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Farkas-type results for fractional programming problems

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Abstract. Considering a constrained fractional programming problem, within the present paper we present some necessary and sufficient conditions which ensure that the optimal objective value of the considered problem is greater than or equal to a given real constant. The desired results are obtained using the Fenchel-Lagrange duality approach applied to an optimization problem with convex or difference of convex (DC) objective functions and finitely many convex constraints. Moreover, it is shown that our general results encompass as special cases some recently obtained Farkas-type results.

Key Words. Farkas-type results, fractional programming, DC functions, conjugate functions, conjugate duality

1 Introduction

Since many optimization problems which arise from the practical needs turn out to be of fractional type, more and more papers treating this kind of problems have appeared during the last decades. Although many papers are oriented more in the practical field, as they present techniques of solving such problems (see, for example, [9], [11], [16]), the theoretical side has not

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been neglected. In papers like [1], [2], [8], [14] dual problems of various fractional programming problems are constructed and weak and strong duality assertions are also given.

The problem we work with consists in minimizing a fractional function when its variable covers a nonempty convex subset of $\mathbb{R}^n$ and finitely many convex constrains are non-positive. Considering λ an arbitrary real number, our aim is to give some necessary and sufficient conditions which ensure that the optimal objective value of the considered problem is greater than or equal to λ. More precisely, we give necessary and sufficient conditions which ensure that

$$x \in X, h(x) \leqq 0 \Rightarrow \frac{f(x)}{g(x)} \geq \lambda,$$

where the nonempty convex set $X \subseteq \mathbb{R}^n$ and the proper convex functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are given. As usual for a fractional programming problem, the condition $g(x) > 0$ for all x feasible is also assumed.

The approach we use is the following. To an initial fractional programming problem we attach a new one, whose objective function is a convex function or the difference of two convex functions, while the constraints remain the ones of the initial problem. We would like to mention that the objective function of the new problem depends on a real parameter λ. Namely, it is a convex function for λ non-negative and a difference of convex functions for λ strictly negative. To the new problem we determine its Fenchel-Lagrange-type dual problem, a type of dual problems recently introduced by Wanka and Boţ (cf. [15]). The construction of the dual is described in detail and a constraint qualification which assures strong duality is presented. Using the relations between the optimal objective values of the attached problem and its dual, the desired result is presented in the form of a Farkas-type result.

Recently, Boţ and Wanka [6] have presented some Farkas-type results for inequality systems involving finitely many convex functions using an approach based on the theory of conjugate duality for convex optimization problems. In this paper their results are naturally extended to the problem we treat and, moreover, it is shown that some other recent statements can be derived as special cases of our general result.

The paper has the following structure. The second section presents some definitions and results which are used later within the paper. In Section 3 we give a dual for the optimization problem with a convex objective function and finitely many convex constraints. Using the acquired duality one of our main results is presented. The fourth section of the paper presents some results similar to the ones presented in the third section. The difference arises as a consequence of the fact that the objective function of the problem we treat is the difference of two convex functions. Within the last section of the paper it is shown that some recent statements from the literature are actually particular instances of our main results.

2 Notations and preliminaries

For the sake of completeness some well-known definitions and results are recalled in the following. As usual, by $\mathbb{R}^n$ is denoted the n-dimensional real space for any positive integer n. All vectors are considered to be column vectors. Any column vector can be transposed to a row vector by an upper index ^{T}. By $x^T y = \sum_{i=1}^n x_i y_i$ is denoted the usual inner product of two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in $\mathbb{R}^n$. As usual, the space $\mathbb{R}^n$ is partially ordered by its positive orthant $\mathbb{R}^n_+$, namely

$$x \leqq y \Leftrightarrow y - x \in \mathbb{R}^n_+, \quad \forall x, y \in \mathbb{R}^n.$$

Let us consider an arbitrary set $X \subseteq \mathbb{R}^n$. By $\mathrm{ri}(X)$ and $\mathrm{co}(X)$ are denoted the *relative interior* and the *convex hull* of the set X, respectively. Furthermore, the *cone* and the *convex cone* generated by the set X are denoted by $\mathrm{cone}(X) = \bigcup_{\lambda \geq 0} \lambda X$ and, respectively, $\mathrm{coneco}(X) = \bigcup_{\lambda \geq 0} \lambda \,\mathrm{co}(X)$. By $v(P)$ we denote the optimal objective value of an optimization problem (P).

If $X \subseteq \mathbb{R}^n$ is given, we consider the following two functions, the *indicator function*

$$\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}, \quad \delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & \text{otherwise}, \end{cases}$$

and the *support function*

$$\sigma_X : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad \sigma_X(u) = \sup_{x \in X} u^T x,$$

respectively.

For a given function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$ its *effective domain* and by $\mathrm{epi}(f) = \big\{(x, r) : x \in \mathbb{R}^n, r \in$

$\mathbb{R}, f(x) \leq r\big\}$ its *epigraph*, respectively. The function f is called *proper* if its effective domain is a nonempty set and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for the function f the *conjugate relative to the set* X by

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f_X^*(p) = \sup_{x \in X} \big\{p^T x - f(x)\big\}.$$

It is easy to observe that for $X = \mathbb{R}^n$ the conjugate relative to the set X is actually the *(Fenchel-Moreau) conjugate function* of f usually denoted by f^*. Even more, it is trivial to prove that

$$f_X^* = (f + \delta_X)^* \text{ and } \delta_X^* = \sigma_X.$$

Further we adopt the following conventions (cf. [12])

$$(+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty,$$

$$0(+\infty) = +\infty \text{ and } 0(-\infty) = 0.$$

It is easy to see that the last two conventions imply

$$0f = \delta_{\mathrm{dom}(f)}.$$

Definition 2.1 A function $h : \mathbb{R}^n \to \mathbb{R}^m$ is called *convex* if for all $x, y \in X$ and for all $t \in [0, 1]$ one has

$$h\big(tx + (1-t)y\big) \leqq th(x) + (1-t)h(y).$$

A function h is called *concave* if $-h$ is convex.

