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New regularity conditions for Lagrange and Fenchel-Lagrange duality in infinite dimensional spaces

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Abstract. We give new regularity conditions based on epigraphs that assure strong duality between a given primal convex optimization problem and its Lagrange and Fenchel-Lagrange dual problems, respectively, in infinite dimensional spaces. Moreover we completely characterize through equivalent statements the so-called stable strong duality between the initial problem and the mentioned duals.

Keywords. Conjugate functions, Lagrange dual, Fenchel-Lagrange dual, constraint qualifications, epigraphs

1 Introduction

Duality is an important and powerful tool in optimization, where it is present subject to several approaches. Among the most used and known duality concepts there are the ones named after J.L. Lagrange and, respectively, W. Fenchel. Finding weaker conditions under which there is strong duality, i.e. the situation when the optimal objective values of the primal and dual problem coincide and the dual has, moreover, an optimal solution is one of the most interesting and challenging problems in optimization. Many authors have dealt with this kind of problems improving and extending the previous results both in finitely and infinitely dimensional spaces. We recall here some recent works dealing with this subject, namely [3], [4], [5], [6], [11], [12] and [16]. Some of these conditions, usually called regularity conditions or constraint qualifications, guarantee also

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strong duality for any linear perturbation of the objective function of the primal problem, a situation called stable strong duality.

In a recent paper ([2]) we have delivered new and weak conditions under which some formulae for the subdifferential of composed convex functions in infinite dimensional spaces are valid. Using them we have derived a new regularity condition that guarantees strong duality between a convex optimization problem and its Fenchel dual problem, rediscovering another recent result due to two of the authors in [5]. This new regularity condition is, to the best of our knowledge, the weakest condition so far that guarantees strong duality for the Fenchel dual problem in infinite dimensional spaces. Within the present paper we use some results from the same article ([2]) in order to determine weaker regularity conditions assuring strong duality between a convex optimization problem and its Lagrange and Fenchel-Lagrange dual problems, respectively, in infinite dimensional spaces. Moreover we give equivalent statements for the so-called stable strong duality between the initial problem and the mentioned duals. We also show that our constraint qualification for the Lagrange duality is weaker than some others recently given in the literature.

As the Lagrange and Fenchel dual problems are widely known and used we do not write much about them here, but the same does not apply for the Fenchel-Lagrange dual problem. It has been introduced by two of the present authors, Boţ and Wanka, first in finite dimensional spaces, then also for problems having their variables lying in infinite dimensional spaces. As its name suggests, the Fenchel-Lagrange dual problem is a "combination" of the well-known Fenchel and Lagrange dual problems. The interested reader is referred to [3], [4] or [6] for more on the way the Fenchel-Lagrange dual problem is constructed.

The paper is structured as follows. After this introduction follow some necessary preliminaries where we introduce the context we work in and we recall the previous results used within this paper. Section 3 contains the new results we give concerning Lagrange duality, while the fourth part does the same for the ones regarding Fenchel-Lagrange duality. Then come the conclusions, followed by a short appendix dedicated to the same kind of results as in Sections 3 and 4, concerning this time Fenchel duality. The list of references closes the paper.

2 Preliminaries

Consider two nontrivial locally convex vector spaces X and Z and their continuous dual spaces X^* and Z^*, endowed with the weak* topologies $w(X^*, X)$ and, respectively, $w(Z^*, Z)$. Let the non-empty closed convex cone $K \subseteq Z$ and its dual cone $K^* = \{z^* \in Z^* : \langle z^*, z\rangle \geq 0 \ \forall z \in Z\}$ be given, where we denote by $\langle z^*, z\rangle = z^*(z)$ the value at z of the continuous linear functional z^*. On Z we consider the partial order induced by K, "$\leq_K$", defined by $x \leq_K y \Leftrightarrow y - x \in K$, $x, y \in Z$. To Z we attach a greatest element with respect to "$\leq_K$" denoted by ∞

which does not belong to Z and let $Z^\bullet = Z \cup \{\infty\}$. Then for any $z \in Z^\bullet$ one has $z \leq_K \infty$ and we consider on $Z^\bullet$ the following operations: $z + \infty = \infty + z = \infty$ and $t\infty = \infty$ for all $z \in Z$ and all $t \geq 0$.

Given a subset U of X, by $\mathrm{cl}(U)$ we denote its *closure* in the corresponding topology, while its *indicator* function is $\delta_U : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\delta_U(x) = \begin{cases} 0, & \text{if } x \in U, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 1. ([2]) A set $U \subseteq X$ is said to be *closed regarding the subspace* $W \subseteq X$ if $U \cap W = \mathrm{cl}(U) \cap W$.

Now we give some notions regarding functions used within our paper.

For a function $f : X \to \overline{\mathbb{R}}$ we have

- the *domain*: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$,
- the *epigraph*: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$,
- the *conjugate regarding the set* $U \subseteq X$: $f_U^* : X^* \to \overline{\mathbb{R}}$ given by $f_U^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in U\}$,
- f is *proper*: $f(x) > -\infty\ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$,
- f is *C-increasing*: $f(x) \geq f(y)\ \forall x, y \in X$ such that $y \leq_C x$, with C a non-empty closed convex cone in X,
- f is *lower-semicontinuous regarding the subspace* $W \subseteq X$: $\mathrm{epi}(f) \cap (W \times \mathbb{R}) = \mathrm{cl}(\mathrm{epi}(f)) \cap (W \times \mathbb{R})$, i.e. $\mathrm{epi}(f)$ is closed regarding the subspace $W \times \mathbb{R}$.

When $U = X$ the conjugate regarding the set U is the classical conjugate function. Between a function and its conjugate regarding some set $U \subseteq X$ there is *Young's inequality*

$$f_U^*(x^*) + f(x) \geq \langle x^*, x\rangle\ \forall x \in U\ x^* \in X^*.$$

Given two proper functions $f, g : X \to \overline{\mathbb{R}}$, we have the *infimal convolution* of f and g defined by

$$f \Box g : X \to \overline{\mathbb{R}},\ \big(f \Box g\big)(a) = \inf\{f(x) + g(a - x) : x \in X\},$$

which is called *exact* at some $a \in X$ when there is an $x \in X$ such that $\big(f \Box g\big)(a) = f(x) + g(a - x)$.

There are notions given for functions with extended real values that can be formulated also for functions having their ranges in infinite dimensional spaces.

For a function $h : X \to Z^\bullet$ one has

· the *domain*: $\mathrm{dom}(h) = \{x \in X : h(x) \in Z\}$,

· h is *proper*: $\mathrm{dom}(h) \neq \emptyset$,

· h is *K-convex*: $h(tx+(1-t)y) \leq_K th(x)+(1-t)h(y)\ \forall x, y \in X\ \forall t \in [0,1]$,

· h is *K-lower-semicontinuous* at $x \in X$: for any neighborhood V of zero in Z and for any $b \in Z$ satisfying $b \leq_K h(x)$, there exists a neighborhood V_x of x in X such that $h(V_x) \subseteq b + V + K \cup \{\infty\}$,

· for $\alpha \in K^*$, $(\alpha h) : X \to \overline{\mathbb{R}}$, $(\alpha h)(x) = \langle \alpha, h(x) \rangle$ for $x \in \mathrm{dom}(h)$ and $(\alpha h)(x) = +\infty$ otherwise,

· h is *star K-lower-semicontinuous* at $x \in X$: (αh) is lower-semicontinuous at $x\ \forall \alpha \in K^*$,

· for a subset $W \subseteq Z$: $h^{-1}(W) = \{x \in X : \exists z \in W \text{ s.t. } h(x) = z\}$.

Remark 1. ([10]) If, for some $x \in X$, $h(x) \in Z$ the definition of K-lower-semicontinuity of h at x amounts to asking for any neighborhood V of zero (in Z) the existence of a neighborhood V_x of x such that $h(V_x) \subseteq h(x)+V+K \cup \{\infty\}$.

