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Some new regularity conditions for Fenchel duality in real linear spaces

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Abstract. In this paper we give a new regularity condition for Fenchel duality concerning convex optimization problems in real linear spaces. Then we prove that this condition is implied by some regularity conditions given so far in the literature for this general class of optimization problems. By giving an appropriate example we show that the new regularity condition is indeed weaker than the aforementioned ones.

Key Words. conjugate functions, regularity condition, Fenchel duality, subdifferential sum formula

AMS subject classification. 49N15, 90C25, 90C46

1 Introduction

Having an optimization problem, one can attach to it (for example, by using the perturbation theory developed in [6]), a dual problem such that between these two problems weak duality always holds. That is, the optimal objective value of the dual problem is less than or equal to the optimal objective value of the primal problem. In most cases there is a so-called duality gap between the optimal objective values of these problems. The challenge is to give weak regularity conditions in order to have strong duality, which means that the

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two optimal objective values are equal and the dual problem has an optimal solution. From the beginning of this theory, many mathematicians tried to give such conditions in finite dimensional spaces (see [13]), in Banach spaces (see [1]), in Hilbert spaces (see [5]), in Fréchet spaces (see [15]), in locally convex spaces (see [3], [4], [6], [12], [18]), or even in real linear spaces (see [7], [10], [19]). An overview on these conditions was given in [9] and [17].

In this paper we consider a convex optimization problem in a real linear space. By using a conjugacy notion for the functions involved, which extends the similar notion in a locally convex space, we introduce a Fenchel dual problem to it. Then we give a new weak regularity condition which ensures strong duality between the primal problem and its Fenchel dual problem. To this end, we employ some abstract convexity notions, the necessary theory being developed in [2].

In contrast to other conditions given in the literature by Elster and Nehse (see [7]) and Lassonde (see [10]), written in terms of the core, respectively, the intrinsec core of the effective domains of the functions involved, the one proposed by us is formulated by using the epigraphs of their conjugate functions. We prove that our condition is weaker than the aforementioned regularity conditions. Also, it is a generalization of the regularity condition introduced by Boţ and Wanka in the framework of locally convex spaces (see [3]) and turns out to be a sufficient condition for the subdifferential sum formula of a convex function with the precomposition of another convex function with a linear mapping.

The paper is organized as follows. In the next section we present some definitions, notations and include some results concerning c-convexity of functions and sets that will be used later in the paper. In Section 3 we deal with the theory of conjugate functions in real linear spaces, giving some important properties. In Section 4 we prove some Moreau-Rockafellar-type theorems in real linear spaces, generalizing the one existing in locally convex spaces. In Section 5 we give the announced regularity condition for Fenchel duality and treat some particular cases of it. We establish also some results concerning subdifferential calculus. Finally, a list of references closes the paper.

2 Preliminaries

Let us consider a real linear space X and $X^{\#}$ its algebraic dual space. Let $f : X \to \overline{\mathbb{R}}$ be a given function, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

We have

- the *domain* of f: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$,
- the *epigraph* of f: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$,
- f is *proper* if $f(x) > -\infty \ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$,
- $g : X \to \mathbb{R}$ is *affine* if $\exists (x^\#, \alpha) \in X^\# \times \mathbb{R}$ such that $g(x) = x^\#(x) + \alpha, \forall x \in X$,
- $g \leq f \Leftrightarrow g(x) \leq f(x), \forall x \in X$,
- $\langle x^\#, x \rangle := x^\#(x)$, for $x^\# \in X^\#$ and $x \in X$,
- the *subdifferential* of f at x ($f(x) \in \mathbb{R}$) is the set

$$\partial f(x) = \{x^\# \in X^\# : f(y) - f(x) \geq \langle x^\#, y - x \rangle, \forall y \in X\},$$

- the *indicator function* of a subset A of X, defined by

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Definition 1. Let $f_i : X \to \overline{\mathbb{R}}, i = 1, ..., m$, be given proper functions. The function $f_1 \Box ... \Box f_m : X \to \overline{\mathbb{R}}$ defined by

$$f_1 \Box ... \Box f_m(x) = \inf \left\{ \sum_{i=1}^{m} f_i(x_i) : \sum_{i=1}^{m} x_i = x \right\}$$

is called the **infimal convolution** function of $f_1, ..., f_m$. We say that the infimal convolution function $f_1 \Box ... \Box f_m$ is exact at x if there exist some $x_i \in X, i = 1, ..., m, \sum_{i=1}^{m} x_i = x$ such that $f_1 \Box ... \Box f_m(x) = f_1(x_1) + ... + f_m(x_m)$.

Definition 2. Let X and Y be real linear spaces, $A : X \to Y$ be a linear mapping and $f : X \to \overline{\mathbb{R}}$ be a given function.

(a) The function $Af : Y \to \overline{\mathbb{R}}$ defined by

$$Af(y) = \inf\{f(x) : Ax = y\},$$

is called the **marginal function** of f through A.

(b) The set $A \times id_{\mathbb{R}}(\text{epi}(f))$ is the image of the set $\text{epi}(f)$ through the function $A \times id_{\mathbb{R}} : X \times \mathbb{R} \to Y \times \mathbb{R}$, that is $A \times id_{\mathbb{R}}(\text{epi}(f)) = \{(Ax, r) : f(x) \leq r\}$, where $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is the identity mapping, $id_{\mathbb{R}}(r) = r, \forall r \in \mathbb{R}$.

(c) The adjoint operator of A is defined by $A^{\#} : Y^{\#} \to X^{\#}$, $\langle A^{\#} y^{\#}, x \rangle = \langle y^{\#}, Ax \rangle, \forall y^{\#} \in Y^{\#}, \forall x \in X$.

Now we recall some definitions and results given in [2].

Definition 3. We say that $f : X \to \overline{\mathbb{R}}$ is **c-convex on X** if

$$\forall x \in X, f(x) = \sup\{g(x) : g \text{ is an affine minorant of } f\}.$$

The set of all c-convex functions defined on X will be denoted by $\Gamma(X)$.

Definition 4. We define **the c-convex hull** of f as being the function

$$\text{cc}(f) : X \to \overline{\mathbb{R}}, \text{cc}(f)(x) = \sup\{g(x) : g \in \Gamma(X),\ g \leq f\}, \forall x \in X.$$

We have the following characterization for the c-convex hull of a function.

Lemma 1. ([2]) *For $f : X \to \overline{\mathbb{R}}$ we have*

$$\text{cc}(f) = \sup\{g : g \text{ affine, } g \leq f\}.$$

Definition 5. For $M \subseteq X$ we define **the c-convex hull** of M by

$$\text{cc}(M) = \bigcap_{(x^{\#}, \alpha) \in (X^{\#} \setminus \{0\}) \times \mathbb{R}} \left\{ H^{\leq}(x^{\#}, \alpha) : M \subseteq H^{\leq}(x^{\#}, \alpha) \right\},$$

where $H^{\leq}(x^{\#}, \alpha) = \{x \in X : x^{\#}(x) \leq \alpha\}$ is a so-called **c-half-space** ([2]). We say that M is **c-convex** if and only if $M = \text{cc}(M)$.