Definition 2.2 Let the proper functions $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be given. The infimal convolution function of $f_1, ..., f_m$ is the function

$$f_1 \Box ... \Box f_m : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad (f_1 \Box ... \Box f_m)(x) = \inf \left\{ \sum_{i=1}^m f_i(x_i) : x = \sum_{i=1}^m x_i \right\}.$$

The following statements close this preliminary section.

Theorem 2.1 ([13]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(f_i))$ is nonempty, then

$$\left(\sum_{i=1}^m f_i \right)^*(p) = (f_1^* \Box ... \Box f_m^*)(p) = \inf \left\{ \sum_{i=1}^m f_i^*(p_i) : p = \sum_{i=1}^m p_i \right\},$$

and for each $p \in \mathbb{R}^n$ the infimum is attained.

Corollary 2.2 ([4]) Let $f_1, ..., f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. If the set $\bigcap_{i=1}^m \text{ri}(\text{dom}(f_i))$ is nonempty, then

$$\text{epi}\left(\left(\sum_{i=1}^m f_i\right)^*\right) = \sum_{i=1}^m \text{epi}(f_i^*).$$

Proposition 2.3 ([4]) Let $f : \mathbb{R}^k \to \overline{\mathbb{R}}$ be a proper function and $\alpha > 0$ a real number. One has

$$\text{epi}\big((\alpha f)^*\big) = \alpha \, \text{epi}\big(f^*\big).$$

General framework

In the following we present some assumptions which we consider fulfilled throughout the entire paper. Let X be a nonempty convex subset of $\mathbb{R}^n$. The problem we work with is

$$(P) \qquad \inf_{\substack{x \in X, \\ h(x) \leqq 0}} \frac{f(x)}{g(x)},$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper and convex function, $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a concave function such that $-g$ is proper and $h : \mathbb{R}^n \to \mathbb{R}^m$ is a convex function such that

$$X \cap \text{dom}(f) \cap h^{-1}(-\mathbb{R}^m_+) \neq \emptyset, \tag{1}$$

where $h^{-1}(-\mathbb{R}^m_+) = \{x \in \mathbb{R}^n : h(x) \leqq 0\}$. Moreover, we suppose that $g(x) > 0$ for all x feasible to the problem (P), i.e., for all $x \in X \cap h^{-1}(-\mathbb{R}^m_+)$.

Before going further, we would like to underline some conclusions which can be easily extracted from the conditions already imposed. The first concerns the objective value of the problem (P). Namely, since the relation (1) is fulfilled, it is easy to see that $v(P) < +\infty$. The second result we would like to mention regards the properness of the function $-g$. Although we suppose that $g(x) > 0$ over the feasible set and the later is considered nonempty, it is not hard to see that the fulfillment of this condition does not necessarily imply the properness of the function $-g$. Therefore the conditions imposed are not superfluous.

Lastly, for an arbitrary real number λ let us consider the attached problem (cf. [9])

$$(P^\lambda) \qquad \inf_{\substack{x\in X,\\ h(x)\leqq 0}} \big(f(x)-\lambda g(x)\big).$$

Then the following result, whose proof is skipped because of its simplicity, can also be proved.

Lemma 2.4 The following equivalence holds

$$v(P)\geq\lambda \Leftrightarrow v(P^\lambda)\geq 0.$$

Our next step is to construct a dual problem to (P^λ) and to give sufficient conditions in order to achieve strong duality, i.e., the situation when the optimal objective value of the primal coincides with the optimal objective value of the dual and the dual has an optimal solution. Since the objective function of the problem (P^λ) depends on the sign of λ, we have to treat two different cases. First, we assume that λ is a non-negative value. In this case the objective function of the problem (P^λ) is convex and therefore the theory already developed for convex programming can be used. The second case occurs for λ strictly negative. In this case the objective function of the problem (P^λ) becomes the difference of two convex functions and therefore we have to use a slightly different approach inspired from DC programming.

3 The case $\lambda\geq 0$

A look at the objective function of the problem (P^λ) shows us that the function $f-\lambda g$ is a convex function and, using the methods of convex programming, a dual problem can be easily established.

To the problem (P^λ) we associate its Lagrange dual problem

$$(D^\lambda) \qquad \sup_{q\geqq 0}\inf_{x\in X}\big\{\big(f+\lambda(-g)\big)(x)+(q^T h)(x)\big\}.$$

But for our aims it is important to point out the idea of reformulation the inner infimum of the Lagrange dual problem by using conjugate functions.

Regarding this infimum concerning x, the definition of the conjugate relative to a set allows it to be rewritten as

$$\begin{aligned}
& \inf_{x\in X} \big\{ \big(f+\lambda(-g)\big)(x) + (q^T h)(x) \big\} \\
= & -\sup_{x\in X} \big\{ -f(x) - \lambda(-g)(x) - (q^T h)(x) \big\} \\
= & -\sup_{x\in \mathbb{R}^n} \big\{ -f(x) - \lambda(-g)(x) - (q^T h)(x) - \delta_X(x) \big\} \\
= & -\sup_{x\in \mathbb{R}^n} \big\{ -f(x) - \lambda(-g)(x) - (q^T h + \delta_X)(x) \big\} \\
= & -\big((f+\lambda(-g)) + q^T h + \delta_X\big)^*(0).
\end{aligned}$$

Assuming $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(-g)) \cap \mathrm{ri}(X) \neq \emptyset$, by Theorem 2.1 we get further

$$\begin{aligned}
& \inf_{x\in X} \big\{ \big(f+\lambda(-g)\big)(x) + (q^T h)(x) \big\} \\
= & -\inf_{u,v\in \mathbb{R}^n,} \big\{ f^*(u) + \big(\lambda(-g)\big)^*(v) + (q^T h)^*_X(-u-v) \big\} \\
= & \sup_{u,v\in \mathbb{R}^n} \big\{ -f^*(u) - \big(\lambda(-g)\big)^*(v) - (q^T h)^*_X(-u-v) \big\},
\end{aligned}$$

and the dual (D^λ) becomes

$$(D^\lambda) \qquad \sup_{\substack{u,v\in \mathbb{R}^n,\\ q\geqq 0}} \big\{ -f^*(u) - \big(\lambda(-g)\big)^*(v) - (q^T h)^*_X(-u-v) \big\}.$$