Remark 2. There are also other extensions of the lower-semicontinuity to functions taking values in infinite dimensional spaces, like the K-epi-closedness, the level-closedness or the K-sequentially lower-semicontinuity. We refer the interested reader to [1], [10] or [13] for more on the subject.

Proposition 1. *When $\alpha \in K^*$ and $h : X \to Z^\bullet$ is proper, K-convex and K-lower-semicontinuous, then (αh) is proper, convex and lower-semicontinuous.*

Proof. The properness of (αh) follows immediately, as well as its convexity, from the definition. The lower-semicontinuity of (αh), i.e. the star K-lower-semicontinuity of h, follows from Lemma 1.7 in [14]. □

In the following we recall the results given in [2] used within this paper. Consider the proper convex lower-semicontinuous function $F : X \to \overline{\mathbb{R}}$, the K-increasing proper convex lower-semicontinuous function $G : Z \to \overline{\mathbb{R}}$ and the proper K-convex K-lower-semicontinuous function $H : X \to Z^\bullet$ such that $H(\mathrm{dom}(F) + K)$ and $\mathrm{dom}(G)$ have at least a point in common. Moreover, let us consider the following regularity conditions

(CQ) $\quad \{0_{X^*}\} \times \mathrm{epi}(G^*) + \bigcup\limits_{\alpha \in K^*} \{(a, -\alpha, r) : (a, r) \in \mathrm{epi}(F + (\alpha H))^*\}$ is closed regarding the subspace $X^* \times \{0_{Z^*}\} \times \mathbb{R}$,

$(\overline{CQ})$ $\{0_{X^*}\} \times \operatorname{epi}(G^*) + \{(p, 0_{Z^*}, r) : (p, r) \in \operatorname{epi}(F^*)\} + \underset{\alpha \in K^*}{\cup} \{(p, -\alpha, r) : (p, r) \in \operatorname{epi}(\alpha H)^*\}$ is closed regarding to the subspace $X^* \times \{0_{Z^*}\} \times \mathbb{R}$,

(CQD) the function $(p, q) \mapsto \inf\limits_{\alpha \in K^* + q} \{G^*(\alpha) + (F + ((\alpha - q)H))^*(p)\}$ is lower-semicontinuous regarding the subspace $X^* \times \{0_{Z^*}\}$ and at $(0_{X^*}, 0_{Z^*})$ the infimum is attained,

$(\overline{CQD})$ the function $(p, q) \mapsto \inf\limits_{\alpha \in K^* + q} \{G^*(\alpha) + (F + ((\alpha - q)H))^*(p)\}$ is lower-semicontinuous regarding the subspace $X^* \times \{0_{Z^*}\}$ and $\operatorname{epi}(A^* \square B^*) \cap (\{0_{X^*}\} \times \{0_{Z^*}\} \times \mathbb{R}) = (\{0_{X^*}\} \times \operatorname{epi}(G^*) + \{(p, 0_{Z^*}, r) : (p, r) \in \operatorname{epi}(F^*)\} + \underset{\alpha \in K^*}{\cup} \{(p, -\alpha, r) : (p, r) \in \operatorname{epi}((\alpha H)^*)\}) \cap (\{0_{X^*}\} \times \{0_{Z^*}\} \times \mathbb{R})$,

where $A, B : X \times Z \to \overline{\mathbb{R}}$ are the functions defined by $A(x, z) = G(z)$ and $B(x, z) = F(x) + \delta_{\{(x,z) \in X \times Z : H(x) - z \in -K\}}(x, z)$, for $(x, z) \in X \times Z$. We have established in [2] the following statements. Let us mention that for an attained infimum (supremum) instead of inf (sup) we write min (max).

Remark 3. (CQ) yields (CQD) and $(\overline{CQ})$ delivers $(\overline{CQD})$, but the reverse implications are not always true. See [2] for examples.

Theorem 1. *(CQ) is fulfilled if and only if for any $p \in X^*$ one has*

$$(F + G \circ H)^*(p) = \min_{\alpha \in K^*} \{G^*(\alpha) + (F + (\alpha H))^*(p)\}.$$

Theorem 2. *$(\overline{CQ})$ is fulfilled if and only if for any $p \in X^*$ it holds*

$$(F + G \circ H)^*(p) = \min_{\substack{\alpha \in K^*,\\ \beta \in X^*}} \{G^*(\alpha) + F^*(\beta) + (\alpha H)^*(p - \beta)\}.$$

Remark 4. The fulfillment of $(\overline{CQ})$ implies the validity of (CQ), while $(\overline{CQ})$ does not always hold when (CQ) is satisfied, see [2] for an example.

Theorem 3. *Assume (CQD) valid. Then*

$$\inf_{x \in X} [F(x) + G \circ H(x)] = \max_{\alpha \in K^*} \{-G^*(\alpha) - (F + (\alpha H))^*(0_{X^*})\}.$$

Theorem 4. *Assume $(\overline{CQD})$ valid. Then*

$$\inf_{x \in X} [F(x) + G \circ H(x)] = \max_{\substack{\alpha \in K^*,\\ \beta \in X^*}} \{-G^*(\alpha) - F^*(\beta) - (\alpha H)^*(-\beta)\}.$$

Remark 5. The fulfillment of $(\overline{CQD})$ guarantees the satisfaction of (CQD), while $(\overline{CQD})$ does not always hold when (CQD) is valid (cf. [2]).

As announced, this paper deals with Lagrange and Fenchel-Lagrange duality for convex optimization problems. Thus we consider the primal convex optimization problem

$$(P) \qquad \inf_{\substack{x \in U, \\ g(x) \in -C}} f(x),$$

where Y is a nontrivial locally convex vector space, U is a non-empty convex subset of X, C is a non-empty closed convex cone in Y, $f : X \to \overline{\mathbb{R}}$ is a proper convex lower-semicontinuous function and $g : X \to Y^\bullet$ is a proper C-convex C-lower-semicontinuous function. Moreover, we need to impose the condition

$$\mathrm{dom}(f) \cap g^{-1}(-C) \cap U \neq \emptyset.$$

To this problem we attach both the Lagrange and Fenchel-Lagrange dual problems. For each of these dual problems we completely characterize the so-called stable strong duality and we give weak conditions under which strong duality occurs.

For the convex optimization problem (P) we denote by $v(P)$ its optimal objective value and we use this notation also for the optimal objective values of the other problems that appear within our paper. Let us state also that by strong duality we understand the situation when the optimal objective values of the primal and dual problem coincide and the dual problem has an optimal solution, while stable strong duality (cf. [12]) takes place when strong duality holds for any linear perturbation of the objective function f.

3 Lagrange duality

In this section we introduce a new constraint qualification, derived from (CQD), which guarantees strong duality between the given primal optimization problem (P) and its Lagrange dual problem,

$$(D^L) \qquad \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x)],$$

while using (CQ) we give an equivalent formulation (complete characterization) of the stable strong duality for this pair of problems.

