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Conjugate Duality in Vector Optimization and Some Applications to the Vector Variational Inequality

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Abstract: The aim of this paper is to investigate an approach dealing with the construction of gap functions for the vector variational inequality with regard to conjugate duality in vector optimization. In order to introduce new gap functions for the vector variational inequality, we consider some special perturbation functions. To verify the properties of a gap function, duality results are used.

Key words: conjugate duality, conjugate map, vector optimization, perturbation function, vector variational inequality, gap function

AMS subject classification: 49N15, 54C60, 58E35, 90C29

1 Introduction

The conjugate duality for vector optimization problems has been investigated by many authors. Especially, in [17] (see also [15]) Tanino and Sawaragi developed the conjugate duality by introducing new concepts of conjugate maps and set-valued subgradients based on Pareto efficiency. Furthermore, by using the concept of supremum of a set (cf. [18]) on the basis of weak orderings, the conjugate duality theory has been extended to a partially ordered topological vector space (see [19]) and to set-valued vector optimization problems (see [16]), respecively.

In the case of scalar optimization the construction of a gap function for variational inequalities has been associated to Lagrange duality (see [8]). By applying the duality results for scalar optimization problems introduced in [21], different gap functions for variational inequalities have been proposed (see [2]).

Since the vector variational inequality in a finite-dimensional space was introduced first in [7], several papers concerning the relations between vector optimization and vector variational inequalities have been published (see for instance, [9], [13] and [14]).

In this paper we consider the extension of the approach dealing with the construction of gap functions from variational inequalities to the vector variational inequality by using the conjugate duality in vector optimization.

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This paper is organized as follows. In Section 2 we recall the definition of the maximum of a set in a finite-dimensional Euclidean space and some of its properties. Also, the concepts of conjugate maps, set-valued subgradients and the duality results in vector optimization are discussed. Section 3 is devoted to the presentation of perturbation functions associated to the conjugate duality in vector optimization. Similar functions have been considered in [21] for the scalar case. They allow us to propose different dual vector optimization problems having set-valued objective maps. In addition, duality assertions for these problems are obtained. Considering the conjugate maps with vector variables in Section 4 we discuss further dual problems with vector variables. These can be seen as special cases of the results in Section 3. Finally, the dual problems introduced in Section 3 and in Section 4 allow us to define some new gap functions for the vector variational inequality. In order to prove the properties in the definition of a gap function, the duality assertions discussed in Section 3 and in Section 4 are used.

2 Mathematical preliminaries

Let C be a pointed closed and convex cone in $\mathbb{R}^n$. For any $\xi, \mu \in \mathbb{R}^n$, we use the following ordering relations:

$$\begin{aligned} \xi \underset{C}{\leq} \mu &\Leftrightarrow \mu - \xi \in C; \\ \xi \underset{C\backslash\{0\}}{\leq} \mu &\Leftrightarrow \mu - \xi \in C\backslash\{0\}; \\ \xi \underset{C\backslash\{0\}}{\nleq} \mu &\Leftrightarrow \mu - \xi \notin C\backslash\{0\}. \end{aligned}$$

The notions $\underset{C}{\geq}$, $\underset{C\backslash\{0\}}{\geq}$ and $\underset{C\backslash\{0\}}{\ngeq}$ are used in an alternative way.

Definition 2.1 *A point $y \in \mathbb{R}^n$ is said to be a maximal point of a set $Y \subseteq \mathbb{R}^n$ if $y \in Y$ and there is no $y' \in Y$ such that $y \underset{C\backslash\{0\}}{\leq} y'$.*

The set of all maximal points of Y is called the maximum of Y and is denoted by $\max\limits_{C\backslash\{0\}} Y$. The minimum of Y is defined analogously. Further we take the cone C being the nonnegative orthant

$$\mathbb{R}^n_+ = \left\{ x = (x_1, ..., x_n)^T \in \mathbb{R}^n |\ x_i \geq 0,\ i = \overline{1, n} \right\}.$$