Theorem 1. ([2]) *Let $f : X \to \overline{\mathbb{R}}$ be such that $\{g : g \text{ affine, } g \leq f\} \neq \emptyset$. Then*

(a) $\text{epi}(\text{cc}(f)) = \text{cc}\left(\text{epi}(f)\right)$,

(b) $f \in \Gamma(X) \Leftrightarrow \text{epi}(f) \subseteq X \times \mathbb{R}$ *is c-convex.*

Theorem 2. ([2]) *Let A be a subset of X. Then*

$$\delta_A \in \Gamma(X), \text{ i.e. } \delta_A \text{ is c-convex, if and only if } A \text{ is c-convex.}$$

3 Conjugate functions

This section is devoted to the theory of conjugate functions. We introduce the conjugate of a function defined on a real linear space and also we establish the connection between this notion and the classical one, given in the framework of locally convex spaces.

Definition 6. For $f : X \to \overline{\mathbb{R}}$ we call

(a) $f^{\#} : X^{\#} \to \overline{\mathbb{R}}, f^{\#}(x^{\#}) = \sup\limits_{x \in X}[x^{\#}(x) - f(x)], \forall x^{\#} \in X^{\#}$, the **conjugate function** of f,

(b) $f^{\#\#} : X \to \overline{\mathbb{R}}, f^{\#\#}(x) = \sup\limits_{x^{\#} \in X^{\#}} [x^{\#}(x) - f^{\#}(x^{\#})], \forall x \in X$, the **biconjugate function** of f.

Remark 1. (a) For a function $g : X^{\#} \to \overline{\mathbb{R}}$, the conjugate and the biconjugate of g are analogously defined as follows

$$g^{\#} : X \to \overline{\mathbb{R}}, g^{\#}(x) = \sup_{x^{\#} \in X^{\#}} [x^{\#}(x) - g(x^{\#})], \forall x \in X,$$

$$g^{\#\#} : X^{\#} \to \overline{\mathbb{R}}, g^{\#\#}(x^{\#}) = \sup_{x \in X}[x^{\#}(x) - g^{\#}(x)], \forall x^{\#} \in X^{\#}.$$

(b) For a function $f : X \to \overline{\mathbb{R}}$ defined on a locally convex space X, we also have the classical Fenchel conjugate of f, defined by

$$f^* : X^* \to \overline{\mathbb{R}}, f^*(x^*) = \sup_{x \in X}[x^*(x) - f(x)],$$

where X^* is the topological dual of X. We have that $X^* \subseteq X^{\#}$ and $f^{\#} |_{X^*} = f^*$. One can see that $f^{\#\#}(x) = \sup\limits_{x^{\#} \in X^{\#}} [x^{\#}(x) - f^{\#}(x^{\#})] \geq \sup\limits_{x^* \in X^*} [x^*(x) - f^*(x^*)] = f^{**}(x), \forall x \in X$, so $f^{\#\#} \geq f^{**}$. In finite dimensional spaces, $f^{\#}$ and f^* are identical, as in this case $X^{\#} = X^*$.

Proposition 1. *Let the function $f : X \to \overline{\mathbb{R}}$ be given. Then*

(a) $f^{\#}(x^{\#}) + f(x) \geq x^{\#}(x), \forall x \in X, \forall x^{\#} \in X^{\#}$ (Young-Fenchel inequality);

(b) $f^{\#} \in \Gamma(X^{\#})$;

(c) $f \le g \Rightarrow g^{\#} \le f^{\#}$*;*

(d) $f^{\#} = \big(\mathrm{cc}(f)\big)^{\#}$.

Proof. We prove just (b) and (d) since (a) and (c) are trivial consequences of Definition 6.

(b) If f is not proper, one can see that $f^{\#} \equiv +\infty$ or $f^{\#} \equiv -\infty$, and these are c-convex functions (see Definition 3). If f is proper then $f^{\#}(x^{\#}) = \sup_{x\in \mathrm{dom}(f)}[x^{\#}(x) - f(x)]$, $\forall x^{\#} \in X^{\#}$. We have to show that

$$f^{\#}(x^{\#}) = \sup\big\{g(x^{\#}) : g \text{ affine}, \, g \le f^{\#}\big\}.$$

Let be $x_0^{\#} \in X^{\#}$. Obviously,

$$\sup\big\{g(x_0^{\#}) : g \text{ affine}, \, g \le f^{\#}\big\} \le f^{\#}(x_0^{\#}).$$

If we suppose that in the relation above the inequality is strict, then one can find a real number r such that

$$\sup\big\{g(x_0^{\#}) : g \text{ affine}, \, g \le f^{\#}\big\} < r < f^{\#}(x_0^{\#}).$$

This means that $g(x_0^{\#}) < r, \forall g$ affine, $g \le f^{\#}$ and $\exists x_0 \in \mathrm{dom}(f)$ such that

$$x_0^{\#}(x_0) - f(x_0) > r.$$

If we define $h : X^{\#} \to \mathbb{R}, h(x^{\#}) := x^{\#}(x_0) - f(x_0)$, then (by (a)), $h(x^{\#}) \le f^{\#}(x^{\#}), \forall x^{\#} \in X^{\#}$, that is h is an affine minorant of $f^{\#}$, so

$$r < h(x_0^{\#}) < r,$$

which is a contradiction.

(d) Using (c), we get $f^{\#} \le \big(\mathrm{cc}(f)\big)^{\#}$, as $\mathrm{cc}(f) \le f$ (see Proposition 1(a) in [2]). We prove now the opposite inequality. Let $x^{\#} \in X^{\#}$. If $f^{\#}(x^{\#}) = +\infty$, then $\big(\mathrm{cc}(f)\big)^{\#}(x^{\#}) = f^{\#}(x^{\#}) = +\infty$. If not, there exists $\alpha \in \mathbb{R}$ such that $f^{\#}(x^{\#}) \le \alpha$. Then $x^{\#}(x) - f(x) \le \alpha, \forall x \in X$, that is $x^{\#}(x) - \alpha \le f(x), \forall x \in X$. We obtain by Lemma 1 that $x^{\#}(x) - \alpha \le \mathrm{cc}(f)(x), \forall x \in X$, so $x^{\#}(x) - \mathrm{cc}(f)(x) \le \alpha, \forall x \in X$, implying that $\big(\mathrm{cc}(f)\big)^{\#}(x^{\#}) \le \alpha$. But this is true for every $\alpha \ge f^{\#}(x^{\#})$. By allowing α converging to $f^{\#}(x^{\#})$, we conclude that $\big(\mathrm{cc}(f)\big)^{\#}(x^{\#}) \le f^{\#}(x^{\#})$. As $x^{\#}$ was arbitrary, the equality follows. ∎

The following theorem is extending a well-known result in locally convex spaces (see also [11]).