In order to state the mentioned duality assertions by using the cited results from [2], let us take the following choice for the functions and sets involved

$$K = \{0_X\} \times C, Z = X \times Y, F(x) = f(x) \forall x \in X, G(x,y) = \delta_{-C}(y) \forall (x,y) \in X \times Y,$$

and
$$H(x)=\begin{cases}(0_X,g(x)), & x\in U,\\ \infty, & \text{otherwise.}\end{cases}$$

Thus $K^*=X^*\times C^*$ and the conjugates of F and G are in this case $F^*=f^*$ and, as $\delta^*_{-C}(y^*)=\sup_{v\in -C}\langle y^*,v\rangle=0$ when $y^*\in C^*$ and it is $+\infty$ otherwise,

$$G^*(x^*,y^*)=\begin{cases}0, & x^*=0_{X^*},\ y^*\in C^*,\\ +\infty, & \text{otherwise.}\end{cases}$$

It is easy to notice that one gets immediately for any $(x^*,y^*)\in X^*\times C^*$, when $x\in X$,

$$((x^*,y^*)H)(x)=\begin{cases}(y^*g)(x), & x\in U,\\ +\infty, & \text{otherwise,}\end{cases}=(y^*g)(x)+\delta_U(x)$$

and
$$(F+G\circ H)(x)=\begin{cases}f(x), & x\in U,\ g(x)\in -C,\\ +\infty, & \text{otherwise.}\end{cases}$$

In order to obtain the stable strong duality statement regarding (P) and (D^L) we must perturb the objective function f of (P) with a linear perturbation function. Thus, taking some $p\in X^*$, the linearly perturbed primal problem is

$$(P_p)\qquad \inf_{\substack{x\in U,\\ g(x)\in -C}}[f(x)-\langle p,x\rangle].$$

Using the functions F, G and H as chosen above, one gets

$$v(P_p)=-\sup_{\substack{x\in U,\\ g(x)\in -C}}\{\langle p,x\rangle-f(x)\}=-\sup_{x\in X}\{\langle p,x\rangle-(F+G\circ H)(x)\},$$

thus $v(P_p)=-(F+G\circ H)^*(p)$. As asserted in Theorem 1, the validity of (CQ) is equivalent to

$$(F+G\circ H)^*(p)=\min_{\alpha\in K^*}\{G^*(\alpha)+(F+(\alpha H))^*(p)\}\ \forall p\in X^*,$$

thus moreover to

$$\begin{aligned}v(P_p)&=-\min_{\alpha\in K^*}\{G^*(\alpha)+(F+(\alpha H))^*(p)\}\\ &=\max_{(x^*,\lambda)\in X^*\times C^*}\{-G^*(x^*,\lambda)-(F+((x^*,\lambda)H))^*(p)\}\ \forall p\in X^*.\end{aligned}$$

Using the formula of the conjugate of G, we get that (CQ) is further equivalent to
$$v(P_p)=\max_{\lambda\in C^*}\{-(F+((0_{X^*},\lambda)H))^*(p)\}=\max_{\lambda\in C^*}\{-(f+(\lambda g))^*_U(p)\},$$

for all $p \in X^*$. Since

$$(f + (\lambda g))_U^*(p) = \sup_{x \in U} [\langle p, x \rangle - f(x) - (\lambda g)(x)] = - \inf_{x \in U} [-\langle p, x \rangle + f(x) + (\lambda g)(x)],$$

we get

$$(CQ) \Leftrightarrow v(P_p) = \max_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x) - \langle p, x \rangle] \ \forall p \in X^*. \tag{1}$$

In order to avoid any confusions the regularity conditions (CQ) and (CQD) will become (CQ^L), respectively (CQD^L), for the special choices of F, G and H announced earlier, i.e. when written using f, g, U and C. As $\text{epi}(G^*) = \{0_{X^*}\} \times C^* \times [0, +\infty)$, we get that (CQ^L) means that the set

$$\begin{aligned} M &= \{0_{X^*}\} \times \{0_{X^*}\} \times C^* \times [0, +\infty) \\ &+ \bigcup_{\substack{x^* \in X^*, \\ \lambda \in C^*}} \{(a, -x^*, -\lambda, r) : (a, r) \in \text{epi}((f + (\lambda g) + \delta_U)^*)\} \end{aligned}$$

is closed regarding the subspace $S = X^* \times \{0_{X^*}\} \times \{0_{Y^*}\} \times \mathbb{R}$. Consequently, the set M can be characterized as follows

$$(a, b, c, r) \in M \Leftrightarrow b \in X^* \text{ and } (a, r) \in \bigcup_{\lambda \in C^* \cap (C^* - c)} \text{epi}((f + (\lambda g) + \delta_U)^*). \tag{2}$$

The proof is quite elementary. If $(a, b, c, r) \in M$ then there are some $x^* \in X^*$, $\bar{\lambda} \in C^*$ and $s \geq 0$ such that $(a, b, c, r) = (0_{X^*}, 0_{X^*}, \bar{\lambda} + c, s) + (a, -x^*, -\bar{\lambda}, r - s)$ and $(a, r - s) \in \text{epi}((f + (\bar{\lambda} g) + \delta_U)^*)$. Thus $b = -x^*$, $\bar{\lambda} + c \in C^*$, which means $\bar{\lambda} \in C^* - c$. Moreover

$$(a, r) \in \text{epi}((f + (\bar{\lambda} g) + \delta_U)^*) \subseteq \bigcup_{\lambda \in C^* \cap (C^* - c)} \text{epi}((f + (\lambda g) + \delta_U)^*),$$

so the implication left to right in (2) is secured. On the other hand, taking (a, b, c, r) in the set described in the right-hand side of (2), there is a $\bar{\lambda} \in C^* \cap (C^* - c)$ such that the quadruple can be written as follows

$$(a, b, c, r) = (0_{X^*}, 0_{X^*}, \bar{\lambda} + c, 0) + (a, b, -\bar{\lambda}, r),$$

and it is clear that the first member of this sum belongs to $\{0_{X^*}\} \times \{0_{X^*}\} \times C^* \times [0, +\infty)$, while the second to

$$\bigcup_{\substack{\lambda \in C^*, \\ x^* \in X^*}} \{(a, -x^*, -\lambda, r) : (a, r) \in \text{epi}((f + (\lambda g) + \delta_U)^*)\},$$

i.e. $(a, b, c, r) \in M$. Let us see now how we can write equivalently that M is closed regarding the subspace S.

Lemma 1. *The regularity condition* (CQ^L) *is equivalent to the fact that*

$$\bigcup_{\lambda\in C^*} \operatorname{epi}((f+(\lambda g)+\delta_U)^*) \text{ is closed.} \tag{3}$$

Proof. First we recall that $M\cap S \subseteq \operatorname{cl}(M\cap S) \subseteq \operatorname{cl}(M)\cap\operatorname{cl}(S) = \operatorname{cl}(M)\cap S$. Now let us prove that $\operatorname{cl}(M\cap S) = \operatorname{cl}(M)\cap S$. Take some $(a,b,c,r)\in\operatorname{cl}(M)\cap S$. Then $b=0_{X^*}$ and $c=0_{Y^*}$ and let us consider the neighborhoods V of a in X^*, U of 0_{X^*} in X^* and W of 0_{Y^*} in Y^* and some $\varepsilon>0$ such that $V\times U\times W\times(r-\varepsilon,r+\varepsilon)$ is a neighborhood of $(a,0_{X^*},0_{Y^*},r)$. This yields the existence of a quadruple $(\bar a,\bar b,\bar c,\bar r)\in V\times U\times W\times(r-\varepsilon,r+\varepsilon)$ such that $(\bar a,\bar b,\bar c,\bar r)\in M$. This means actually, by (2), $\bar b\in X^*$ and $(\bar a,\bar r)\in\cup_{\lambda\in C^*\cap(C^*-c)}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$. The latter gives $(\bar a,\bar r)\in\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$, thus $(\bar a,0_{X^*},0_{Y^*},\bar r)\in M\cap S$. As $(\bar a,0_{X^*},0_{Y^*},\bar r)$ is also in $V\times U\times W\times(r-\varepsilon,r+\varepsilon)$ and this neighborhood has been arbitrarily chosen, it follows $(a,b,c,r)\in\operatorname{cl}(M\cap S)$. Consequently $\operatorname{cl}(M\cap S)\supseteq\operatorname{cl}(M)\cap S$, therefore $\operatorname{cl}(M\cap S)=\operatorname{cl}(M)\cap S$. Thus the fact that M is closed regarding the subspace S means in this case that $M\cap S=\operatorname{cl}(M\cap S)$, i.e. $M\cap S$ is closed.

Consider the mapping $T: X^*\times X^*\times Y^*\times\mathbb{R}\to X^*\times\mathbb{R}\times X^*\times Y^*$ defined by $T(a,b,c,r)=(a,r,b,c)$. It is clear that T is a homeomorphism, so $M\cap S$ is closed if and only if $T(M\cap S)$ is closed. As $T(M\cap S)=\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)\times\{0_{X^*}\}\times\{0_{Y^*}\}$, one gets that $M\cap S$ is closed if and only if $\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$ is closed. □

As (3) is equivalent to (CQ^L) it will be used further instead of the latter. Using the discussion given in the beginning of the section, especially (1), we establish now the stable strong duality statement concerning (P) and (D^L).