We consider (D^λ) first for $\lambda > 0$. Using the definition of the conjugate it can be easily proved that

$$\big(\lambda(-g)\big)^*(v) = \lambda(-g)^*\big(\frac{1}{\lambda}v\big).$$

Introducing new variables $x^* = u$ and $y^* = \frac{1}{\lambda}v$ allows to write (D^λ) for $\lambda > 0$ in the new form

$$(D^\lambda) \qquad \sup_{\substack{x^*,y^*\in \mathbb{R}^n,\\ q\geqq 0}} \big\{ -f^*(x^*) - \lambda(-g)^*(y^*) - (q^T h)^*_X(-x^* - \lambda y^*) \big\}.$$

Now we look at (D^λ) for $\lambda = 0$ realizing

$$(D^0) \qquad \sup_{\substack{u,v\in \mathbb{R}^n,\\ q\geqq 0}} \big\{ -f^*(u) - \big(0(-g)\big)^*(v) - (q^T h)^*_X(-u-v) \big\}.$$

For the second term in the objective function we get by definition

$$-\big(0(-g)\big)^*(v) = \begin{cases} 0, & v = 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore (D^0) may be rewritten omitting v, namely

$$(D^0) \qquad \sup_{\substack{u\in\mathbb{R}^n,\\ q\geqq 0}} \big\{ - f^*(u) - (q^T h)^*_X(-u)\big\}.$$

But setting formally $\lambda = 0$ in the new form of (D^λ) we obtain

$$\sup_{\substack{x^*,y^*\in\mathbb{R}^n,\\ q\geqq 0}} \big\{ - f^*(x^*) - 0(-g)^*(y^*) - (q^T h)^*_X(-x^* - 0y^*)\big\}$$
$$= \sup_{\substack{u\in\mathbb{R}^n,\\ q\geqq 0}} \big\{ - f^*(u) - (q^T h)^*_X(-u)\big\}.$$

This is indeed the above problem (D^0).

Thus, from the beginning we may write (D^λ) in the new form also for $\lambda = 0$.

Taking a closer look at the new form of the dual problem (D^λ), it is easy to see that it is actually the Fenchel-Lagrange dual problem of (P^λ) (more information regarding this type of a dual are to be found in [5] and [15]).

Since the optimal objective value of the problem (P^λ) is always greater than or equal to the optimal objective value of its Fenchel-Lagrange dual, i.e., $v(P^\lambda) \geq v(D^\lambda)$, the next result follows at hand.

Theorem 3.1 Between the primal problem (P^λ) and the dual problem (D^λ) weak duality always holds, i.e., $v(P^\lambda) \geq v(D^\lambda)$.

In order to secure strong duality, the following constraint qualification is considered

$$(CQ) \quad \exists x' \in \mathrm{ri}\,\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}\,\big(\mathrm{dom}(-g)\big) \cap \mathrm{ri}(X) \text{ s.t. } \begin{cases} h_i(x') \leq 0, & i \in L, \\ h_i(x') < 0, & i \in N, \end{cases}$$

where $L := \big\{ i \in \{1,...,m\} : h_i \text{ is an affine function} \big\}$ and $N := \{1,...,m\}\setminus L$.

Theorem 3.2 Assume that $v(P^\lambda)$ is finite. If (CQ) is fulfilled, then between (P^λ) and (D^λ) strong duality holds, namely $v(P^\lambda) = v(D^\lambda)$ and the dual problem has an optimal solution.

Proof. To the problem (P^λ) we associate its Lagrange dual problem (D^λ). Since the condition (CQ) is fulfilled, it is well-known from the existent literature (see [13]) that between (P^λ) and (D^λ) strong duality holds. This means nothing but the fact that the optimal objective values of (P^λ) and (D^λ) are equal and, moreover, there exists $\overline{q} \geqq 0$ such that

$$\begin{aligned}
v(D^\lambda) &= \sup_{q \geqq 0} \inf_{x \in X} \left\{ \big(f + \lambda(-g)\big)(x) + (q^T h)(x) \right\} \\
&= \inf_{x \in X} \left\{ f(x) + \lambda(-g)(x) + (\overline{q}^T h)(x) \right\} \\
&= -\sup_{x \in \mathbb{R}^n} \left\{ -f(x) - \lambda(-g)(x) - (\overline{q}^T h)(x) - \delta_X(x) \right\} \\
&= -\big(f + \lambda(-g) + (\overline{q}^T h + \delta_X)\big)^*(0).
\end{aligned}$$

Since $\mathrm{dom}(h) = \mathbb{R}^n$ the equality $\mathrm{dom}(\overline{q}^T h + \delta_X) = X$ follows at hand. Moreover, as $\mathrm{dom}(\lambda(-g)) = \mathrm{dom}(-g)$, the fulfillment of the condition (CQ) implies

$$\mathrm{ri}\,\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}\,\big(\mathrm{dom}(\lambda(-g))\big) \cap \mathrm{ri}\,\big(\mathrm{dom}((\overline{q}^T h + \delta_X))\big) \neq \emptyset.$$

By Theorem 2.1 we get further

$$v(D^\lambda) = -\inf_{u,v \in \mathbb{R}^n} \left\{ f^*(u) + \big(\lambda(-g)\big)^*(v) + \big(\overline{q}^T h + \delta_X\big)^*(-u - v) \right\},$$

and there exist some $\overline{u}, \overline{v} \in \mathbb{R}^n$ such that the infimum is attained, i.e.,

$$v(D^\lambda) = -f^*(\overline{u}) - \big(\lambda(-g)\big)^*(\overline{v}) - \big(\overline{q}^T h\big)^*_X(-\overline{u} - \overline{v}).$$

If we consider $\overline{x}^* = \overline{u}$ and $\overline{y}^* = \frac{1}{\lambda}\overline{v}$ for $\lambda > 0$ and using for $\lambda = 0$ the same arguments as above where we have derived the new formulation for (D^λ) we get

$$v(D^\lambda) = -f^*(\overline{x}^*) - \lambda(-g)^*(\overline{y}^*) - (\overline{q}^T h)^*_X(-\overline{x}^* - \lambda\overline{y}^*).$$

Indeed for $\lambda = 0$ we have $\overline{y}^* = 0$ in the optimum. Since $v(P^\lambda) = v(D^\lambda)$ and $(\overline{x}^*, \overline{y}^*, \overline{q})$ is an optimal solution for (D^λ), the proof is complete. □

The results presented above are the backbone in the demonstration of the following Farkas-type results.