Theorem 3. *For $f : X \to \overline{\mathbb{R}}$ we have $f^{\#\#} = \mathrm{cc}(f)$.*

Proof. Let $x \in X$ be arbitrary. It holds

$$f^{\#\#}(x) = \sup_{x^\# \in X^\#} \{x^\#(x) - f^\#(x^\#)\} = \sup_{x^\# \in \mathrm{dom}(f^\#)} \{x^\#(x) - f^\#(x^\#)\}$$

$$= \sup\{x^\#(x) + \alpha : x^\# \in \mathrm{dom}(f^\#), \alpha \in \mathbb{R}, \alpha \leq -f^\#(x^\#)\}$$

$$= \sup\left\{x^\#(x) + \alpha : x^\# \in \mathrm{dom}(f^\#), \alpha \in \mathbb{R}, \alpha \leq \inf_{y \in X}\{f(y) - x^\#(y)\}\right\}$$

$$= \sup\{x^\#(x) + \alpha : x^\# \in \mathrm{dom}(f^\#), \alpha \in \mathbb{R}, \alpha + x^\#(y) \leq f(y), \forall y \in X\}$$

$$= \sup\{x^\#(x) + \alpha : x^\# \in X^\#, \alpha \in \mathbb{R}, \alpha + x^\#(y) \leq f(y), \forall y \in X\},$$

and this delivers the desired result, as the last term in this sequence of equalities is exactly $\mathrm{cc}(f)(x)$ (see Lemma 1). ∎

By the above theorem and Proposition 1(a) in [2], we get $f(x) \geq f^{\#\#}(x)$, $\forall x \in X$. Moreover, if X is a locally convex space then we have the following sequence of inequalities

$$f(x) \geq f^{\#\#}(x) \geq f^{**}(x), \forall x \in X.$$

If f is proper, convex and lower semi-continuous, then $f = f^{**}$ (see [6]), so in this case we obtain that $f = f^{\#\#} = f^{**}$.

The following two results follow from Theorem 3.

Corollary 1. *A function $f : X \to \overline{\mathbb{R}}$ is c-convex if and only if $f^{\#\#} = f$.*

Proof. This is a direct consequence of the above theorem and Proposition 1(b) in [2].

$$f \in \Gamma(X) \Leftrightarrow f = \mathrm{cc}(f) \Leftrightarrow f = f^{\#\#}.$$

∎

Corollary 2. *For a function $f : X \to \overline{\mathbb{R}}$ we have $f^{\#\#\#} = f^{\#}$.*

Proof. We apply Theorem 3 and Proposition 1(d) to obtain

$$f^{\#\#\#} = (f^{\#\#})^{\#} = \big(\mathrm{cc}(f)\big)^{\#} = f^{\#}.$$

$\square$

Remark 2. It is easy to prove that if f is proper and c-convex, then $f^{\#}$ is also proper.

In a locally convex space X, a proper function f is convex and lower semi-continuous if and only if $f^{**} = f$, where X^* is endowed with the weak* topology $\omega(X^*, X)$ (see [6]). In the following we give an example of a function f for which $f^{**} \neq f$ but $f^{\#\#} = f$.

Example 1. ([2]) Consider X an infinite dimensional normed space and let $\{e_i : i \in I\}$ be a vector basis of it. We may suppose that $\mathbb{N} \subseteq I$. Obviously, $\{\big(1/\|e_i\|\big)e_i : i \in I\}$ is again a vector basis, so without lose of generality we may suppose that $\|e_i\| = 1, \forall i \in I$. Define $f_0 : \{e_i : i \in I\} \to \mathbb{R}$,

$$f_0(e_i) = \begin{cases} i, & \text{if } i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is well known from the linear algebra that f_0 can be extended uniquely to a linear function on X, let us call it $x_0^{\#}$. We claim that $x_0^{\#} \in X^{\#} \setminus X^*$. Indeed, if we suppose that $x_0^{\#}$ is continuous, then $\exists L \geq 0$ s.t. $|x_0^{\#}(x)| \leq L\|x\|, \forall x \in X$. But this implies, for $x = e_i, i \in \mathbb{N}$, that $i \leq L, \forall i \in \mathbb{N}$, which is a contradiction. Now consider the following set

$$M := \ker(x_0^{\#}) = \{x \in X : x_0^{\#}(x) = 0\}.$$

Since

$$M = \{x \in X : x_0^{\#}(x) \leq 0\} \bigcap \{x \in X : -x_0^{\#}(x) \leq 0\},$$

M is c-convex (see Lemma 3 in [2])). Let be $x_n = e_1 - (1/n)e_n, \forall n \in \mathbb{N}$. It is easy to see that $x_n \in M, \forall n \in \mathbb{N}$. Because of $\|x_n - e_1\| = 1/n, \forall n \in \mathbb{N}$, we get that the limit of the sequence $\{x_n\}$ is e_1, but this element does not belong to M, so M is a c-convex set which is not topologically closed.

Taking the indicator function of M, we have, by Theorem 2, that δ_M is c-convex, so in view of Corollary 1, $\delta_M^{\#\#} = \delta_M$. Because M is not closed, δ_M

is not lower semi-continuous, hence $\delta_M^{**} \neq \delta_M$.

We close this section giving a result concerning the epigraph of the conjugate of the sum of two functions. The proof is similar to the one given in [3] for locally convex spaces.

Proposition 2. *Let $f, g : X \to \overline{\mathbb{R}}$ be proper functions which satisfy the condition* $\mathrm{dom}(f) \bigcap \mathrm{dom}(g) \neq \emptyset$. *The following statements are equivalent*

(i) $\mathrm{epi}\left((f+g)^{\#}\right) = \mathrm{epi}(f^{\#}) + \mathrm{epi}(g^{\#})$,

(ii) $(f+g)^{\#} = f^{\#} \square g^{\#}$ *and* $f^{\#} \square g^{\#}$ *is exact at every* $x^{\#} \in X^{\#}$.

4 Moreau-Rockafellar-type theorems

Using some results given in [2], we show in this section that in real linear spaces we have some similar results with the one which exist in locally convex spaces.