Lemma 2.1 *[15, cf. Proposition 3.1.3] Let $Y_1, Y_2 \subseteq \mathbb{R}^n$. Then*

(i) $\max\limits_{\mathbb{R}^n_+\backslash\{0\}} (Y_1 + Y_2) \subseteq \max\limits_{\mathbb{R}^n_+\backslash\{0\}} Y_1 + \max\limits_{\mathbb{R}^n_+\backslash\{0\}} Y_2$;

(ii) $\min\limits_{\mathbb{R}^n_+\backslash\{0\}} (Y_1 + Y_2) \subseteq \min\limits_{\mathbb{R}^n_+\backslash\{0\}} Y_1 + \min\limits_{\mathbb{R}^n_+\backslash\{0\}} Y_2$.

Definition 2.2 *[10, cf. Definition 8.2.2]*

(i) Let $Y \subseteq \mathbb{R}^n$ be a given set. The set $\min\limits_{\mathbb{R}^n_+\backslash\{0\}} Y$ is said to be externally stable if

$$Y \subseteq \min_{\mathbb{R}^n_+\backslash\{0\}} Y + \mathbb{R}^n_+.$$

(ii) Similarly, the set $\max_{\mathbb{R}^n_+\setminus\{0\}} Y$ *is said to be externally stable if*

$$Y \subseteq \max_{\mathbb{R}^n_+\setminus\{0\}} Y - \mathbb{R}^n_+.$$

Lemma 2.2 *[15, Lemma 6.1.1] Let* $F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *and* $F_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be set-valued maps and* $X \subseteq \mathbb{R}^n$*. Then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big[F_1(x) + F_2(x)\Big] \subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big[F_1(x) + \max_{\mathbb{R}^p_+\setminus\{0\}} F_2(x)\Big].$$

If $\max_{\mathbb{R}^p_+\setminus\{0\}} F_2(x)$ *is externally stable for every* $x \in X$*, then the converse inclusion also holds.*

Corollary 2.1 *[15, Corollary 6.1.3] Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be a set-valued map and* $X \subseteq \mathbb{R}^n$*. If* $\max_{\mathbb{R}^p_+\setminus\{0\}} F(x)$ *is externally stable for every* $x \in X$*, then*

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} F(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \max_{\mathbb{R}^p_+\setminus\{0\}} F(x).$$

Before describing the conjugate duality for vector optimization, let us recall the concepts of conjugate maps and the set-valued subgradient.

Definition 2.3 *[10, Definition 8.2.1]*
Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ *be a set-valued map.*

(i) The set-valued map $h^* : \mathbb{R}^{p\times n} \rightrightarrows \mathbb{R}^p$ *defined by*

$$h^*(U) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} \Big[Ux - h(x)\Big],\ U \in \mathbb{R}^{p\times n}$$

is called the conjugate map of h*.*

(ii) The conjugate map of h^**,* h^{**} *is called the biconjugate map of* h*, i.e.*

$$h^{**}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{U\in\mathbb{R}^{p\times n}} \Big[Ux - h^*(U)\Big],\ x \in \mathbb{R}^n.$$

(iii) U *is said to be a subgradient of the set-valued map* h *at* $(\bar{x};\bar{y})$ *if* $\bar{y} \in h(\bar{x})$ *and*

$$\bar{y} - U\bar{x} \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} [h(x) - Ux].$$

The set of all subgradients of h at $(x;y)$ is denoted by $\partial h(x;y)$ and is called the *subdifferential of* h *at* $(x;y)$. If $\partial h(x;y) \neq \varnothing$, $\forall y \in h(x)$, then h is said to be *subdifferentiable at* x.