Theorem 3.3 Take λ a non-negative real number and suppose that (CQ) is fulfilled. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leq 0 \Rightarrow \frac{f(x)}{g(x)} \geq \lambda$;

(ii) there exist $x^*, y^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)^*_X(-x^* - \lambda y^*) \leq 0. \tag{2}$$

Proof. Since

$$(i) \Leftrightarrow v(P) \geq \lambda \Leftrightarrow v(P^\lambda) \geq 0,$$

our aim is to prove that the last relation of the previous equivalences holds if and only if (ii) holds, too.

"$\Rightarrow$" As the assumptions of Theorem 3.2 are achieved, strong duality holds between (P^λ) and (D^λ), namely $v(P^\lambda) = v(D^\lambda)$ and the dual (D^λ) has an optimal solution. But this means actually that there exist some $x^*, y^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$0 \leq v(P^\lambda) = v(D^\lambda) = -f^*(x^*) - \lambda(-g)^*(y^*) - (q^T h)^*_X(-x^* - \lambda y^*),$$

and relation (2) follows as a consequence.

"$\Leftarrow$" As we can find some $x^*, y^* \in \mathbb{R}^n$ and $q \geqq 0$ such that relation (2) holds, it is obvious that

$$v(D^\lambda) = \sup_{\substack{x^*, y^* \in \mathbb{R}^n, \\ q \geqq 0}} \big\{ - f^*(x^*) - \lambda(-g)^*(y^*) - (q^T h)^*_X(-x^* - \lambda y^*)\big\} \geq 0.$$

Since weak duality between (P^λ) and (D^λ) always holds, we get $v(P^\lambda) \geq 0$, too, and the desired equivalence has been proved. $\square$

Let us reformulate the previous statement as a theorem of the alternative.

Corollary 3.4 Assume that $\lambda \geq 0$ is a real number and that (CQ) is fulfilled. Then either the inequality system

(I) $\quad x \in X, h(x) \leqq 0, \frac{f(x)}{g(x)} < \lambda$

has a solution or the system

$$
\begin{array}{ll}
(II) & f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)^*_X(-x^* - \lambda y^*) \leq 0, \\
 & x^*, y^* \in \mathbb{R}^n, q \geqq 0,
\end{array}
$$

has a solution, but never both.

Inspired by the results presented in [6] and [10], the following theorem presents an equivalent assertion to the statement (ii) in Theorem 3.3 using only the epigraphs of the functions involved. Moreover, we show in the last part of the present paper that the results presented in the papers mentioned above are actually particular cases of the next result.

Theorem 3.5 The statement (ii) in Theorem 3.3 is equivalent to

$$
(0,0) \in \operatorname{epi}(f^*) + \lambda \operatorname{epi}((-g)^*) + \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X). \qquad (3)
$$

Proof. "$\Rightarrow$" As statement (ii) in Theorem 3.3 is fulfilled, there exist some $x^*, y^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$
f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)^*_X(-x^* - \lambda y^*) \leq 0.
$$

Further we deal with two cases.

First let us suppose that $q = 0$. In this case the previous relation becomes $f^*(x^*) + \lambda(-g)^*(y^*) + \sigma_X(-x^* - \lambda y^*) \leq 0$ and from here we get $\sigma_X(-x^* - \lambda y^*) \leq -f^*(x^*) - \lambda(-g)^*(y^*)$. This assures $(-x^* - \lambda y^*, -f^*(x^*) - \lambda(-g)^*(y^*)) \in \operatorname{epi}(\sigma_X)$, and, as $(0,0) \in \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(h_i^*)\right)$, we have

$$
\begin{aligned}
(0,0) &= \big(x^*, f^*(x^*)\big) + \lambda\big(y^*, (-g)^*(y^*)\big) + \big(-x^* - \lambda y^*, -f^*(x^*) - \lambda(-g)^*(y^*)\big) \\
&\in \operatorname{epi}(f^*) + \lambda \operatorname{epi}((-g)^*) + \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X).
\end{aligned}
$$

Suppose now $q \neq 0$. The set $I_q = \{i \in \{1, ..., m\} : q_i \neq 0\}$ is obviously nonempty and the relation (2) can be rewritten as

$$
f^*(x^*) + \lambda(-g)^*(y^*) + \left(\sum_{i \in I_q} q_i h_i\right)^*_X (-x^* - \lambda y^*) \leq 0.
$$

By definition, the previous and the next relations are equivalent to each other

$$\big(-x^* - \lambda y^*, -f^*(x^*) - \lambda(-g)^*(y^*)\big) \in \operatorname{epi}\left(\left(\sum_{i \in I_q} q_i h_i\right)^*_X\right).$$

Using Corollary 2.2 and Proposition 2.3 we get

$$\begin{aligned}
&\operatorname{epi}\left(\left(\sum_{i \in I_q} q_i h_i\right)^*_X\right) = \sum_{i \in I_q} \operatorname{epi}\big((q_i h_i)^*\big) + \operatorname{epi}(\sigma_X) \\
= \;& \sum_{i \in I_q} q_i \operatorname{epi}\big(h_i^*\big) + \operatorname{epi}(\sigma_X) = \left(\sum_{i \in I_q} q_i\right) \sum_{i \in I_q} \frac{q_i}{\sum\limits_{i \in I_q} q_i} \operatorname{epi}\big(h_i^*\big) + \operatorname{epi}(\sigma_X) \\
\subseteq \;& \left(\sum_{i \in I_q} q_i\right) \operatorname{co}\left(\bigcup_{i=1}^m \operatorname{epi}\big(h_i^*\big)\right) + \operatorname{epi}(\sigma_X) \subseteq \operatorname{coneco}\left(\bigcup_{i \in I_q} \operatorname{epi}\big(h_i^*\big)\right) + \operatorname{epi}(\sigma_X) \\
\subseteq \;& \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}\big(h_i^*\big)\right) + \operatorname{epi}(\sigma_X).
\end{aligned}$$