Theorem 4. (see Theorem 2.4 in [3]) *Let X and Y be real linear spaces, $A : X \to Y$ a linear mapping and $g : Y \to \overline{\mathbb{R}}$ a proper function. Then*

$$\mathrm{cc}\left(\mathrm{epi}(A^{\#} g^{\#})\right) = \mathrm{cc}\left(A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))\right).$$

Proof. First, let be $(x^{\#}, r) \in A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))$. Then there exists $y^{\#} \in Y^{\#}$ s.t. $A^{\#} y^{\#} = x^{\#}$ and $(y^{\#}, r) \in \mathrm{epi}(g^{\#})$. It follows

$$A^{\#} g^{\#}(x^{\#}) = \inf\{g^{\#}(y^{\#}) : A^{\#} y^{\#} = x^{\#}\} \leq r,$$

thus $(x^{\#}, r) \in \mathrm{epi}(A^{\#} g^{\#})$. So the inclusion $A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#})) \subseteq \mathrm{epi}(A^{\#} g^{\#})$ is true. We show that $\mathrm{epi}(A^{\#} g^{\#}) \subseteq \mathrm{cc}\left(A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))\right)$ and this will lead us to the desired result. Let $(x^{\#}, r) \in \mathrm{epi}(A^{\#} g^{\#})$. Let H be a c-half-space such that $A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#})) \subseteq H$. Take an arbitrary $\varepsilon > 0$. We have $A^{\#} g^{\#}(x^{\#}) \leq r < r + \varepsilon \Leftrightarrow \inf\{g^{\#}(y^{\#}) : A^{\#} y^{\#} = x^{\#}\} < r + \varepsilon \Leftrightarrow \exists y_{\varepsilon}^{\#}$ such that $A^{\#} y_{\varepsilon}^{\#} = x^{\#}, g^{\#}(y_{\varepsilon}^{\#}) < r + \varepsilon \Leftrightarrow (x^{\#}, r + \varepsilon) \in A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#})) \subseteq H$. Thus

$$(x^{\#}, r + \varepsilon) \in H, \forall \varepsilon > 0. \tag{1}$$

H cannot be a lower half-space (see the discussion on the half-spaces in $X \times \mathbb{R}$ in [2], Section 4). Thus H must be a vertical or an upper half-space. Using (1) and Lemma 5 from [2] we get $(x^{\#}, r) \in H$. But H was arbitrary, so we conclude that $(x^{\#}, r) \in \mathrm{cc}\left(A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))\right)$. □

Theorem 5. (see Theorem 2.7 in [8]) *Let X and Y be real linear spaces, $A : X \to Y$ a linear mapping and $g \in \Gamma(Y)$ a proper function such that $g \circ A$ is proper on X. Then*

$$\mathrm{epi}((g \circ A)^{\#}) = \mathrm{cc}\left(\mathrm{epi}(A^{\#} g^{\#})\right).$$

Proof. We prove first the following equality

$$g \circ A = (A^{\#} g^{\#})^{\#}. \tag{2}$$

We have $A^{\#} g^{\#} : X^{\#} \to \overline{\mathbb{R}}$. Then

$$\begin{aligned}
(A^{\#} g^{\#})^{\#}(x) &= \sup_{x^{\#} \in X^{\#}} [\langle x^{\#}, x \rangle - (A^{\#} g^{\#})(x^{\#})] \\
&= \sup_{x^{\#} \in X^{\#}} [\langle x^{\#}, x \rangle - \inf_{A^{\#} y^{\#} = x^{\#}} g^{\#}(y^{\#})] \\
&= \sup_{\substack{(x^{\#}, y^{\#}) \in X^{\#} \times Y^{\#} \\ A^{\#} y^{\#} = x^{\#}}} [\langle x^{\#}, x \rangle - g^{\#}(y^{\#})] \\
&= \sup_{y^{\#} \in Y^{\#}} [\langle A^{\#} y^{\#}, x \rangle - g^{\#}(y^{\#})] \\
&= \sup_{y^{\#} \in Y^{\#}} [\langle y^{\#}, Ax \rangle - g^{\#}(y^{\#})] = g^{\#\#}(Ax) = g(Ax), \forall x \in X,
\end{aligned}$$

where the last equality is given by Corollary 1. Using (2) and Theorem 3 we obtain $(g \circ A)^{\#} = (A^{\#} g^{\#})^{\#\#} = \mathrm{cc}(A^{\#} g^{\#})$. This implies that

$$\mathrm{epi}((g \circ A)^{\#}) = \mathrm{epi}(\mathrm{cc}(A^{\#} g^{\#})). \tag{3}$$

Next we show that the function $A^{\#} g^{\#}$ has at least one affine minorant. Because $g \circ A$ is proper, there exists $x_0 \in X$ such that $g(Ax_0) \in \mathbb{R}$. Using the Young-Fenchel inequality, we get $(A^{\#} g^{\#})(x^{\#}) = \inf_{A^{\#} y^{\#} = x^{\#}} [g^{\#}(y^{\#})] \geq \inf_{A^{\#} y^{\#} = x^{\#}} [\langle y^{\#}, Ax_0 \rangle - g(Ax_0)] = \inf_{A^{\#} y^{\#} = x^{\#}} [\langle A^{\#} y^{\#}, x_0 \rangle - g(Ax_0)] = \langle x^{\#}, x_0 \rangle - g(Ax_0), \forall x^{\#} \in X^{\#}$. If we define $h : X^{\#} \to \mathbb{R}$ by

$$h(x^{\#}) = \langle x^{\#}, x_0 \rangle - g(Ax_0),$$

then h is an affine minorant of $A^{\#}g^{\#}$. By Theorem 1(a) follows that

$$\text{epi}(\text{cc}(A^{\#}g^{\#})) = \text{cc}(\text{epi}(A^{\#}g^{\#}))$$

and (3) leads to the desired conclusion. $\square$

Combining the last two theorems we have the following result.

Theorem 6. *Let X and Y be real linear spaces, $A : X \to Y$ a linear mapping and $g \in \Gamma(Y)$ a proper function such that $g \circ A$ is proper on X. Then*

$$\text{epi}((g \circ A)^{\#}) = \text{cc}\left(\text{epi}(A^{\#}g^{\#})\right) = \text{cc}\left(A^{\#} \times id_{\mathbb{R}}(\text{epi}(g^{\#}))\right).$$

The next theorem is similar to the one established by Rockafellar and Moreau in locally convex spaces (see [14], [16]). It gives a characterization of the epigraph of the conjugate of the sum of two functions.

Theorem 7. (Moreau-Rockafellar) *Let $f, h \in \Gamma(X)$ be proper functions such that* $\text{dom}(f) \bigcap \text{dom}(h) \neq \emptyset$. *Then*

$$\text{epi}((f+h)^{\#}) = \text{cc}(\text{epi}(f^{\#}\Box h^{\#})) = \text{cc}(\text{epi}(f^{\#}) + \text{epi}(h^{\#})).$$

Proof. We obtain this result applying Theorem 6 for the particular case $Y = X \times X, g : X \times X \to \overline{\mathbb{R}}, g(x,y) = f(x) + h(y)$ and $A : X \to X \times X, Ax = (x,x)$. The adjoint operator of A is $A^{\#} : X^{\#} \times X^{\#} \to X^{\#}, A^{\#}(p,q) = p + q$. To show that g is also c-convex, we compute its biconjugate. We have

$$g^{\#}(p,q) = \sup_{(x,y) \in X \times X} \{p(x) + q(y) - f(x) - h(y)\} =$$

$$\sup_{x \in X}\{p(x) - f(x)\} + \sup_{y \in X}\{q(y) - h(y)\} = f^{\#}(p) + h^{\#}(q).$$

In a similar way, by Corollary 1 we have

$$g^{\#\#}(x,y) = f^{\#\#}(x) + h^{\#\#}(y) = f(x) + h(y) = g(x,y), \forall (x,y) \in X \times X.$$

So $g^{\#\#} = g$, hence g is c-convex, in view of Corollary 1. $\square$

5 Fenchel duality

Let be X and Y two real linear spaces, $A : X \to Y$ a linear mapping, $f \in \Gamma(X), g \in \Gamma(Y)$ proper functions such that $A(\mathrm{dom}(f)) \bigcap \mathrm{dom}(g) \neq \emptyset$. We consider the following convex optimization problem

$$(P_A) \inf_{x \in X} \{f(x) + g(Ax)\}.$$

In this section we give a regularity condition which ensures strong duality between (P_A) and its Fenchel dual problem and we show that it is weaker than other conditions given so far in the literature. Also we give some results related to subdifferential calculus.

