(x)-\mu_i h_i(x)) + \chi_{\mathcal{A}}(x)\right)$$

$$= -\left(\left[\sum_{i=1}^{m}\bar{\lambda}_i f_i + \chi_{\mathcal{A}}\right] + \left[\sum_{i=1}^{m}\bar{\lambda}_i(-\mu_i)h_i + \chi_{\mathcal{A}}\right]\right)^*(0).$$

Using again the constraint qualification (CQ), it follows by Theorem 16.4 in [11] that there exists $\bar{p}\in\mathbb{R}^n$ such that

$$\left(\left[\sum_{i=1}^{m}\bar{\lambda}_i f_i + \chi_{\mathcal{A}}\right] + \left[\sum_{i=1}^{m}\bar{\lambda}_i(-\mu_i)h_i + \chi_{\mathcal{A}}\right]\right)^*(0) =$$

$$\left(\sum_{i=1}^{m}\bar{\lambda}_i f_i + \chi_{\mathcal{A}}\right)^*(\bar{p}) + \left(\sum_{i=1}^{m}\bar{\lambda}_i(-\mu_i)h_i + \chi_{\mathcal{A}}\right)^*(-\bar{p}).$$

This means that

$$\sum_{i=1}^{m}\bar{\lambda}_i d_i = -\left(-\sum_{i=1}^{m}\bar{\lambda}_i\mu_i h_i\right)^*_{\mathcal{A}}(-\bar{p}) - \left(\sum_{i=1}^{m}\bar{\lambda}_i f_i\right)^*_{\mathcal{A}}(\bar{p}),$$

which is nothing else than $(\bar{p},\bar{\lambda},d)\in\mathcal{B}^O_\mu$. Therefore $d\in\mathcal{D}^{(\mu)}_O$.

Assuming that $d\notin \text{vmax}(\mathcal{D}^{(\mu)}_O)$, there must exists $\bar{d}\in\mathcal{D}^{(\mu)}_O$ such that $d\in\bar{d}-\{\mathbb{R}^m_+\backslash\{0\}\}$. According to Theorem 3.3 we have that $\bar{d}\in\mathcal{D}^{(\mu)}_O\subseteq\mathcal{D}^{(\mu)}_F$ and this contradicts the maximality of d in $\mathcal{D}^{(\mu)}_F$. In conclusion d must belong to $\text{vmax}(\mathcal{D}^{(\mu)}_O)$. □

We conclude the paper by giving the relation which exists under the stated assumptions between the sets of maximal elements of the image sets of the multiobjective dual problems treated, namely

$$\text{vmax}(\mathcal{D}_\mu) = \text{vmax}(\mathcal{D}^{(\mu)}_O) = \text{vmax}(\mathcal{D}^{(\mu)}_F) = \text{vmax}(\mathcal{D}^{(\mu)}_{FL}),\ \forall\mu\in\mathbb{R}^m_+.$$

In other words, the image sets of the efficiency sets of all multiobjective dual problems (D_μ), (D^μ_F), (D^μ_{FL}) and (D^μ_O) to the primal problem (P_μ) coincide for all $\mu\in\mathbb{R}^m_+$.

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