As a remark, let us mention that this calculation requires the assumption $\cap_{i \in I_q} \operatorname{ri}(\operatorname{dom}(h_i)) \cap \operatorname{ri}(X) \neq \emptyset$, which is automatically satisfied. Thus

$$\begin{aligned}
(0,0) &= \big(x^*, f^*(x^*)\big) + \lambda\big(y^*, (-g)^*(y^*)\big) + \big(-x^* - \lambda y^*, -f^*(x^*) - \lambda(-g)^*(y^*)\big) \\
&\in \operatorname{epi}(f^*) + \lambda \operatorname{epi}((-g)^*) + \operatorname{coneco}\left(\bigcup_{i=1}^m \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X),
\end{aligned}$$

and the necessity is proved.

"$\Leftarrow$" As relation (3) is fulfilled, there exist $(x^*, r) \in \operatorname{epi}(f^*)$, $(y^*, s) \in \operatorname{epi}((-g)^*)$, $(z^*, t) \in \operatorname{coneco}\big(\bigcup_{i=1}^m \operatorname{epi}(h_i^*)\big)$ and $(w^*, p) \in \operatorname{epi}(\sigma_X)$ such that

$$(0,0) = (x^*, r) + \lambda(y^*, s) + (z^*, t) + (w^*, p). \tag{4}$$

Moreover, there exist $\alpha \geq 0$, $\mu_i \geq 0$ and $(u_i, v_i) \in \operatorname{epi}(h_i^*)$, $i = 1, ..., m$, such that $\sum_{i=1}^m \mu_i = 1$ and

$$(z^*, t) = \alpha \sum_{i=1}^m \mu_i (u_i, v_i). \tag{5}$$

If $\alpha = 0$ we have $(z^*, t) = (0, 0)$ and relation (4) becomes

$$(0, 0) = (x^*, r) + \lambda(y^*, s) + (w^*, p).$$

Since $f^*(x^*) \leq r$, $(-g)^*(y^*) \leq s$ and $\sigma_X(w^*) = \delta_X^*(w^*) \leq p$, the equality from above implies

$$w^* = -x^* - \lambda y^* \text{ and } f^*(x^*) + \lambda(-g)^*(y^*) + \delta_X^*(w^*) \leq 0.$$

Considering $q = (0, ..., 0) \in \mathbb{R}^m$ the inequality

$$f^*(x^*) + \lambda(-g)^*(y^*) + \left(\sum_{i=1}^m q_i h_i + \delta_X\right)^*(-x^* - \lambda y^*) \leq 0$$

follows at hand. Using the definition of the conjugate relative to a set, the conclusion is straightforward in this case.

If $\alpha > 0$ let us consider $q = (\alpha\mu_1, ..., \alpha\mu_m) \in \mathbb{R}^m$. As $\sum_{i=1}^m \mu_i = 1$, the set I_q is nonempty and relation (5) becomes in this case

$$(z^*, t) = \sum_{i \in I_q} q_i(u_i, v_i).$$

From the previous relation the relations

$$z^* = \sum_{i \in I_q} q_i u_i \text{ and } t = \sum_{i \in I_q} q_i v_i \geq \sum_{i \in I_q} q_i h_i^*(u_i)$$

can be easily deduced. Combining these with relation (4) and with the inequalities $f^*(x^*) \leq r$, $(-g)^*(y^*) \leq s$ and $\sigma_X(w^*) = \delta_X^*(w^*) \leq p$ we obtain

$$\sum_{i \in I_q} q_i u_i + w^* = -x^* - \lambda y^*$$

and

$$f^*(x^*) + \lambda(-g)^*(y^*) + \sum_{i \in I_q} q_i h_i^*(u_i) + \delta_X^*(w^*) \leq 0.$$

As

$$
\begin{aligned}
&\sum_{i\in I_q} q_i h_i^*(u_i) + \delta_X^*(w^*) = \sum_{i\in I_q} (q_i h_i)^*(q_i u_i) + \delta_X^*(w^*) \\
\geq\ & \left(\sum_{i\in I_q} q_i h_i + \delta_X\right)^* \left(\sum_{i\in I_q} q_i u_i + w^*\right) = \left(\sum_{i\in I_q} q_i h_i\right)_X^* \left(\sum_{i\in I_q} q_i u_i + w^*\right) \\
=\ & \left(\sum_{i=1}^{m} q_i h_i\right)_X^* \left(\sum_{i\in I_q} q_i u_i + w^*\right) = \left(q^T h\right)_X^* (-x^* - \lambda y^*),
\end{aligned}
$$

the desired conclusion arises immediately. □

4 The case $\lambda < 0$

If λ is a strictly negative real number, it is not hard to see that the objective function of the problem (P^λ), namely $f - \lambda g$, it is not necessarily a convex function. Therefore, in order to determine a dual problem, the approach used in the previous section cannot be directly employed. Still, as the function $f + \lambda(-g)$ is actually the difference of two convex functions ($f - \lambda g = f - (\lambda g)$ and f and λg are convex functions), it is well-known from the existent literature that for such kind of problem a dual can be established, provided that some necessary assumptions are fulfilled. That is why, in addition to the conditions imposed at the beginning, we suppose further that the function $-g$ is lower semicontinuous over the feasible set of the problem (P). As a last remark, we would like to mention that the approach we use further is based on a result presented by Martínez-Legaz and Volle in [12].