When $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ is a vector-valued function, then the conjugate map φ^* of φ is defined by

$$\varphi^*(T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{Tx - \varphi(x)|\ x \in \mathbb{R}^n\Big\},\ T \in \mathbb{R}^{p\times n}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\}$ be an extended vector-valued function. Here $+\infty$ is the imaginary point whose every component is $+\infty$. We consider the following unconstrained vector optimization problem

$$(P_u) \qquad \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ f(x) | \ x \in \mathbb{R}^n \Big\}.$$

In other words, (P_u) is the problem of finding $\bar{x} \in \mathbb{R}^n$ such that

$$f(x) \underset{\mathbb{R}^p_+\backslash\{0\}}{\not\leq} f(\bar{x}), \ \forall x \in \mathbb{R}^n.$$

Let $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\}$ be another vector-valued function such that

$$\Phi(x, 0) = f(x), \ \forall x \in \mathbb{R}^n,$$

which is the so-called perturbation function. *The value function* is a set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^p \cup \{+\infty\}$ defined by

$$\Psi(y) = \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ \Phi(x, y) | \ x \in \mathbb{R}^n \Big\}.$$

Clearly $\Psi(0) = \min_{\mathbb{R}^p_+\backslash\{0\}} f(\mathbb{R}^n)$ is the minimal frontier of the problem (P_u). The problem (P_u) can be stated as the primal optimization problem

$$(P_u) \qquad \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ \Phi(x, 0) | \ x \in \mathbb{R}^n \Big\}.$$

The conjugate map of Φ, denoted by $\Phi^* : \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m} \rightrightarrows \mathbb{R}^p \cup \{+\infty\}$, is a set-valued map defined in the usual manner:

$$\Phi^*(U, V) = \max_{\mathbb{R}^p_+\backslash\{0\}} \Big\{ Ux + Vy - \Phi(x, y) | \ x \in \mathbb{R}^n, \ y \in \mathbb{R}^m \Big\}.$$

Then the conjugate dual optimization problem can be defined as being

$$(D_u) \qquad \max_{\mathbb{R}^p_+\backslash\{0\}} \bigcup_{V \in \mathbb{R}^{p\times m}} \Big[-\Phi^*(0, V) \Big].$$

Since $-\Phi^*$ is a set-valued map, the problem (D_u) is not an ordinary vector optimization problem. In other words, it can be reformulated as follows.

Find $V^* \in \mathbb{R}^{p\times m}$ such that

$$-\Phi^*(0, V^*) \cap \max_{\mathbb{R}^p_+\backslash\{0\}} \bigcup_{V \in \mathbb{R}^{p\times m}} \Big[-\Phi^*(0, V) \Big] \neq \varnothing.$$

Theorem 2.1 *[15, Proposition 6.1.12] (Weak duality)*

$$\Phi(x, 0) \notin -\Phi^*(0, V) - \mathbb{R}^p_+\backslash\{0\}, \ \forall x \in \mathbb{R}^n, \ \forall \ V \in \mathbb{R}^{p\times m}.$$

Definition 2.4 *The primal problem (P_u) is said to be stable with respect to the perturbation function Φ if the value function Ψ is subdifferentiable at $y = 0$.*

Theorem 2.2 *[15, Theorem 6.1.1] (Strong duality)*

(i) The primal problem (P_u) is stable with respect to Φ if and only if for each solution x^* to the primal problem (P_u) there exists a solution V^* to the dual problem (D_u) such that

$$\Phi(x^*, 0) \in -\Phi^*(0, V^*). \tag{2.1}$$

(ii) Conversely, if $x^* \in \mathbb{R}^n$ and $V^* \in \mathbb{R}^{p\times m}$ satisfy (2.1), then x^* is a solution to (P_u) and V^* is a solution to (D_u).