Obviously, the function $g \circ A : X \to \overline{\mathbb{R}}$ is also c-convex. Indeed, by the proof of Theorem 5 we already know that $g \circ A = (A^{\#} g^{\#})^{\#}$, hence

$$(g \circ A)^{\#\#} = (A^{\#} g^{\#})^{\#\#\#}.$$

Now we use Corollary 2 to conclude that $(g \circ A)^{\#\#} = (A^{\#} g^{\#})^{\#} = g \circ A$, thus, by Corollary 1, $g \circ A$ is c-convex. So we have (cf. Theorem 5 and Theorem 7)

$$\begin{aligned} \mathrm{epi}((f + g \circ A)^{\#}) &= \mathrm{cc}\left(\mathrm{epi}(f^{\#}) + \mathrm{epi}((g \circ A)^{\#})\right) \\ &= \mathrm{cc}\Big(\mathrm{epi}(f^{\#}) + \mathrm{cc}\left(A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))\right)\Big), \end{aligned}$$

which is nothing else than (because $\mathrm{cc}(E + \mathrm{cc}(F)) = \mathrm{cc}(E + F)$, see Proposition 2 in [2])

$$\mathrm{epi}((f + g \circ A)^{\#}) = \mathrm{cc}(\mathrm{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))).$$

We introduce the following regularity condition

$$(RC_A) : \ \ \mathrm{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#})) \text{ is c-convex in } X^{\#} \times \mathbb{R}.$$

It is easy to see that (RC_A) is equivalent to

$$\mathrm{epi}((f + g \circ A)^{\#}) = \mathrm{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#})). \tag{4}$$

Theorem 8. *Let X and Y be real linear spaces, $A : X \to Y$ a linear mapping, $f \in \Gamma(X), g \in \Gamma(Y)$ proper functions such that $A(\mathrm{dom}(f)) \bigcap \mathrm{dom}(g) \neq \emptyset$. Then (RC_A) is fulfilled if and only if $\forall x^{\#} \in X^{\#}$,*

$$(f + g \circ A)^{\#}(x^{\#}) = \inf\{f^{\#}(x^{\#} - A^{\#} y^{\#}) + g^{\#}(y^{\#}) : y^{\#} \in Y^{\#}\}$$

and the infimum is attained.

Proof. " $\Rightarrow$ " Let $x^{\#} \in X^{\#}$. For all $x \in X$ and $y^{\#} \in Y^{\#}$ we have (by the Young-Fenchel inequality)

$$f^{\#}(x^{\#} - A^{\#}y^{\#}) + g^{\#}(y^{\#}) \geq \langle x^{\#} - A^{\#}y^{\#}, x\rangle - f(x) + \langle y^{\#}, Ax\rangle - g(Ax)$$
$$= \langle x^{\#}, x\rangle - f(x) - g(Ax)$$

and therefore

$$\inf\{f^{\#}(x^{\#} - A^{\#}y^{\#}) + g^{\#}(y^{\#}) : y^{\#} \in Y^{\#}\} \geq (f + g \circ A)^{\#}(x^{\#}) \qquad (5)$$

If $(f+g\circ A)^{\#}(x^{\#}) = +\infty$, then the conclusion follows. If $(f+g\circ A)^{\#}(x^{\#}) < +\infty$, we have $(x^{\#}, (f + g \circ A)^{\#}(x^{\#})) \in \text{epi}((f + g \circ A)^{\#})$. The regularity condition (RC_A) being fulfilled, there exist $(u^{\#}, r) \in \text{epi}(f^{\#})$ and $(v^{\#}, s) \in A^{\#} \times id_{\mathbb{R}}(\text{epi}(g^{\#}))$ such that $x^{\#} = u^{\#} + v^{\#}$ and $(f + g \circ A)^{\#}(x^{\#}) = r + s$. So there exists $y^{\#} \in Y^{\#}$ such that $A^{\#}y^{\#} = v^{\#}$ and $g^{\#}(y^{\#}) \leq s$, which implies

$$f^{\#}(x^{\#} - A^{\#}y^{\#}) + g^{\#}(y^{\#}) = f^{\#}(u^{\#}) + g^{\#}(y^{\#})$$
$$\leq r + s = (f + g \circ A)^{\#}(x^{\#}).$$

This delivers the desired result.
" $\Leftarrow$ " It is enough to prove that the equality in (4) is fulfilled. Let $(u^{\#}, r) \in \text{epi}(f^{\#})$ and $(v^{\#}, s) \in A^{\#} \times id_{\mathbb{R}}(\text{epi}(g^{\#}))$. Hence there exists $y^{\#} \in Y^{\#}$ such that $A^{\#}y^{\#} = v^{\#}$ and $g^{\#}(y^{\#}) \leq s$. By (5) we have

$$(f + g \circ A)^{\#}(u^{\#} + v^{\#}) \leq f^{\#}(u^{\#}) + g^{\#}(y^{\#}) \leq r + s,$$

so $(u^{\#} + v^{\#}, r + s) \in \text{epi}((f + g \circ A)^{\#})$. Thus the inclusion

$$\text{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\text{epi}(g^{\#})) \subseteq \text{epi}((f + g \circ A)^{\#})$$

is always satisfied. For the opposite inclusion, let $(x^{\#}, r) \in \text{epi}((f+g\circ A)^{\#})$. Then $(f+g\circ A)^{\#}(x^{\#}) \leq r \Rightarrow \exists y^{\#} \in Y^{\#}$ such that $f^{\#}(x^{\#}-A^{\#}y^{\#})+g^{\#}(y^{\#}) \leq r$. The element $(x^{\#}, r)$ can be written in the following way

$$(x^{\#}, r) = (x^{\#} - A^{\#}y^{\#}, f^{\#}(x^{\#} - A^{\#}y^{\#})) + (A^{\#}y^{\#}, r - f^{\#}(x^{\#} - A^{\#}y^{\#})),$$

which belongs to $\text{epi}(f^{\#})+A^{\#}\times id_{\mathbb{R}}(\text{epi}(g^{\#}))$ and so the conclusion follows.$\square$

In a similar way like for locally convex spaces (see [3]), by using Theorem 8, one can prove the following formula for the subdifferential of the sum of a convex function with the precomposition of another convex function with a linear mapping.