Lemma 4.1 For all x feasible to the problem (P^λ) we have

$$
(-g)(x) = \sup_{y^* \in \mathrm{dom}((-g)^*)} \{{y^*}^T x - (-g)^*(y^*)\}.
$$

Proof. Since $-g$ is a proper and convex function, for each x feasible to (P) the lower semicontinuity of the function $-g$ at x implies

$$
(-g)(x) = (-g)^{**}(x) = \sup_{y^* \in \mathrm{dom}((-g)^*)} \{{y^*}^T x - (-g)^*(y^*)\}. \qquad \square
$$

Remark As $g(x) > 0$ for all feasible x, we have that $X \cap h^{-1}(-\mathbb{R}^n_+) \subseteq \mathrm{dom}(-g)$. Since $-g$ is proper and convex it follows that $-g$ is continuous over $\mathrm{ri}(\mathrm{dom}(-g))$. Nevertheless, this is not sufficient, as the result in Lemma 4.1 does not necessarily holds if the function $-g$ is not lower semicontinuous over the feasible set. Without this assumption the equality $g(x) = g^{**}(x)$ must not be fulfilled for all $x \in X \cap h^{-1}(-\mathbb{R}^n_+)$. As an example, let us consider $m = n = 1$, $X = [0, +\infty)$ and the functions

$$g : \mathbb{R} \to \overline{\mathbb{R}}, \quad g(x) = \begin{cases} -\infty, & x < 0, \\ 1, & x = 0, \\ 2, & x > 0, \end{cases}$$

and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = -x$. The previous conditions are fulfilled, namely $-g$ is a proper and convex function such that $g(x) > 0$ for all feasible x and $X \cap h^{-1}(-\mathbb{R}_+) \subseteq \mathrm{dom}(-g)$. It is not hard to see that the function $(-g)^*$ takes the value 2 for $y^* \leq 0$ and $+\infty$ otherwise. Using this we get further

$$(-g)^{**}(0) = \sup_{y^* \in \mathrm{dom}((-g)^*)} \{{y^*}^T 0 - (-g)^*(y^*)\} = \sup_{y^* \leq 0} \{{y^*}^T 0 - 2\} = -2 < -g(0).$$

Regarding our case, there exist situations when the conditions imposed at the very beginning are enough to secure the lower semicontinuity of the function $-g$ over the feasible set of the problem (P). As an example let us suppose that the feasible set is a subset of the relative interior of the domain of the function $-g$. Then the lower semicontinuity of the function $-g$ over the feasible set arises as a consequence of its convexity and the fact that $-g(x) < 0$ for all x feasible (for details see [13]).

Making use of Lemma 4.1, the problem (P^λ) can be rewritten as

$$(P^\lambda) \qquad \inf_{\substack{x \in X, \\ h(x) \leqq 0}} \Big\{ f(x) + \lambda \sup_{y^* \in \mathrm{dom}((-g)^*)} \{{y^*}^T x - (-g)^*(y^*)\} \Big\}.$$

After some minor calculations the following form is obtained

$$(P^\lambda) \qquad \inf_{y^* \in \mathrm{dom}((-g)^*)} \inf_{\substack{x \in X, \\ h(x) \leqq 0}} \Big\{ f(x) + \lambda {y^*}^T x - \lambda (-g)^*(y^*) \Big\}.$$

Obviously, the inner infimum of this formulation is a convex optimization problem. Therefore for any $y^* \in \mathrm{dom}((-g)^*)$ we consider the problem

$$(P^\lambda_{y^*}) \qquad \inf_{\substack{x \in X, \\ h(x) \leqq 0}} \big(f(x) + (-\lambda)(-\widetilde{g})(x) \big)$$

with $\widetilde{g} : \mathbb{R}^n \to \mathbb{R}$, $\widetilde{g}(x) = y^{*T}x - (-g)^*(y^*)$.

Let us fix $y^* \in \mathrm{dom}((-g)^*)$. Since the functions f and $-\widetilde{g}$ are convex functions and $-\lambda > 0$, the results provided within the previous section allow us to affirm that the problem

$$(D^\lambda_{y^*}) \qquad \sup_{\substack{x^*,z^* \in \mathbb{R}^n,\\ q \geqq 0}} \Big\{ - f^*(x^*) - (-\lambda)(-\widetilde{g})^*(z^*) - (q^T h)^*_X(-x^* - (-\lambda)z^*) \Big\}$$

is a Fenchel-Lagrange-type dual problem to $(P^\lambda_{y^*})$. Using only the definition of the conjugate of a function it is easy to calculate that

$$(-\widetilde{g})^*(z^*) = \begin{cases} -(-g)^*(y^*), & z^* = -y^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus the dual $(D^\lambda_{y^*})$ becomes

$$(D^\lambda_{y^*}) \qquad \sup_{\substack{q \geqq 0,\\ x^* \in \mathbb{R}^n}} \Big\{ - f^*(x^*) - \lambda(-g)^*(y^*) - (q^T h)^*_X(-x^* - \lambda y^*) \Big\}.$$

As in the previous section, our aim is to give weak and strong duality assertions regarding the problems $(P^\lambda_{y^*})$ and its dual $(D^\lambda_{y^*})$. Therefore we impose the following constraint qualifications

$$(\widetilde{CQ}) \quad \exists x' \in \mathrm{ri}\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}(X) \text{ s.t. } \begin{cases} h_i(x') \leq 0, & i \in L, \\ h_i(x') < 0. & i \in N, \end{cases}$$

Since $\mathrm{dom}(-\widetilde{g}) = \mathbb{R}^n$, the following two results can be easily proved using Theorem 3.1 and Theorem 3.2, respectively.

Theorem 4.2 Let $y^* \in \mathrm{dom}((-g)^*)$ be fixed. Between the primal problem $(P^\lambda_{y^*})$ and its dual $(D^\lambda_{y^*})$ weak duality always holds.

Theorem 4.3 Consider an arbitrary $y^* \in \mathrm{dom}((-g)^*)$. If $v(P^\lambda_{y^*})$ is finite and $(\widetilde{CQ})$ is fulfilled, then strong duality holds between $(P^\lambda_{y^*})$ and $(D^\lambda_{y^*})$, i.e. $v(P^\lambda_{y^*}) = v(D^\lambda_{y^*})$ and the dual $(D^\lambda_{y^*})$ has an optimal solution.