3 Conjugate duality for the constrained vector optimization problem

In this section some special perturbation functions investigated for scalar optimization in [21] are applied to the constrained vector optimization problem. As a consequence, we obtain different dual problems having set-valued objective maps. In analogy to the scalar case, let us call them the Lagrange, the Fenchel and the Fenchel-Lagrange dual problem, respectively. Let $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions and $X \subseteq \mathbb{R}^n$. Consider the vector optimization problem

$$(VO) \qquad \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f(x) |\ x \in G \Big\},$$

where

$$G = \Big\{ x \in X |\ g(x) \leqq_{\mathbb{R}^m_+} 0 \Big\}.$$

Let us introduce now the following perturbation functions (cf. [4] and [21])

$$\Phi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},\ \ \Phi_1(x,u) = \begin{cases} f(x), & x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise}; \end{cases}$$

$$\Phi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\},\ \ \Phi_2(x,v) = \begin{cases} f(x+v), & x \in G, \\ +\infty, & \text{otherwise}; \end{cases}$$

$$\Phi_3 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},$$

$$\Phi_3(x,v,u) = \begin{cases} f(x+v), & x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise}. \end{cases}$$

Then the corresponding value functions can be written as follows.

$$\begin{aligned} \Psi_1 : \mathbb{R}^m \rightrightarrows \mathbb{R}^p,\ \Psi_1(u) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \Phi_1(x,u) |\ x \in \mathbb{R}^n \Big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f(x) |\ x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u \Big\}; \\ \Psi_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^p,\ \Psi_2(v) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \Phi_2(x,v) |\ x \in \mathbb{R}^n \Big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f(x+v) |\ x \in G \Big\}; \\ \Psi_3 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p,\ \Psi_3(v,u) &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \Phi_3(x,v,u) |\ x \in \mathbb{R}^n \Big\} \\ &= \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f(x+v) |\ x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u \Big\}. \end{aligned}$$

(ii) If $\forall v \in \mathbb{R}^n$ the set $\Psi_2(v)$ is externally stable, then the problem (VO) is stable with respect to Φ_2.

(iii) If $\forall (v,u) \in \mathbb{R}^n \times \mathbb{R}^m$ the set $\Psi_3(v,u)$ is externally stable and there exists $x_0 \in X$ such that $-g(x_0) \in int\ \mathbb{R}^m_+$, then the problem (VO) is stable with respect to Φ_3.

Later for the applications we have to consider the vector optimization problem with linear objective function (cf. Section 5). Since the objective function is linear and not strictly convex, we can not apply the above stability criteria. But the following result deals with this case. Let $A \in \mathbb{R}^{p \times n}$. Consider the vector optimization problem

$$(P_A) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ Ax |\ x \in G \Big\}.$$

Before giving a stability criterion for (P_A) with respect to Φ_2, let us mention the following trivial properties.

Remark 3.1 Let $h : \mathbb{R}^n \to \mathbb{R}^p$ be a vector-valued function and $Z \subseteq \mathbb{R}^n$. The following assertions are true:

(i) $\{h(x)|\ x \in Z\} = \bigcup\limits_{x \in Z} \{h(x)\}$.

(ii) For any $t \in \mathbb{R}^p$ it holds $\{h(x) + t|\ x \in Z\} = \{h(x)|\ x \in Z\} + t$.

(iii) For any set $A \subseteq \mathbb{R}^p$ it holds $\bigcup\limits_{x \in Z} \Big\{ A + h(x) \Big\} = A + \bigcup\limits_{x \in Z} \{h(x)\}$.

For the problem (P_A) we can state the following assertion.

Proposition 3.2 *Let the set $\min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\}$ be externally stable. Then the problem (P_A) is stable with respect to Φ_2.*

Proof: Let $f(x) = Ax,\ A \in \mathbb{R}^{p \times n}$. Then, in view of Remark 3.1, one has

$$\begin{aligned} -\Psi_2^*(T) &= \min_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{v \in \mathbb{R}^n} \Big[\min_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax + Av|\ x \in G\} - Tv \Big] \\ &= \min_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{v \in \mathbb{R}^n} \Big[Av - Tv + \min_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\} \Big] \\ &= \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big[\{(A - T)v | v \in \mathbb{R}^n\} + \min_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\} \Big]. \end{aligned}$$