Theorem 5. *The set* $\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$ *is closed if and only if for any* $p\in X^*$ *one has*

$$\inf_{\substack{x\in U,\\ g(x)\in -C}} [f(x)-\langle p,x\rangle] = \max_{\lambda\in C^*}\inf_{x\in U}[f(x)+(\lambda g)(x)-\langle p,x\rangle].$$

Remark 6. One may notice that in the previous statement we have rediscovered Theorem 3.2 in [12]. The difference between our result and the cited one consists in the fact that there g is taken star C-lower-semicontinuous and here it is C-lower-semicontinuous. We could consider from the beginning g star C-lower-semicontinuous, too, as the results we call need actually this property for g in order to be valid. We prefer to use a notion more present in the literature (cf. [1], [2], [10], [13], [14], [15]), though. Thus we work with g C-lower-semicontinuous, mentioning that the results we give remain true also when g is taken a bit more general, i.e. star C-lower-semicontinuous.

Regarding (CQD^L), we know that it means that the function

$$(p,b,c) \mapsto \inf_{(x^*,\lambda)\in X^*\times C^*+(b,c)} \{G^*(x^*,\lambda) + (F + (x^* - b, \lambda - c)H))^*(p)\}$$

is lower-semicontinuous regarding the subspace $X^* \times \{0_{X^*}\} \times \{0_{Y^*}\}$ and at $(0_{X^*}, 0_{X^*}, 0_{Y^*})$ the infimum therein is attained. Taking into account the formulae of F, G^* and H, (CQD^L) requires that the function

$$\varphi : X^* \times X^* \times Y^* \to \overline{\mathbb{R}},\ \varphi(p,b,c) = \inf_{\lambda\in C^*\cap(C^*+c)} (f + ((\lambda - c)g) + \delta_U)^*(p)$$

is lower-semicontinuous regarding the subspace $X^* \times \{0_{X^*}\} \times \{0_{Y^*}\}$ and at $(0_{X^*}, 0_{X^*}, 0_{Y^*})$ the infimum therein is attained. The following statement gives a simpler formulation for (CQD^L) and it is followed by the strong duality assertion regarding the primal problem (P) and its Lagrange dual problem (D^L).

Lemma 2. *(CQD^L) turns out to mean that the function*

$$\eta : X^* \to \overline{\mathbb{R}},\ \eta(p) = \inf_{\lambda\in C^*} (f + (\lambda g) + \delta_U)^*(p)$$

is lower-semicontinuous and at 0_{X^*} the infimum within is attained.

Proof. One may easily notice that at $(0_{X^*}, 0_{X^*}, 0_{Y^*})$ the infimum within φ is attained if and only if there is some $\bar{\lambda} \in C^*$ such that $(f + (\bar{\lambda}g) + \delta_U)^*(0_{X^*}) = \inf_{\lambda\in C^*}(f + (\bar{\lambda}g) + \delta_U)^*(0_{X^*})$, i.e. at 0_{X^*} the infimum within η is attained.

On the other hand, the fact that φ is lower-semicontinuous regarding the subspace $X^* \times \{0_{X^*}\} \times \{0_{Y^*}\}$ means actually that $\text{epi}(\varphi) \cap S = \text{cl}(\text{epi}(\varphi)) \cap S$. Let us prove that $\text{cl}(\text{epi}(\varphi) \cap S) = \text{cl}(\text{epi}(\varphi)) \cap S$. As the inclusion "$\subseteq$" is known to be true, let us take some quadruple $(p,b,c,r) \in \text{cl}(\text{epi}(\varphi)) \cap S$. By definition one gets immediately $b = 0_{X^*}$ and $c = 0_{Y^*}$. As $(p, 0_{X^*}, 0_{Y^*}, r) \in \text{cl}(\text{epi}(\varphi))$, by considering the neighborhoods V of p in X^*, U of 0_{X^*} in X^* and W of 0_{Y^*} in Y^* and some $\varepsilon > 0$, there follows the existence of some quadruple $(\bar{p}, \bar{b}, \bar{c}, \bar{r}) \in (V \times U \times W \times (r-\varepsilon, r+\varepsilon)) \cap \text{epi}(\varphi)$. Thus $\inf_{\lambda\in C^*\cap(C^*+\bar{c})}(f + (\lambda g) + \delta_U)^*(\bar{p}) \le \bar{r}$. Let $\bar{\bar{r}} \in (\bar{r}, r+\varepsilon)$. There is at least a $\bar{\lambda} \in C^* \cap (C^* + \bar{c})$ such that $(f + (\bar{\lambda}g) + \delta_U)^*(\bar{p}) < \bar{\bar{r}}$. This leads to $\inf_{\lambda\in C^*}(f + (\lambda g) + \delta_U)^*(\bar{p}) < \bar{\bar{r}}$, so $(\bar{p}, 0_{X^*}, 0_{Y^*}, \bar{\bar{r}}) \in \text{epi}(\varphi) \cap S$. As $(\bar{p}, 0_{X^*}, 0_{Y^*}, \bar{\bar{r}}) \in V \times U \times W \times (r-\varepsilon, r+\varepsilon)$ it follows that (p,b,c,r) belongs to $\text{cl}(\text{epi}(\varphi) \cap S)$, too. Thus also the inclusion "$\supseteq$" is valid, therefore $\text{cl}(\text{epi}(\varphi) \cap S) = \text{cl}(\text{epi}(\varphi)) \cap S$, i.e. φ is lower-semicontinuous regarding the subspace $X^* \times \{0_{X^*}\} \times \{0_{Y^*}\}$ if and only if $\text{epi}(\varphi) \cap S$ is closed.

Using the homeomorphism T introduced within the proof of the previous lemma, one has $T(\text{epi}(\varphi) \cap S) = \text{epi}(\eta) \times \{0_{X^*}\} \times \{0_{Y^*}\}$ as proven further. Taking $(p, 0_{X^*}, 0_{Y^*}, r) \in \text{epi}(\varphi) \cap S$ it follows $\eta(p) \le r$, so $(p,r) \in \text{epi}(\eta)$. As $T(p, 0_{X^*}, 0_{Y^*}, r) = (p, r, 0_{X^*}, 0_{Y^*})$ the inclusion "$\subseteq$" is secured. Viceversa, if $(p, r, 0_{X^*}, 0_{Y^*}) \in \text{epi}(\eta) \times \{0_{X^*}\} \times \{0_{Y^*}\}$, there is $(p, 0_{X^*}, 0_{Y^*}, r) \in \text{epi}(\varphi) \cap S$ fulfilling $T(p, 0_{X^*}, 0_{Y^*}, r) = (p, r, 0_{X^*}, 0_{Y^*})$, so the reverse inclusion stands, too.

Therefore $\text{epi}(\varphi) \cap S$ is closed if and only if $\text{epi}(\eta)$ is closed, i.e. η is lower-semicontinuous. Consequently (CQD^L) is fulfilled if and only if η is a lower-semicontinuous function having at 0_{X^*} the infimum in its definition attained. □

Theorem 6. *If the function $p \mapsto \inf_{\lambda \in C^*}(f + (\lambda g) + \delta_U)^*(p)$ is lower-semicontinuous and at 0_{X^*} the infimum within is attained, i.e. (CQD^L) is valid, then there is strong duality between (P) and (D^L), i.e.*

$$\inf_{\substack{x \in U, \\ g(x) \in -C}} f(x) = \max_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x)].$$

Proof. The assertion arises from Theorem 3 via the discussion in the beginning of the section. □

Remark 7. Usually in the literature (see [6], [11], [12]) the strong duality statement for (P) and (D^L) is given under the assumption of continuity for f, while we give it for f lower-semicontinuous. In the following we show that even assuming f continuous our regularity condition (CQD^L) is weaker than a very recent condition introduced in [12] which is implied by many other regularity conditions in the literature.