Taking into consideration the results presented in the last two theorems, it is natural to introduce the following dual problem to (P^λ)

$$(D^\lambda) \qquad \inf_{y^* \in \mathrm{dom}((-g)^*)} \sup_{\substack{x^* \in \mathbb{R}^n,\\ q \geqq 0}} \Big\{ - f^*(x^*) - \lambda(-g)^*(y^*) - (q^T h)^*_X(-x^* - \lambda y^*) \Big\}.$$

By the construction of the dual problem (D^λ) there are weak and strong duality statements for (P^λ) and (D^λ) as follows.

Theorem 4.4 It holds $v(P^\lambda) \geq v(D^\lambda)$.

Theorem 4.5 If $(\widetilde{CQ})$ is fulfilled, then $v(P^\lambda) = v(D^\lambda)$.

Remark Because of the way the problems (P^λ) and (D^λ) are defined, it is not hard to see that the the equalities $v(P^\lambda) = \inf_{y^* \in \mathrm{dom}((-g)^*)} v(P^\lambda_{y^*})$ and $v(D^\lambda) = \inf_{y^* \in \mathrm{dom}((-g)^*)} v(D^\lambda_{y^*})$ are always fulfilled.

As in the previous section we use further the weak and strong duality assertions presented in the previous theorems to prove the following Farkas-type result.

Theorem 4.6 Take λ a strictly negative number and suppose that $(\widetilde{CQ})$ is fulfilled. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leq 0 \Rightarrow \frac{f(x)}{g(x)} \geq \lambda$;

(ii) for each $y^* \in \mathrm{dom}((-g)^*)$, there exist $x^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)^*_X(-x^* - \lambda y^*) \leq 0. \tag{6}$$

Proof. The proof is similar to the one of Theorem 3.3. Since the following equivalences $(i) \Leftrightarrow v(P) \geq \lambda \Leftrightarrow v(P^\lambda) \geq 0$ hold, we prove that the last inequality is fulfilled if and only if (ii) is fulfilled, too.

"$\Rightarrow$" Take $y^* \in \mathrm{dom}((-g)^*)$. As $0 \leq v(P^\lambda) = \inf_{y^* \in \mathrm{dom}((-g)^*)} v(P^\lambda_{y^*})$ we get $0 \leq v(P^\lambda_{y^*})$, too. By Theorem 4.3, whose hypotheses are fulfilled, strong duality holds between $(P^\lambda_{y^*})$ and $(D^\lambda_{y^*})$, and this implies the existence of some $x^* \in \mathbb{R}^n$ and $q \geqq 0$ which satisfy the relation (3).

"$\Leftarrow$" Consider $y^* \in \mathrm{dom}((-g)^*)$. As we can find some $x^* \in \mathbb{R}^n$ and $q \geqq 0$ such that relation (6) holds, it is obvious that $v(D^\lambda_{y^*}) \geq 0$. Since y^* was arbitrarily taken we get $v(D^\lambda) = \inf_{y^* \in \mathrm{dom}((-g)^*)} v(D^\lambda_{y^*}) \geq 0$. As weak duality between (P^λ) and (D^λ) always holds, we get $v(P^\lambda) \geq v(D^\lambda) \geq 0$, too, and the proof is complete. $\square$

The previous result can be reformulated as a theorem of the alternative in the following way.

Corollary 4.7 Assume that $\lambda < 0$ is a real number and that $(\widetilde{CQ})$ is fulfilled. Then either the inequality system

$$(I) \quad x \in X, h(x) \leqq 0, \tfrac{f(x)}{g(x)} < \lambda$$

has a solution or each of the following systems

$$(II_{y^*}) \quad \begin{array}{l} f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)_X^*(-x^* - \lambda y^*) \leq 0, \\ x^* \in \mathbb{R}^n, q \geqq 0, \end{array}$$

where $y^* \in \operatorname{dom}((-g)^*)$, has a solution, but never both.

As before, our next step is to provide an equivalent assertion to statement (ii) of Theorem 4.6 using only the epigraphs of the involved functions.

Theorem 4.8 The statement (ii) in Theorem 4.6 is equivalent to

$$-\lambda \operatorname{epi}((-g)^*) \subseteq \operatorname{epi}(f^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X). \tag{7}$$

Proof. "$\Rightarrow$" Take an arbitrary pair $(y^*, r) \in \operatorname{epi}((-g)^*)$. Then $y^* \in \operatorname{dom}((-g)^*)$ and assertion (ii) implies the existence of $x^* \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$ such that

$$f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)_X^*(-x^* - \lambda y^*) \leq 0.$$

As the last inequality allows us to affirm that

$$-\lambda r - f^*(x^*) \geq (q^T h)_X^*(-x^* - \lambda y^*),$$

we finally get

$$-\lambda(y^*, r) = (x^*, f^*(x^*)) + (-x^* - \lambda y^*, -\lambda r - f^*(x^*)) \in \operatorname{epi}(f^*) + \operatorname{epi}((q^T h)_X^*).$$

Using the same method as in the necessity of the proof of Theorem 3.5 it can be shown that

$$\operatorname{epi}((q^T h)_X^*) \subseteq \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X).$$

From the previous relations we acquire

$$-\lambda(y^*, r) \in \mathrm{epi}(f^*) + \mathrm{coneco}\left(\bigcup_{i=1}^m \mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X)$$

and, since (y^*, r) is arbitrarily taken, the desired conclusion follows at hand.