Theorem 9. *Let X and Y be real linear spaces, $A : X \to Y$ a linear mapping, $f \in \Gamma(X), g \in \Gamma(Y)$ proper functions such that $A(\text{dom}(f)) \bigcap \text{dom}(g) \neq \emptyset$. If (RC_A) is fulfilled, then $\forall x \in \text{dom}(f) \cap A^{-1}(\text{dom}(g))$,*

$$\partial(f + g \circ A)(x) = \partial f(x) + A^{\#}\partial g(Ax).$$

Consider now the Fenchel dual problem to (P_A)

$$(D_A) \sup_{y^{\#}\in Y^{\#}} \big\{ - f^{\#}(-A^{\#}y^{\#}) - g^{\#}(y^{\#})\big\}.$$

Let us denote by $v(P_A)$ and $v(D_A)$ the optimal objective values of (P_A) and (D_A), respectively. It is easy to show, using the properties of the conjugate functions, that weak duality holds, that is $v(D_A) \leq v(P_A)$. Because $v(P_A) = -(f + g \circ A)^{\#}(0)$, then, in view of Theorem 8 (taking $x^{\#} = 0$), it can be proved that if (RC_A) is fulfilled then we have strong duality, that is $v(P_A) = v(D_A)$ and (D_A) has an optimal solution. Now we give a weaker constraint qualification under which strong duality holds. Let this be defined by

$$\begin{aligned}(FRC_A): \quad & f^{\#}\Box A^{\#}g^{\#} \in \Gamma(X^{\#}) \text{ and } \text{epi}(f^{\#}\Box A^{\#}g^{\#}) \bigcap(\{0\} \times \mathbb{R}) \\ & = (\text{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\text{epi}(g^{\#}))) \bigcap(\{0\} \times \mathbb{R}).\end{aligned}$$

Theorem 10. *If (FRC_A) is fulfilled, then $v(P_A) = v(D_A)$ and (D_A) has an optimal solution.*

Proof. Taking in (5) $x^{\#} = 0$, we have

$$v(P_A) = -(f + g \circ A)^{\#}(0) \geq v(D_A) \geq -f^{\#}(-A^{\#}y^{\#}) - g^{\#}(y^{\#}), \forall y^{\#} \in Y^{\#}.$$

If $v(P_A) = -\infty$, then the conclusion follows. Assume that $v(P_A) > -\infty$. Using Theorems 5 and 7, as well as Proposition 2 in [2], we obtain

$$\text{epi}((f + g \circ A)^{\#}) = \text{cc}\left(\text{epi}(f^{\#}) + \text{epi}((g \circ A)^{\#})\right)$$

$$= \mathrm{cc}\left(\mathrm{epi}(f^{\#}) + \mathrm{cc}(\mathrm{epi}(A^{\#}g^{\#}))\right) = \mathrm{cc}(\mathrm{epi}(f^{\#}) + \mathrm{epi}(A^{\#}g^{\#}))$$
$$= \mathrm{cc}(\mathrm{epi}(f^{\#}\Box A^{\#}g^{\#})),$$

which is nothing else than (see Theorem 1(a)) $(f + g \circ A)^{\#} = \mathrm{cc}(f^{\#}\Box A^{\#}g^{\#})$ (we can use the same argument as in the proof of Theorem 5 to show that the function $f^{\#}\Box A^{\#}g^{\#}$ has at least one affine minorant). The regularity condition (FRC_A) being fulfilled, we have actually that $(f + g \circ A)^{\#} = f^{\#}\Box A^{\#}g^{\#}$ and, because of $-v(P_A) = (f + g \circ A)^{\#}(0)$, we get

$$(0, -v(P_A)) \in (\mathrm{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))) \bigcap (\{0\} \times \mathbb{R}).$$

Hence, there exist $(u^{\#}, r) \in \mathrm{epi}(f^{\#})$ and $(v^{\#}, s) \in A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))$ such that $u^{\#} + v^{\#} = 0$ and $r + s = -v(P_A)$. Further there exists $y^{\#} \in Y^{\#}$ such that $A^{\#}y^{\#} = v^{\#}$ and $g^{\#}(y^{\#}) \leq s$. Thus $u^{\#} = -A^{\#}y^{\#}$ and

$$v(P_A) = -r - s \leq -f^{\#}(u^{\#}) - g^{\#}(y^{\#}) = -f^{\#}(-A^{\#}y^{\#}) - g^{\#}(y^{\#}) \leq v(D_A),$$

which delivers the desired conclusion. □

Remark 3. We prove that (RC_A) implies (FRC_A). Indeed, if (RC_A) is fulfilled, then $\mathrm{epi}((f + g \circ A)^{\#}) = \mathrm{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#}))$ (cf. (4)). As we have seen in the proof of Theorem 10, the following relations hold (see also the proof of Theorem 4)

$$\mathrm{epi}((f + g \circ A)^{\#}) = \mathrm{cc}(\mathrm{epi}(f^{\#}\Box A^{\#}g^{\#})) \supseteq \mathrm{epi}(f^{\#}\Box A^{\#}g^{\#})$$
$$\supseteq \mathrm{epi}(f^{\#}) + \mathrm{epi}(A^{\#}g^{\#}) \supseteq \mathrm{epi}(f^{\#}) + A^{\#} \times id_{\mathbb{R}}(\mathrm{epi}(g^{\#})).$$

Because of (RC_A), for all these inclusions equality holds. Thus the set $\mathrm{epi}(f^{\#}\Box A^{\#}g^{\#})$ is c-convex and now use Theorem 1(b) to conclude that (FRC_A) is true. Boţ and Wanka gave in [3] an example of an optimization problem in $\mathbb{R}^2$ for which (FRC_A) is fulfilled, but (RC_A) not. Obviously, this example applies also for the regularity conditions introduced in this paper.

We treat now some particular cases of the above results. In case $X = Y$ and $A = id_X$, the identity mapping of X, the problems (P_A) and (D_A) become

$$(P) \inf_{x \in X} \{f(x) + g(x)\}$$

and

$$(D) \sup_{x^\# \in X^\#} \big\{ - f^\#(-x^\#) - g^\#(x^\#) \big\},$$

respectively. The functions $A^\# g^\#$ and $A^\# \times id_{\mathbb{R}}$ will be nothing else than $g^\#$ and the identity mapping of $X^\# \times \mathbb{R}$, respectively. The relation (4) and the regularity conditions (RC_A) and (FRC_A) turn out to be

$$\text{epi}\left((f+g)^\#\right) = \text{epi}(f^\#) + \text{epi}(g^\#), \tag{6}$$

$$(RC) : \text{epi}(f^\#) + \text{epi}(g^\#) \text{ is c-convex in } X^\# \times \mathbb{R}$$

and, respectively,

$$\begin{aligned}(FRC): \quad & f^\# \Box g^\# \in \Gamma(X^\#) \text{ and } \text{epi}(f^\# \Box g^\#) \bigcap (\{0\} \times \mathbb{R}) \\ & = (\text{epi}(f^\#) + \text{epi}(g^\#)) \bigcap (\{0\} \times \mathbb{R}).\end{aligned}$$

The latter is nothing else than

$$(FRC): \ f^\# \Box g^\# \text{ is a c-convex function and is exact at 0.}$$

From Theorems 8, 9 and 10 we get the following corollaries.