Proposition 2. *If X is a Fréchet space and $f : X \to \mathbb{R}$ is moreover continuous, the fulfillment of the so-called dual CQ (cf. [12])*

$$(dCQ) \qquad \bigcup_{\lambda \in C^*} \text{epi}(\delta_U + (\lambda g))^* \text{ is closed,}$$

guarantees the validity of (CQD^L).

Proof. By (3.3) in [12], (dCQ) is valid if and only if

$$\bigcup_{\lambda \in C^*} \text{epi}(\delta_U + (\lambda g))^* = \text{epi}(\delta^*_{U \cap g^{-1}(-C)}).$$

As f is continuous it follows (cf. [12])

$$\text{epi}(f^*) + \text{epi}(\delta^*_{U \cap g^{-1}(-C)}) = \text{epi}((f + \delta_{U \cap g^{-1}(-C)})^*) = \bigcup_{\lambda \in C^*} \text{epi}((f + (\lambda g) + \delta_U)^*),$$

so the latter is a closed set, too.

Next we show that $\cup_{\lambda \in C^*} \text{epi}((f + (\lambda g) + \delta_U)^*)$ is closed if and only if $p \mapsto \eta(p) = \inf_{\lambda \in C^*}(f + (\lambda g) + \delta_U)^*(p)$ is lower-semicontinuous and the infimum therein is always attained.

Take first some pair $(p, r) \in \cup_{\lambda \in C^*} \text{epi}((f + (\lambda g) + \delta_U)^*)$. This means that there is some $\bar{\lambda} \in C^*$ satisfying $(f + (\bar{\lambda} g) + \delta_U)^*(p) \leq r$, so $\inf_{\lambda \in C^*}(f + (\lambda g) + \delta_U)^*(p) \leq r$. Thus $(p, r) \in \text{epi}(\eta)$, therefore $\cup_{\lambda \in C^*} \text{epi}((f + (\lambda g) + \delta_U)^*) \subseteq \text{epi}(\eta)$.

Consider now a pair $(p,r)\in \operatorname{epi}(\eta)$. For any $n\in\mathbb{N}$ there is at least a $\lambda_n\in C^*$ such that $(p,r+(1/n))\in \operatorname{epi}((f+(\lambda_n g)+\delta_U)^*)\subseteq \cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$. Letting n converge towards the positive infinity we obtain $(p,r)\in \operatorname{cl}(\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*))$, so $\operatorname{epi}(\eta)\subseteq \operatorname{cl}(\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*))$. Therefore we have obtained

$$\bigcup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)\subseteq \operatorname{epi}(\eta)\subseteq \operatorname{cl}\Big(\bigcup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)\Big), \tag{4}$$

which delivers, by taking the closures of the sets involved

$$\operatorname{cl}(\operatorname{epi}(\eta))=\operatorname{cl}\Big(\bigcup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)\Big).$$

If $\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$ is closed it follows by (4) that $\operatorname{epi}(\eta)$ is closed, so η is lower-semicontinuous.

Fix arbitrarily some $p\in X^*$. Since $\operatorname{dom}(f)\cap g^{-1}(-C)\cap U\neq\emptyset$ one gets $\eta(p)>-\infty$. If $\eta(p)=+\infty$ it is clear that the infimum within η is attained at any $\lambda\in C^*$. The other possible situation is $\eta(p)\in\mathbb{R}$. If this occurs, one has $(p,\eta(p))\in\operatorname{epi}(\eta)=\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$. Thus there is some $\bar\lambda\in C^*$ such that $(f+(\bar\lambda g)+\delta_U)^*(p)=\eta(p)=\inf_{\lambda\in C^*}(f+(\lambda g)+\delta_U)^*(p)$, i.e. at p the infimum within η is attained at $\bar\lambda$. Therefore the infimum within η is always attained.

On the other hand, let $p\mapsto\eta(p)$ be lower-semicontinuous and the infimum therein is always attained. Observe that η is lower-semicontinuous if and only if its epigraph is closed. Taking any $(p,r)\in\operatorname{cl}(\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*))$ it follows $\eta(p)\leq r$ and there is some $\bar\lambda\in C^*$ where the infimum within the formula of η is attained. Thus $(p,r)\in\operatorname{epi}((f+(\bar\lambda g)+\delta_U)^*)\subseteq\cup_{\lambda\in C^*}\operatorname{epi}((f+(\lambda g)+\delta_U)^*)$. Therefore the latter set is closed.

We have shown that if (dCQ) holds, η is lower-semicontinuous and the infimum therein is always attained. By Lemma 2 it follows that (CQD^L) is fulfilled.□

An example showing that (CQD^L) does not necessarily imply (dCQ) follows.

Example 1. Let $X=U=Y=\mathbb{R}$, $C=[0,+\infty)$, $f(x)=0$ for any $x\in\mathbb{R}$ and $g(x)=x^2$ whenever $x\in\mathbb{R}$. We have $C^*=[0,+\infty)$ and $\cup_{\lambda\in C^*}\operatorname{epi}((f+\delta_U+(\lambda g))^*)=\cup_{\lambda\geq 0}\operatorname{epi}((\lambda g)^*)$.

For $\lambda=0$ we have $(\lambda g)^*(p)=0$ if $p=0$ and $(\lambda g)^*(p)=+\infty$ otherwise, so $\operatorname{epi}((0g)^*)=\{0\}\times[0,+\infty)$. When $\lambda>0$ one gets $(\lambda g)^*(p)=\sup_{x\in\mathbb{R}}\{px-\lambda x^2\}=p^2/(4\lambda)$ for any $p\in\mathbb{R}$. Thus

$$\bigcup_{\lambda\geq 0}\operatorname{epi}((\lambda g)^*)=\{0\}\times[0,+\infty)\cup\bigcup_{\substack{\lambda>0,\\ p\in\mathbb{R}}}\{p\}\times\Big[\frac{p^2}{4\lambda},+\infty\Big)=\{0\}\times[0,+\infty)\cup\mathbb{R}\times(0,+\infty).$$

As this is clearly not a closed set, (dCQ) is violated.

On the other hand, the function η is now $\eta(p) = \inf_{\lambda \geq 0}(\lambda g)^*(p)$, $p \in \mathbb{R}$ and, as the conjugate inside has already been calculated, we get $\eta(p) = 0 \ \forall p \in \mathbb{R}$. It is easy to notice that this is a lower-semicontinuous function and the infimum regarding $\lambda \geq 0$ is attained at 0 when $p = 0$. Therefore (CQD^L) is valid in this case, unlike (dCQ). □

Remark 8. Thus our regularity condition (CQD^L) turns out to be weaker than all the constraint qualifications that assure strong duality for (P) and (D^L) mentioned in [12], as there is proven that they imply (dCQ). Another constraint qualification that guarantees strong duality between (P) and (D^L) when f and g are continuous is $(CCCQ)$ in [11] (see also [6]), mentioned later within this paper, too. Since $(CCCQ) \Rightarrow (dCQ)$ (cf. [12]) it is clear that (CQD^L) is valid when $(CCCQ)$, too. As the mentioned papers are very recent, to the best of our knowledge (CQD^L) is the weakest regularity condition in the literature guaranteeing strong duality between (P) and (D^L).

4 Fenchel-Lagrange duality

This part of the paper is dedicated to the introduction of a new constraint qualification (CQD^{FL}) derived from $(\overline{CQD})$ which guarantees strong duality between the given primal optimization problem (P) and its Fenchel-Lagrange dual problem,

$$(D^{FL}) \qquad \sup_{\substack{\lambda \in C^*,\\ \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*_U(-\beta)\}.$$

The Fenchel-Lagrange dual problem has been introduced and intensively studied by Boţ and Wanka. More on the way it is introduced and its relations to Fenchel and Lagrange duals may be found in [4] and [6], while in [3] it is proven to swallow as special case the still used geometric dual problem. Despite being recently introduced, the Fenchel-Lagrange duality has already some interesting applications, we remind here those in multiobjective convex optimization (cf. [7] and [8]) and Farkas-type results and theorems of the alternative (cf. [9]). Let us also mention that between the primal problem and its Lagrange and Fenchel-Lagrange duals one has the so-called weak duality statement (cf. [6])

$$v(D^{FL}) \leq v(D^L) \leq v(P). \tag{5}$$

Thus any condition that is sufficient to guarantee strong duality between (P) and (D^{FL}) yields strong duality for (P) and (D^L), too.