As the set $\min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\}$ is externally stable, for $T = A$ one has (cf. Corollary 2.1)

$$-\Psi_2^*(A) = \min_{\mathbb{R}^p_+ \setminus \{0\}} \min_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\} = \min_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\}.$$

In other words, $\forall z \in \min\limits_{\mathbb{R}^p_+ \setminus \{0\}} \{Ax|\ x \in G\}$, it holds $z \in -\Psi_2^*(A)$. This means that $\partial \Psi_2(0; z) \neq \emptyset$. □

Lagrange duality. In the remainder of this section we obtain different dual problems associated to the mentioned perturbation functions. First we show how to construct the dual problem to (VO) relative to the perturbation function Φ_1. Let us prove now the following preliminary result.

Proposition 3.3 *Let* $\Lambda \in \mathbb{R}^{p\times m}$. *Then*

(i) $\Phi_1^*(0,\Lambda) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{\Lambda u|\ u \in \mathbb{R}^m_+\} + \{\Lambda g(x) - f(x)|\ x \in X\}\Big\}.$

(ii) If the set $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u|\ u \in \mathbb{R}^m_+\}$ *is externally stable, then it holds*

$$\Phi_1^*(0,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \max_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u|\ u \in \mathbb{R}^m_+\} + \{\Lambda g(x) - f(x)|\ x \in X\}\Big\}.$$

Proof:

(i) Let $\Lambda \in \mathbb{R}^{p\times m}$. Taking into account Remark 3.1

$$\begin{aligned}
\Phi_1^*(0,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Lambda u - \Phi_1(x,u)|\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\Lambda u - f(x)|\ x \in X,\ g(x) \leq_{\mathbb{R}^m_+} u\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda u - f(x)|\ g(x) \leq_{\mathbb{R}^m_+} u\Big\}.
\end{aligned}$$

Setting $\bar{u} := u - g(x)$, we have

$$\begin{aligned}
\Phi_1^*(0,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) - f(x) + \Lambda\bar{u}|\ \bar{u} \in \mathbb{R}^m_+\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) - f(x) + \{\Lambda\bar{u}|\ \bar{u} \in \mathbb{R}^m_+\}\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{\Lambda u|\ u \in \mathbb{R}^m_+\} + \{\Lambda g(x) - f(x)|\ x \in X\}\Big\}.
\end{aligned}$$

(ii) Follows from Lemma 2.2. □

According to Proposition 3.3, we can propose the following dual problem to (VO)

$$\begin{aligned}
(D_L^{VO}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \Big[-\Phi_1^*(0,\Lambda)\Big] \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{-\Lambda u|\ u \in \mathbb{R}^m_+\} + \{f(x) - \Lambda g(x)|\ x \in X\}\Big\}.
\end{aligned}$$

This dual problem may be considered as a kind of Lagrange-type dual problem. This interpretation appears evident and natural in the context of the following derivation of the classical Lagrange dual problem to (VO) (cf. [15]).
As applications of Theorem 2.1 and Theorem 2.2 we get weak and strong duality results for (VO) and (D_L^{VO}).

Proposition 3.4 *(weak duality)*

$$f(x) + \xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0,\ \forall x \in G,\ \forall \xi \in \Phi_1^*(0,\Lambda),$$

where $\Lambda \in \mathbb{R}^{p\times m}$.

Proposition 3.5 *(Strong duality)*

(i) (VO) *is stable with respect to* Φ_1 *if and only if for each solution* x^* *to* (VO) *there exists a solution* Λ^* *to* (D_L^{VO}) *such that*

$$f(x^*) \in -\Phi_1^*(0, \Lambda^*). \tag{3.1}$$

(ii) Conversely, if $x^* \in G$ *and* $\Lambda^* \in \mathbb{R}^{p\times m}$ *satisfy* (3.1)*, then* x^* *is a solution to* (VO) *and* Λ^* *is a solution to* (D_L^{VO}).