"$\Leftarrow$" Take an arbitrary $y^* \in \mathrm{dom}((-g)^*)$. As $(y^*, (-g)^*(y^*)) \in \mathrm{epi}((-g)^*)$, by relation (7) we have

$$-\lambda(y^*, (-g)^*(y^*)) \in \mathrm{epi}(f^*) + \mathrm{coneco}\left(\bigcup_{i=1}^m \mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

Thus there exist $(x^*, r) \in \mathrm{epi}(f^*)$ and $(u, v) \in \mathrm{coneco}\left(\bigcup_{i=1}^m \mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X)$ such that

$$-\lambda(y^*, (-g)^*(x^*)) = (x^*, r) + (u, v).$$

Following the idea presented in the sufficiency part of the proof of Theorem 3.5 it can be proved that there exists $q \geqq 0$ such that $(q^T h)_X^*(u) \leq v$. Combining the inequality $f^*(x^*) \leq r$ with the previous one and with the equality from above we get

$$-\lambda y^* = x^* + u \text{ and } -\lambda(-g)^*(y^*) \geq f^*(x^*) + (q^T h)_X^*(u),$$

and this completes the proof. $\square$

5 A special case

Within this section we treat a special case of our general result. Two ideas are emphasized. First, within this special case our main statements "merge", i.e. we give a pair of theorems whose conclusions do not depend on the sign of λ. On the other hand, the assertions within this section generalize some recently obtained results.

Throughout this section all X, f, h are considered as before, while the function $g : \mathbb{R}^n \to \mathbb{R}$ is taken constant $g(x) = 1$. Using the definition it is not hard to prove that

$$(-g)^*(y^*) = \begin{cases} 1, & y^* = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Even more, in this special case the constraint qualification (CQ) becomes $(\widetilde{CQ})$.

Theorem 5.1 Suppose that $(\widetilde{CQ})$ holds and let λ be an arbitrary non-negative real number. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leqq 0 \Rightarrow f(x) \geq \lambda$;

(ii) there exists $x^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + (q^T h)^*_X(-x^*) \leq -\lambda. \tag{8}$$

Proof. By Theorem 3.3 we have (i) fulfilled if and only if there exist $x^*, y^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + \lambda(-g)^*(y^*) + (q^T h)^*_X(-x^* - \lambda y^*) \leq 0.$$

Since it is necessary to have $(-g)^*(y^*)$ different from $+\infty$, y^* can take only the value 0 and the previous inequality becomes

$$f^*(x^*) + \lambda + (q^T h)^*_X(-x^*) \leq 0.$$

The equivalence between the previous relation and relation (8) is obvious. □

Theorem 5.2 The statement (ii) in Theorem 5.1 is equivalent to

$$(0, -\lambda) \in \operatorname{epi}(f^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X). \tag{9}$$

Proof. Theorem 3.5 ensures that the statement (ii) of Theorem 5.1 is equivalent to

$$(0, 0) \in \operatorname{epi}(f^*) + \lambda \operatorname{epi}((-g)^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X).$$

But

$$\lambda\,\mathrm{epi}((-g)^*) = \lambda\big(\{0\}\times[1,+\infty)\big) = (0,\lambda) + \{0\}\times[0,+\infty),$$

and the previous relation becomes

$$(0,0) \in \mathrm{epi}(f^*) + (0,\lambda) + \{0\}\times[0,+\infty) + \mathrm{coneco}\left(\bigcup_{i=1}^{m}\mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

Using the definition of the epigraph of a function it can be easily proved that $\mathrm{epi}(\sigma_X)+\{0\}\times[0,+\infty) = \mathrm{epi}(\sigma_X)$ and the relation (9) follows from the previous one. $\square$

Theorem 5.3 Suppose that $(\widetilde{CQ})$ holds and let λ be a strictly negative real number. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leqq 0 \Rightarrow f(x) \geq \lambda$;

(ii) there exists $x^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + (q^T h)^*_X(-x^*) \leq -\lambda.$$

Proof. Theorem 4.6 assures that (i) is fulfilled if and only if there exist $x^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + \lambda + (q^T h)^*(-x^*) \leq 0.$$

Thus $f^*(x^*) + (q^T h)^*(-x^*) \leq -\lambda$ and the proof is complete. $\square$

Theorem 5.4 The statement (ii) in Theorem 5.3 is equivalent with

$$(0,-\lambda) \in \mathrm{epi}(f^*) + \mathrm{coneco}\left(\bigcup_{i=1}^{m}\mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

Proof. By Theorem 4.8 we get (ii) is equivalent to

$$-\lambda\,\mathrm{epi}((-g)^*) \subseteq \mathrm{epi}(f^*) + \mathrm{coneco}\left(\bigcup_{i=1}^{m}\mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

This can be equivalently written as

$$\{0\}\times[-\lambda,+\infty) \subseteq \mathrm{epi}(f^*) + \mathrm{coneco}\left(\bigcup_{i=1}^{m}\mathrm{epi}(h_i^*)\right) + \mathrm{epi}(\sigma_X).$$

As the properties of the epigraph assure that the previous inclusion holds if and only if

$$(0, -\lambda) \in \operatorname{epi}(f^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X),$$

the proof is completed. $\square$

The following statements unify the previous results.

Theorem 5.5 Suppose that $(\widetilde{CQ})$ holds and let λ be an arbitrary real number. Then the following assertions are equivalent:

(i) $x \in X, h(x) \leqq 0 \Rightarrow f(x) \geq \lambda$;

(ii) there exists $x^* \in \mathbb{R}^n$ and $q \geqq 0$ such that

$$f^*(x^*) + (q^T h)_X^*(-x^*) \leq -\lambda. \tag{10}$$

Theorem 5.6 The statement (ii) in Theorem 5.5 is equivalent to

$$(0, -\lambda) \in \operatorname{epi}(f^*) + \operatorname{coneco}\left(\bigcup_{i=1}^{m} \operatorname{epi}(h_i^*)\right) + \operatorname{epi}(\sigma_X). \tag{11}$$

As a last remark, we would like to mention that the previous result has been proved by Boţ and Wanka in [6] for $\lambda = 0$.

6 Conclusions

In this paper we present a Farkas-type result for systems involving finitely many convex functions and one of fraction type. The approach we use is based on conjugate duality for an optimization problem consisting in minimizing a convex/difference of convex (DC) functions subject to finitely many convex inequality constraints. The results we present generalizes some recently obtained Farkas-type results.

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