First we give a stable strong duality type statement derived from Theorem 2. In order to avoid any confusion, $(\overline{CQ})$ will be called further (CQ^{FL}) and it means, after replacing F, G, H and K with their formulations using f, g, U and

C given in the previous section, that the set

$$\begin{aligned} N &= \{0_{X^*}\} \times \{0_{X^*}\} \times C^* \times [0, +\infty) + \{(a, 0_{X^*}, 0_{Y^*}, r) : (a, r) \in \mathrm{epi}(f^*)\} \\ &+ \bigcup_{\substack{x^* \in X^*, \\ \lambda \in C^*}} \{(a, -x^*, -\lambda, r) : (a, r) \in \mathrm{epi}(((\lambda g) + \delta_U)^*)\} \end{aligned}$$

is closed regarding the subspace S. By Theorem 2 we have, taking into account the way F, G, H and K are written using f, g, U and C and the discussion in the beginning of the previous section,

$$(CQ^{FL}) \Leftrightarrow v(P_p) = \max_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*(\beta) - (\lambda g)^*_U(p - \beta)\} \ \forall p \in X^*. \tag{6}$$

Let us notice moreover that N may be rewritten as

$$\begin{aligned} N &= \{(a, 0_{X^*}, c, r) : (a, r) \in \mathrm{epi}(f^*), c \in C^*\} \\ &+ \bigcup_{\substack{x^* \in X^*, \\ \lambda \in C^*}} \{(a, -x^*, -\lambda, r) : (a, r) \in \mathrm{epi}(((\lambda g) + \delta_U)^*)\} \end{aligned}$$

and in the following we give an equivalent formulation of (CQ^{FL}) which is simpler than the one using N.

Lemma 3. *The regularity condition (CQ^{FL}) is valid if and only if the set*

$$\mathrm{epi}(f^*) + \bigcup_{\lambda \in C^*} (((\lambda g) + \delta_U)^*) \tag{7}$$

is closed.

Proof. We know that $\mathrm{cl}(N \cap S) \subseteq \mathrm{cl}(N) \cap S$. Let us show first that the reverse inclusion holds, too. Take the quadruple $(a, b, c, r) \in \mathrm{cl}(N) \cap S$. It is clear that $b = 0_{X^*}$ and $c = 0_{Y^*}$. Moreover, take some neighborhoods V, U and W as in the proof of Lemma 1 and an $\varepsilon > 0$. Then there is a quadruple $(\bar{a}, \bar{b}, \bar{c}, \bar{r}) \in N \cap (V \times U \times W \times (r - \varepsilon, r + \varepsilon))$. Further, taking into consideration the last formulation of N, $\bar{b} \in X^*$ and there are some $\overline{p_1}$ and $\overline{p_2}$ in X^*, $\overline{r_1}$ and $\overline{r_2}$ in $\mathbb{R}$ and $\bar{\lambda} \in C^*$ such that $(\overline{p_1}, \overline{r_1}) \in \mathrm{epi}(f^*)$ and $(\overline{p_2}, \overline{r_2}) \in \mathrm{epi}(((\bar{\lambda} g) + \delta_U)^*)$, satisfying $\bar{c} = -\bar{\lambda}$, $\bar{a} = \overline{p_1} + \overline{p_2}$ and $\bar{r} = \overline{r_1} + \overline{r_2}$. One may notice immediately that $(\bar{a}, 0_{X^*}, 0_{Y^*}, \bar{r}) \in N$, but it belongs also to S and $V \times U \times W \times (r - \varepsilon, r + \varepsilon)$, so $(a, b, c, d) \in \mathrm{cl}(N \cap S)$. Thus N is closed regarding S if and only if $N \cap S$ is closed.

Considering now the homeomorphism T defined in the proof of Lemma 1 we have that the set $T(N \cap S)$ is closed if and only if $N \cap S$ is closed. Let us prove that

$$T(N \cap S) = \left(\mathrm{epi}(f^*) + \bigcup_{\lambda \in C^*} (((\lambda g) + \delta_U)^*)\right) \times \{0_{X^*}\} \times \{0_{Y^*}\}.$$

We know that $(a, 0_{X^*}, 0_{Y^*}, r) \in N \cap S$ if and only if there are some p_1 and p_2 in X^*, r_1 and r_2 in $\mathbb{R}$ and $\lambda \in C^*$ such that $a = p_1 + p_2$, $r = r_1 + r_2$, $(p_1, r_1) \in \operatorname{epi}(f^*)$ and $(p_2, r_2) \in \operatorname{epi}(((\lambda g) + \delta_U)^*)$. This is equivalent to $(a, r, 0_{X^*}, 0_{Y^*}) \in \Big(\operatorname{epi}(f^*) + \cup_{\lambda \in C^*} \operatorname{epi}(((\lambda g) + \delta_U)^*)\Big) \times \{0_{X^*}\} \times \{0_{Y^*}\}$. Noticing that $T(a, 0_{X^*}, 0_{Y^*}, r) = (a, r, 0_{X^*}, 0_{Y^*})$, the mentioned equality is proved. These considerations above allow us to conclude that (CQ^{FL}) holds if and only if the set in (7) is closed. □

The following statement follows from Theorem 2 via (6) by taking into account Lemma 3. It may be seen as a stable strong duality assertion concerning (P) and its Fenchel-Lagrange dual problem (D^{FL}), as in the left-hand side we have actually $v(P_p)$.

Theorem 7. *The regularity condition (CQ^{FL}) is valid, i.e. the set given in (7) is closed, if and only if for any $p \in X^*$ one has*

$$\inf_{\substack{x \in U, \\ g(x) \in -C}} [f(x) - \langle p, x \rangle] = \max_{\substack{\beta \in X^*, \\ \lambda \in C^*}} \{-f^*(\beta) - (\lambda g)_U^*(p - \beta)\}.$$

In order to give the strong duality theorem for (P) and its Fenchel-Lagrange dual problem (D^{FL}) we will give a simpler formulation for (CQD^{FL}). In the previous section we have proved that the function φ required in $(\overline{CQD})$ to be lower-semicontinuous with respect to $X^* \times \{0_{X^*}\} \times \{0_{Y^*}\}$ enjoys this property if and only if the function η is lower-semicontinuous. The second part in $(\overline{CQD})$ means actually (cf. [2]) $\operatorname{epi}(A^* \Box B^*) \cap (\{0_{X^*}\} \times \{0_{X^*}\} \times \{0_{Y^*}\} \times \mathbb{R}) \subseteq (\{0_{X^*}\} \times \operatorname{epi}(G^*) + \{(p, 0_{Z^*}, r) : (p, r) \in \operatorname{epi}(F^*)\} + \underset{\alpha \in K^*}{\cup} \{(p, -\alpha, r) : (p, r) \in \operatorname{epi}((\alpha H)^*)\}) \cap (\{0_{X^*}\} \times \{0_{X^*}\} \times \{0_{Y^*}\} \times \mathbb{R})$, as the reverse inclusion is always fulfilled. Knowing that $\varphi = A^* \Box B^*$, this turns out to be (see also the discussion before Lemma 3)

$$\forall (p, b, c, r) \in \operatorname{epi}(\varphi) \cap (\{0_{X^*}\} \times \{0_{X^*}\} \times \{0_{Y^*}\} \times \mathbb{R}) \;\Rightarrow\; (p, b, c, r) \in N. \quad (8)$$

Thus we get that (CQD^{FL}) means that η is lower-semicontinuous and (8) holds. The next statement gives a simpler formulation to (CQD^{FL}).