Under the external stability condition of the set $\max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda q |\ q \in \mathbb{R}^m_+\}$, for the dual problem with the objective map defined by Proposition 3.3(ii) we can obtain similar results.

Before considering the next perturbation function, let us, as announced, explain how the problem (D_L^{VO}) turns out to be the classical Lagrange dual problem (cf. [15]) under a certain restriction on the feasible set of the dual. To do this, we assume that

$$\Lambda \in L := \left\{\Lambda \in \mathbb{R}^{p\times m} |\ \Lambda u \underset{\mathbb{R}^p_+}{\geq} 0,\ \forall u \in \mathbb{R}^m_+\right\} = \left\{\Lambda \in \mathbb{R}^{p\times m} |\ \Lambda \mathbb{R}^m_+ \subseteq \mathbb{R}^p_+\right\}.$$

Then we conclude immediately that

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u |\ u \in \mathbb{R}^m_+\} = \{0\},\ \forall \Lambda \in L. \tag{3.2}$$

Because of $\Lambda \in L$, by using (3.2), from Lemma 2.1(i) follows

$$\begin{aligned}
\Phi_1^*(0, -\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \{-\Lambda u |\ u \in \mathbb{R}^m_+\} + \{-\Lambda g(x) - f(x) |\ x \in X\} \Big\} \\
&\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda u |\ u \in \mathbb{R}^m_+\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x) |\ x \in X\} \\
&= -\min_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda u |\ u \in \mathbb{R}^m_+\} + \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x) |\ x \in X\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x) |\ x \in X\}.
\end{aligned}$$

Denoting by $\widetilde{\Phi}(\Lambda) := \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{-\Lambda g(x) - f(x) |\ x \in X\}$, in this case we get the classical Lagrange dual problem to (VO), as follows

$$\begin{aligned}
(\widetilde{D}_L^{VO}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda \in L} \Big[-\widetilde{\Phi}(\Lambda) \Big] \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda \in L} \min_{\mathbb{R}^p_+\setminus\{0\}} \{\Lambda g(x) + f(x) |\ x \in X\}.
\end{aligned}$$

Proposition 3.6 *[15, Theorem 5.2.4] (Weak duality)*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0,\ \forall x \in G,\ \forall \xi \in \widetilde{\Phi}(\Lambda),$$

where $\Lambda \in L$.

Proposition 3.7 *[10, Theorem 8.3.3] (see also [15, Theorem 5.2.5(i)])*
Let $x^* \in G$, $\Lambda^* \in L$ such that $f(x^*) \in -\widetilde{\Phi}(\Lambda^*)$. Then $f(x^*)$ is simultaneously a minimal point to the primal problem (VO) and a maximal point to the dual problem $(\widetilde{D}_L^{VO})$.

Fenchel duality. The following result deals with the dual objective map with respect to the perturbation function Φ_2.

Proposition 3.8 *Let $T \in \mathbb{R}^{p\times n}$. Then*

(i) $\Phi_2^*(0,T) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{Tv - f(v)|\ v \in \mathbb{R}^n\} + \{-Tx|\ x \in G\}\Big\}.$

(ii) If the set $f^*(T) = \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - f(v)|\ v \in \mathbb{R}^n\}$ is externally stable, then it holds

$$\Phi_2^*(0,T) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{f^*(T) + \{-Tx|\ x \in G\}\Big\}.$$

Proof:

(i) Let $T \in \mathbb{R}^{p\times n}$. In view of Remark 3.1

$$\begin{aligned}\Phi_2^*(0,T) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - \Phi_2(x,v)|\ x \in \mathbb{R}^n,\ v \in \mathbb{R}^n\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \{Tv - f(x+v)|\ x \in G,\ v \in \mathbb{R}^n\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \{Tv - f(x+v)|\ v \in \mathbb{R}^n\}.\end{aligned}$$