Corollary 3. *Let X and Y be real linear spaces, $f \in \Gamma(X), g \in \Gamma(Y)$ proper functions such that* $\text{dom}(f) \bigcap \text{dom}(g) \neq \emptyset$. *Then*

(i) (RC) *is fulfilled if and only if* $\forall x^\# \in X^\#$,

$$(f+g)^\#(x^\#) = \inf\{f^\#(x^\# - y^\#) + g^\#(y^\#) : y^\# \in Y^\#\}$$

and the infimum is attained.

(ii) If (RC) *is fulfilled, then* $\forall x \in \text{dom}(f) \cap \text{dom}(g)$,

$$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

Corollary 4. *If* (FRC) *is fulfilled, then* $v(P) = v(D)$ *and* (D) *has an optimal solution.*

Next we show that (RC), and in particular (FRC) too, is implied by some other conditions given in the past in the literature (in the framework of real linear spaces), for having strong duality between the problems (P) and (D).

Let us recall some well-known notions (see, for example, [18]). For a subset $D \subseteq X$ the *core* (or the *algebraic interior*) of D is defined by

$$\text{core}(D) = \{d \in D : \forall x \in X, \exists \varepsilon > 0 \text{ s.t. } \forall \lambda \in [-\varepsilon, \varepsilon], d + \lambda x \in D\}.$$

The core of D relative to $\text{aff}(D - D)$ is called the *intrinsec core* (or the *relative algebraic interior*) of D and is denoted by $\text{icr}(D)$

$$\text{icr}(D) = \{d \in D : \forall x \in \text{aff}(D - D), \exists \varepsilon > 0 \text{ s.t. } \forall \lambda \in [-\varepsilon, \varepsilon], d + \lambda x \in D\}.$$

The following facts are well known: $\text{core}(D) \subseteq \text{icr}(D) \subseteq D$ and if D is convex then

$$x \in \text{core}(D) \Leftrightarrow \bigcup_{\lambda > 0} \lambda(D - x) = X.$$

Consider the following two constraint qualifications

(i) $\text{icr}(\text{dom}(f)) \bigcap \text{icr}(\text{dom}(g)) \neq \emptyset$,

(ii) $0 \in \text{core}(\text{dom}(f) - \text{dom}(g))$.

Let us suppose that one of the above conditions is satisfied. As each of the above conditions ensures strong duality between the problems (P) and (D) (for (i) see [7] and for (ii), [10]), one has

$$(f + g)^{\#}(0) = \sup_{x \in X}\{-f(x) - g(x)\} = -\inf_{x \in X}\{f(x) + g(x)\}$$

$$= -\sup_{y^{\#} \in X^{\#}}\{-f^{\#}(-y^{\#}) - g^{\#}(y^{\#})\} = \inf_{y^{\#} \in X^{\#}}\{f^{\#}(-y^{\#}) + g^{\#}(y^{\#})\}$$

and this infimum is attained. Define the function $h : X \to \overline{\mathbb{R}}$ by $h := g - x^{\#}$, where $x^{\#}$ is an arbitrary element from $X^{\#}$. Because $\text{dom}(h) = \text{dom}(g)$, the above result holds for the functions f and h too, that is

$$(f + h)^{\#}(0) = \inf_{y^{\#} \in X^{\#}}\{f^{\#}(-y^{\#}) + h^{\#}(y^{\#})\}$$

and this infimum is attained. One can easy see that $(f + h)^{\#}(0) = (f + g)^{\#}(x^{\#})$ and $h^{\#}(y^{\#}) = g^{\#}(x^{\#} + y^{\#})$. So we get

$$\begin{aligned}(f + g)^{\#}(x^{\#}) &= \inf\{f^{\#}(-y^{\#}) + g^{\#}(x^{\#} + y^{\#}) : y^{\#} \in X^{\#}\} \\ &= \inf\{f^{\#}(x^{\#} - y^{\#}) + g^{\#}(y^{\#}) : y^{\#} \in Y^{\#}\}\end{aligned}$$

and the infimum is attained. By Corollary 3(i), this is nothing else than (RC) must be fulfilled. Therefore, both the conditions (i) and (ii) imply (RC).

In the following, we prove that (RC) is indeed weaker than (i) and (ii). Let us consider the following two functions $f := \delta_{H^{\leq}(x_0^{\#},0)}$, $g := \delta_{H^{\geq}(x_0^{\#},0)}$, where $H^{\leq}(x_0^{\#},0) = \{x \in X : x_0^{\#}(x) \leq 0\}$, $H^{\geq}(x_0^{\#},0) = \{x \in X : x_0^{\#}(x) \geq 0\}$ and $x_0^{\#}$ belongs to $X^{\#} \setminus \{0\}$. The sets $H^{\leq}(x_0^{\#},0)$ and $H^{\geq}(x_0^{\#},0)$ are c-convex (see Lemma 3 in [2]), so in view of Theorem 2, the functions f and g are c-convex. We show that (RC) is fulfilled. Let us compute the conjugates of these functions. To this end, we need the following result from linear algebra.

Lemma 2. *Let X be a real linear space and $x_1^{\#}, x_2^{\#} \in X^{\#}$. Then*

$$\ker(x_1^{\#}) \subseteq \ker(x_2^{\#}) \text{ if and only if } \exists \alpha \in \mathbb{R} \text{ such that } x_2^{\#} = \alpha x_1^{\#},$$

where by $\ker(x^{\#}) = \{x \in X : x^{\#}(x) = 0\}$ we denote the kernel of $x^{\#} \in X^{\#}$.

We have $f + g = \delta_{H(x_0^{\#},0)}$, where $H(x_0^{\#},0) = \{x \in X : x_0^{\#}(x) = 0\} = \ker(x_0^{\#})$. So $(f+g)^{\#}(x^{\#}) = \sup\limits_{x_0^{\#}(x)=0} x^{\#}(x)$. We show that $(f+g)^{\#} = \delta_{\mathbb{R}x_0^{\#}}$. If $x^{\#} = \alpha x_0^{\#}$, with $\alpha \in \mathbb{R}$, then $(f+g)^{\#}(x^{\#}) = 0$. Let $x^{\#} \in X^{\#} \setminus (\mathbb{R}x_0^{\#})$. We claim that

$$\exists x_0 \in \ker(x_0^{\#}) \text{ such that } x^{\#}(x_0) \neq 0. \tag{7}$$

If not, then $\forall x \in \ker(x_0^{\#}), x^{\#}(x) = 0$ that is $\ker(x_0^{\#}) \subseteq \ker(x^{\#})$ implying by Lemma 2 that $x^{\#} \in \mathbb{R}x_0^{\#}$, which is a contradiction. Hence (7) is true. We may suppose that $x^{\#}(x_0) > 0$. Then $(f+g)^{\#}(x^{\#}) \geq x^{\#}(nx_0) = nx^{\#}(x_0), \forall n \in \mathbb{N}$, thus $(f+g)^{\#}(x^{\#}) = +\infty$.