Lemma 4. *The satisfaction of (CQD^{FL}) means actually the concomitant validity of the following two conditions*

(i) the function η is lower-semicontinuous,

(ii) there is a pair $(\overline{x^*}, \bar{\lambda}) \in X^* \times C^*$ such that

$$f^*(\overline{x^*}) + (\bar{\lambda} g)_U^*(-\overline{x^*}) \leq \inf_{\lambda \in C^*} (f + (\lambda g) + \delta_U)^*(0_{X^*}).$$

Proof. The relation (8) means actually that whenever $r \in \mathbb{R}$ satisfies $\varphi(0_{X^*}, 0_{X^*}, 0_{Y^*}) \leq r$ one has also $(0_{X^*}, 0_{X^*}, 0_{Y^*}, r) \in N$. This is equivalent to the existence of some $p_1, p_2 \in X^*$, $r_1, r_2 \in \mathbb{R}$ and $\bar{\lambda} \in C^*$ such that $0_{X^*} = p_1 + p_2$, $r = r_1 + r_2$, $(p_1, r_1) \in \operatorname{epi}(f^*)$ and $(p_2, r_2) \in \operatorname{epi}((\bar{\lambda} g)_U^*)$. Denoting $\overline{x^*} := p_1$, we get that (8) is equivalent to the existence of the mentioned $\overline{x^*}$, r_1, r_2 and $\bar{\lambda}$ such that $f^*(\overline{x^*}) \leq r_1$ and $(\bar{\lambda} g)_U^*(-\overline{x^*}) \leq r_2$ whenever $r \in \mathbb{R}$ satisfies $\varphi(0_{X^*}, 0_{X^*}, 0_{Y^*}) \leq r$. Further we get that (8) is equivalent to the fact that for any $r \in \mathbb{R}$ satisfying $\varphi(0_{X^*}, 0_{X^*}, 0_{Y^*}) \leq r$ the existence of some $\overline{x^*} \in X^*$ and $\bar{\lambda} \in C^*$ such that $f^*(\overline{x^*}) + (\bar{\lambda} g)_U^*(-\overline{x^*}) \leq r$ is granted. Taking $r = \varphi(0_{X^*}, 0_{X^*}, 0_{Y^*})$ we get that (8) implies

$$\exists \overline{x^*} \in X^* \text{ and } \exists \bar{\lambda} \in C^* : \ f^*(\overline{x^*}) + (\bar{\lambda} g)_U^*(-\overline{x^*}) \leq \inf_{\lambda \in C^*} (f + (\lambda g) + \delta_U)^*(0_{X^*}).$$

Meanwhile, when (ii) holds, for any $r \in \mathbb{R}$ satisfying $\varphi(0_{X^*}, 0_{X^*}, 0_{Y^*}) \leq r$ one obtains that the pair $(\overline{x^*}, \bar{\lambda})$ satisfies also $f^*(\overline{x^*}) + (\bar{\lambda} g)_U^*(-\overline{x^*}) \leq r$, i.e. (8) is valid. The conclusion arises immediately. $\square$

Remark 9. The inequality in (ii) in the previous theorem may be further rewritten as

$$\sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x)] \leq -f^*(\overline{x^*}) - (\bar{\lambda} g)_U^*(-\overline{x^*}).$$

We also have

$$\begin{aligned} -f^*(\overline{x^*}) - (\bar{\lambda} g)_U^*(-\overline{x^*}) &\leq \sup_{\substack{x^* \in X^*, \\ \lambda \in C^*}} \{-f^*(x^*) - (\lambda g)_U^*(-x^*)\} \\ &\leq \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x)], \end{aligned}$$

so (ii) is equivalent to

$$\begin{aligned} \exists (\overline{x^*}, \bar{\lambda}) \in X^* \times C^* : -f^*(\overline{x^*}) - (\bar{\lambda} g)_U^*(-\overline{x^*}) &= \sup_{\substack{x^* \in X^*, \\ \lambda \in C^*}} \{-f^*(x^*) - (\lambda g)_U^*(-x^*)\} \\ &= \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x)]. \end{aligned}$$

Using Theorem 4 and the results above one can easily prove the following strong duality statement for (P) and (D^{FL}).

Theorem 8. *If (CQD^{FL}) is satisfied then there is strong duality between (P) and its Fenchel-Lagrange dual problem (D^{FL}), i.e.*

$$\inf_{\substack{x \in U, \\ g(x) \in -C}} f(x) = \max_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*(\beta) - (\lambda g)_U^*(-\beta)\}.$$

Remark 10. According to Remark 5, (CQD^{FL}) implies (CQD^{L}), thus, as said in the beginning of the section, (CQD^{FL}) guarantees strong duality between (P) and (D^{L}), too. The example in the end of this section gives a situation where (dCQ) fails, while (CQD^{FL}) is valid.

Remark 11. A result similar to the one in the last theorem has been proven in a previous paper of two of the authors (Theorem 4.5 in [6]) under the additional hypotheses X Banach space and $g : X \to Y$ continuous. There the strong duality was shown provided the concomitant fulfillment of the following three conditions

(i) $f^* \Box \delta_D^*$ is lower-semicontinuous,

(ii) $f^* \Box \delta_D^*$ is exact at 0_{X^*},

(iii) $\operatorname{epi}(\delta_D^*) \subseteq \bigcup\limits_{\lambda \in C^*} \operatorname{epi}((\lambda g)_U^*)$,

where $D = \{x \in U : \; g(x) \in -C\}$. As in the original paper (iii) is called $(CCCQ)$ we will maintain this terminology, too. In the following we show that (CQD^{FL}) is indeed weaker than the condition imposed in [6].

Proposition 3. *When X is a Banach space and g is continuous, if $f \Box \delta_D$ is a lower-semicontinuous function, moreover exact at 0_{X^*}, and $(CCCQ)$ holds, then (CQD^{FL}) is valid, too.*

Proof. We know that there is some $\bar{p} \in X^*$ such that

$$(f + \delta_D)^*(0_{X^*}) = (f^* \Box \delta_D^*)(0_{X^*}) = \min_{p \in X^*} [f^*(p) + \delta_D^*(-p)] = f^*(\bar{p}) + \delta_D^*(-\bar{p}).$$

According to the formula of the conjugate we have

$$\delta_D^*(-\bar{p}) = \sup_{x \in X} \{\langle -\bar{p}, x\rangle - \delta_D(x)\} = \sup_{\substack{x \in U, \\ g(x) \in -C}} \langle -\bar{p}, x\rangle = - \inf_{\substack{x \in U, \\ g(x) \in -C}} \langle \bar{p}, x\rangle.$$

Theorem 3.2 in [6] states

$$(CCCQ) \;\Leftrightarrow\; \inf_{\substack{x \in U, \\ g(x) \in -C}} \langle \bar{p}, x\rangle = \max_{\lambda \in C^*} \inf_{x \in U} [\langle \bar{p}, x\rangle + (\lambda g)(x)], \tag{9}$$

so we get

$$(f + \delta_D)^*(0_{X^*}) = \min_{p \in X^*} [f^*(p) - \max_{\lambda \in C^*} \inf_{x \in U} [\langle p, x\rangle + (\lambda g)(x)]] = \min_{\lambda \in C^*} [f^*(\bar{p}) + (\lambda g)^*(-\bar{p})].$$

Therefore there is a pair $(\bar{p}, \bar{\lambda}) \in X^* \times C^*$ such that

$$\inf_{\substack{x \in U, \\ g(x) \in -C}} f(x) = -(f + \delta_D)^*(0_{X^*}) = -f^*(\bar{p}) - (\bar{\lambda} g)_U^*(-\bar{p}) = \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + (\lambda g)(x)],$$

and (ii) in Lemma 4 follows by Remark 9.