For the conjugate of f we have $f^{\#}(x^{\#}) = \sup\limits_{x_0^{\#}(x)\leq 0} x^{\#}(x)$. We prove that $f^{\#} = \delta_{\mathbb{R}_+x_0^{\#}}$. For $x^{\#} \in \mathbb{R}_+x_0^{\#}$ we have $f^{\#}(x^{\#}) = 0$. If $x^{\#} \in \mathbb{R}_-x_0^{\#}$, then it is easy to see that $f^{\#}(x^{\#}) = +\infty$ (use the fact that $x_0^{\#} \neq 0$) and if $x^{\#} \in X^{\#} \setminus (\mathbb{R}x_0^{\#})$, we have $f^{\#}(x^{\#}) \geq \sup\limits_{x_0^{\#}(x)=0} x^{\#}(x) = +\infty$. In a similar way one can prove that $g^{\#} = \delta_{\mathbb{R}_-x_0^{\#}}$. We have

$$\begin{aligned} \operatorname{epi}(f^{\#}) + \operatorname{epi}(g^{\#}) &= \mathbb{R}_+x_0^{\#} \times [0,\infty) + \mathbb{R}_-x_0^{\#} \times [0,\infty) \\ &= \mathbb{R}x_0^{\#} \times [0,\infty) = \operatorname{epi}(f+g)^{\#}, \end{aligned}$$

implying by Theorem 7 that $\mathrm{epi}(f^{\#}) + \mathrm{epi}(g^{\#})$ is c-convex, that is (RC) is fulfilled.

Next we show that the conditions (i) and (ii) fail for these two functions. For (i), we assume that

$$\exists x_0 \in \mathrm{icr}(\mathrm{dom}(f)) \cap \mathrm{icr}(\mathrm{dom}(g))$$

$$= \mathrm{icr}\left(H^{\leq}(x_0^{\#}, 0)\right) \cap \mathrm{icr}\left(H^{\geq}(x_0^{\#}, 0)\right)$$

$$\subseteq H^{\leq}(x_0^{\#}, 0) \cap H^{\geq}(x_0^{\#}, 0) \subseteq H(x_0^{\#}, 0),$$

hence $x_0^{\#}(x_0) = 0$. As $x_0^{\#} \neq 0$, there exists $x_1 \in X$ such that $x_0^{\#}(x_1) \neq 0$. We may suppose that $x_0^{\#}(x_1) < 0$. We have $x_1 = x_1 - 0 \in H^{\leq}(x_0^{\#}, 0) - H^{\leq}(x_0^{\#}, 0) \subseteq \mathrm{aff}(H^{\leq}(x_0^{\#}, 0) - H^{\leq}(x_0^{\#}, 0))$. But $x_0 \in \mathrm{icr}(H^{\leq}(x_0^{\#}, 0))$, so $\exists \varepsilon_1 > 0$ such that $\forall \lambda \in [-\varepsilon_1, \varepsilon_1], x_0 + \lambda x_1 \in H^{\leq}(x_0^{\#}, 0)$, that is $\lambda x_0^{\#}(x_1) \leq 0$. The same x_1 can be written as $x_1 = 0 - (-x_1) \in H^{\geq}(x_0^{\#}, 0) - H^{\geq}(x_0^{\#}, 0) \subseteq \mathrm{aff}(H^{\geq}(x_0^{\#}, 0) - H^{\geq}(x_0^{\#}, 0))$. We obtain that $\exists \varepsilon_2 > 0$ such that $\forall \lambda \in [-\varepsilon_2, \varepsilon_2], x_0 + \lambda x_1 \in H^{\geq}(x_0^{\#}, 0)$, thus $\lambda x_0^{\#}(x_1) \geq 0$. Taking $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ and $\lambda := \varepsilon/2$ we get $x_0^{\#}(x_1) \leq 0$ and $x_0^{\#}(x_1) \geq 0$, hence $x_0^{\#}(x_1) = 0$, which is a contradiction.

Suppose now that (ii) is fulfilled. One can see that

$$\mathrm{dom}(f) - \mathrm{dom}(g) = H^{\leq}(x_0^{\#}, 0) - H^{\geq}(x_0^{\#}, 0) = H^{\leq}(x_0^{\#}, 0),$$

so condition (ii) is equivalent to $0 \in \mathrm{core}\left(H^{\leq}(x_0^{\#}, 0)\right)$, which is nothing else than $\bigcup\limits_{\lambda > 0} \lambda(H^{\leq}(x_0^{\#}, 0)) = X$. As $x_0^{\#} \neq 0$, there exists $x_2 \in X$ such that $x_0^{\#}(x_2) > 0$. For this x_2 we can find $\lambda_1 > 0$ and $x' \in H^{\leq}(x_0^{\#}, 0)$ fulfilling $x_2 = \lambda_1 x'$. Then $0 < x_0^{\#}(x_2) = \lambda_1 x_0^{\#}(x') \leq 0$, which is a contradiction.

We close the paper by considering the following optimization problem

$$(P_0) \inf_{x \in G} f(x) = \inf_{x \in X} [f(x) + \delta_G(x)],$$

where G is a non-empty c-convex subset of X. The Fenchel dual problem of (P_0) becomes

$$(D_0) \sup_{x^{\#} \in X^{\#}} \left\{ -f^{\#}(x^{\#}) - \delta_G^{\#}(-x^{\#}) \right\}$$

or, equivalently

$$(D_0) \sup_{x^{\#} \in X^{\#}} \left\{ -f^{\#}(x^{\#}) + \inf_{x \in G} \langle x^{\#}, x \rangle \right\}.$$

We can give the following strong duality result.

Corollary 5. (see Theorem 4.4 in [4]) *Assume that* $f \in \Gamma(X)$ *is proper such that* $\mathrm{dom}(f) \cap G \neq \emptyset$. *If*

$$f^{\#} \Box \delta_G^{\#} \in \Gamma(X^{\#}) \text{ and is exact at 0,}$$

then $v(P_0) = v(D_0)$ *and* (D_0) *has an optimal solution.*

Proof. By Theorem 2, the function δ_G is c-convex. By Corollary 4 the result follows. □

Remark 4. If we consider $g_i : X \to \overline{\mathbb{R}}, i \in I$, a family of c-convex functions (in particular g_i can be affine functions, see Lemma 1 in [2]), then the set G in the above Corollary can be taken as $G = \{x \in X : g_i(x) \leq a_i, \forall i \in I\}$, where $a_i, i \in I$ are real numbers. This is a c-convex set, see Lemma 7 in [2]. The results given in this paper can be extended to optimization problems with inequality constraints. This can be the issue of future research.

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