On the other hand, taking $(p,r) \in \cup_{\lambda \in C^*} \operatorname{epi}((f+\delta_U+(\lambda g))^*)$ there is some $\bar{\lambda} \in C^*$ such that $(f+\delta_U+(\bar{\lambda} g))^*(p) \leq r$. This delivers

$$\begin{aligned} -r &\leq -(f+\delta_U+(\bar{\lambda} g))^*(p) \leq \sup_{\lambda \in C^*}\{-(f+\delta_U+(\lambda g))^*(p)\} \\ &= \sup_{\lambda \in C^*} \inf_{x \in U}[f(x)+(\lambda g)(x)-\langle p,x\rangle)\} \leq \inf_{\substack{x \in U, \\ g(x) \in -C}} [f(x)-\langle p,x\rangle], \end{aligned}$$

the last relation following because of the weak duality (5). This yields

$$\begin{aligned} r &\geq -\inf_{\substack{x \in U, \\ g(x) \in -C}} [f(x)-\langle p,x\rangle] = \sup_{\substack{x \in U, \\ g(x) \in -C}} \{-f(x)+\langle p,x\rangle\} \\ &= \sup_{x \in X}\{\langle p,x\rangle - f(x) - \delta_D(x)\} = (f+\delta_D)^*(p), \end{aligned}$$

i.e. $(p,r) \in \operatorname{epi}((f+\delta_D)^*)$. Therefore $\cup_{\lambda \in C^*} \operatorname{epi}((f+\delta_U+(\lambda g))^*) \subseteq \operatorname{epi}((f+\delta_D)^*)$, which leads to

$$\operatorname{cl}(\underset{\lambda \in C^*}{\cup} \operatorname{epi}((f+\delta_U+(\lambda g))^*)) \subseteq \operatorname{cl}(\operatorname{epi}((f+\delta_D)^*)) = \operatorname{epi}((f+\delta_D)^*).$$

As (cf. [2])

$$\operatorname{epi}((f+\delta_D)^*) = \operatorname{cl}(\operatorname{epi}(f^* \Box \delta_D^*)) = \operatorname{epi}(f^* \Box \delta_D^*),$$

the latter because of (i), we are allowed to write the following: for any $(p,r) \in \operatorname{cl}(\cup_{\lambda \in C^*} \operatorname{epi}((f+\delta_U+(\lambda g))^*))$ we get $(f^* \Box \delta_D^*)(p) \leq r$, so for each $\varepsilon > 0$ there is an $s_\varepsilon \in \mathbb{R}$ such that $f^*(s_\varepsilon) + \delta_D^*(p-s_\varepsilon) < r+\varepsilon$. By (9) follows the existence of some $\lambda_\varepsilon \in C^*$ such that $(\lambda_\varepsilon g)_U^*(p-s_\varepsilon) \leq \delta_D^*(p-s_\varepsilon) < r+\varepsilon$, thus $f^*(s_\varepsilon)+(\lambda_\varepsilon g)_U^*(p-s_\varepsilon) < r+\varepsilon$, followed by $(f+(\lambda_\varepsilon g))_U^*(p) < r+\varepsilon$. This yields $\inf_{\lambda \in C^*}(f+(\lambda g)+\delta_U)^*(p) \leq r$, i.e. $(p,r) \in \operatorname{epi}(\eta)$, which gives

$$\operatorname{cl}(\underset{\lambda \in C^*}{\cup} \operatorname{epi}((f+\delta_U+(\lambda g))^*)) \subseteq \operatorname{epi}(\eta).$$

As the reverse inclusion has been proven within the proof of Proposition 2, the relation above and (7) imply that $\operatorname{epi}(\eta)$ is closed, i.e. η is lower-semicontinuous, so (i) in (CQD^{FL}) holds, too. $\square$

Remark 12. Example 1 is useful to show that (CQD^{FL}) is indeed weaker than the condition in [6], i.e. the concomitant fulfilment of $(i)-(iii)$ in Remark 11. We notice that $D=\{0\}$, thus $\operatorname{epi}(\delta_D^*) = \mathbb{R} \times [0,+\infty)$, which is not included in $\cup_{\lambda \geq 0} \operatorname{epi}((\lambda g)_U^*) = \{0\} \times [0,+\infty) \cup \mathbb{R} \times (0,+\infty)$, i.e. (iii) in Remark 11 fails. Regarding (CQD^{FL}) we have $(f^* \Box \delta_D^*)(p) = (f+\delta_D)^*(p) = 0 \ \forall p \in \mathbb{R}$, so $f^* \Box \delta_D^*$ is lower-semicontinuous and $\sup_{\lambda \geq 0} \inf_{x \in \mathbb{R}}[f(x)+(\lambda g)(x)] = 0 = -f^*(0)-(0g)^*(0)$, which means, by Remark 9, that it is valid.

5 Conclusions

We have applied some recent constraint qualifications for the formula for the subdifferential of composed convex functions in infinite dimensional spaces to both Lagrange and Fenchel-Lagrange dualities, delivering new regularity conditions that guarantee strong duality in each case. Moreover we completely characterize the stable strong duality in both situations. We prove that these sufficient conditions are weaker than some other very recent ones given in the literature as the weakest so far for both kinds of dualities studied, providing an example where they fail, unlike ours.

6 Appendix: Fenchel duality

For the sake of completeness we give here without proofs some statements concerning Fenchel duality following the scheme used within Sections 3 and 4 for Lagrange, respectively Fenchel-Lagrange duality. These assertions were stated and proven in [5] and then rediscovered in [2] where they are seen as arising from Theorems $1-4$, too.

Take also the proper convex lower-semicontinuous function $h : Y \to \overline{\mathbb{R}}$ and the linear continuous mapping $A : X \to Y$ such that $\mathrm{dom}(f) \cap A(\mathrm{dom}(h)) \neq \emptyset$. We need to recall first some notions. The *identity* function on X is defined by $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x\ \forall x \in X$. As in [5] we introduce also the *product function*

$$(f \times h) : X \times Y \to \overline{\mathbb{R}} \times \overline{\mathbb{R}},\ (f \times h)(x, y) = (f(x), h(y))\ \forall (x, y) \in X \times Y.$$

The *adjoint* of A is A^* given by $\langle x, A^*y^*\rangle = \langle Ax, y^*\rangle$ for any $(x, y^*) \in X \times Y^*$. We have also the *marginal* function of f through A as $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\}$, $y \in Y$. Consider the following regularity conditions

(CQ^F) $\quad$ $\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(h^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$,

and

(CQD^F) $\quad$ $f^*\square A^*h^*$ is lower-semicontinuous and $\mathrm{epi}(f^*\square A^*h^*) \cap (\{0_{X^*}\} \times \mathbb{R}) = (\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(h^*))) \cap (\{0_{X^*}\} \times \mathbb{R})$.

Remark 13. The satisfaction of (CQ^F) guarantees the validity of (CQD^F), while the reverse implication does not always hold, as proved by Example 5.11 in [2].

The pair of problems we are dealing with here consists of

$$(P^F) \qquad \inf_{x\in X}[f(x)+(h\circ A)(x)]$$

and

$$(D^F) \qquad \sup_{y^*\in Y^*}\{-f^*(-A^*y^*)-h^*(y^*)\}.$$

We give first the stable strong duality type statement for (P^F) and (D^F), followed by the strong duality assertion.

Theorem 9. *The condition (CQ^F) is fulfilled if and only if for any $p\in X^*$*

$$\inf_{x\in X}[f(x)+h(Ax)-\langle p,x\rangle] = -(f+h\circ A)^*(p) = \max_{y^*\in Y^*}\{-f^*(p-A^*y^*)-h^*(y^*)\}.$$

Theorem 10. *If (CQD^F) is valid, then*

$$\inf_{x\in X}[f(x)+h(Ax)] = \max_{y^*\in Y^*}\{-f^*(-A^*y^*)-h^*(y^*)\}.$$

Remark 14. As underlined in [2] and [5], (CQD^F) is the weakest sufficient condition known to us in the literature that guarantees strong duality between (P^F) and (D^F) in the given circumstances.

Remark 15. The results within this Appendix may be further particularized by taking $Y=X$ and $A=\mathrm{id}_X$, as shown in [2] and [5].

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