Denoting $\bar{v} := x + v$, one gets

$$\begin{aligned}\Phi_2^*(0,T) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \{T\bar{v} - f(\bar{v}) - Tx|\ \bar{v} \in \mathbb{R}^n\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in G} \Big\{ - Tx + \{T\bar{v} - f(\bar{v})|\ \bar{v} \in \mathbb{R}^n\}\Big\}\\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{Tv - f(v)|\ v \in \mathbb{R}^n\} + \{-Tx|\ x \in G\}\Big\}.\end{aligned}$$

(ii) By using Lemma 2.2, we obtain (ii). □

As a consequence we state the following dual problem to (VO), which will be called the Fenchel dual problem

$$\begin{aligned}(D_F^{VO}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \Big[- \Phi_2^*(0,T)\Big]\\ = \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{f(v) - Tv|\ v \in \mathbb{R}^n\} + \{Tx|\ x \in G\}\Big\}.\end{aligned}$$

Again as consequences of the general theory we have weak and strong duality assertions.

Proposition 3.9 *(weak duality)*

$$f(x)+\xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0,\ \forall x\in G,\ \forall \xi\in \Phi_2^*(0,T)$$

where $T\in\mathbb{R}^{p\times n}$.

Proposition 3.10 *(Strong duality)*

(i) (VO) *is stable with respect to* Φ_2 *if and only if for each solution* x^* *to* (VO), *there exists a solution* T^* *to* (D_F^{VO}) *such that*

$$f(x^*)\in -\Phi_2^*(0,T^*). \tag{3.3}$$

(ii) Conversely, if $x^*\in G$ *and* $T^*\in\mathbb{R}^{p\times n}$ *satisfy (3.3), then* x^* *is a solution to* (VO) *and* T^* *is a solution to* (D_F^{VO}).

As mentioned before, under the external stability of the set $f^*(T)=\max_{\mathbb{R}^p_+\setminus\{0\}}\{Tv-f(v)|\ v\in\mathbb{R}^n\}$, for the dual problem with the objective map defined by Proposition 3.8(ii) we can also show similar dual assertions.

Fenchel-Lagrange duality. Now we consider the dual problem related to the perturbation function Φ_3.

Proposition 3.11 *Let* $\Lambda\in\mathbb{R}^{p\times m}$ *and* $T\in\mathbb{R}^{p\times n}$. *Then*

(i) $\Phi_3^*(0,T,\Lambda)=\max_{\mathbb{R}^p_+\setminus\{0\}}\Big\{\bigcup_{u\in\mathbb{R}^m_+}\{\Lambda u\}+\bigcup_{v\in\mathbb{R}^n}\{Tv-f(v)\}+\bigcup_{x\in X}\{\Lambda g(x)-Tx\}\Big\}.$

(ii) If the sets $\max_{\mathbb{R}^p_+\setminus\{0\}}\{\Lambda u|\ u\in\mathbb{R}^m_+\}$ *and* $f^*(T)$ *are externally stable, then it holds*

$$\Phi_3^*(0,T,\Lambda)=\max_{\mathbb{R}^p_+\setminus\{0\}}\Big\{\max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{u\in\mathbb{R}^m_+}\{\Lambda u\}+f^*(T)+\bigcup_{x\in X}\{\Lambda g(x)-Tx\}\Big\}.$$

Proof:

(i) Let $T\in\mathbb{R}^{p\times n}$ and $\Lambda\in\mathbb{R}^{p\times m}$. By applying Remark 3.1

$$\begin{aligned}
\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}}\Big\{Tv+\Lambda u-\Phi_3(x,v,u)|\ x\in\mathbb{R}^n,\ v\in\mathbb{R}^n,\ u\in\mathbb{R}^m\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\Big\{Tv+\Lambda u-f(x+v)|\ x\in X,\ v\in\mathbb{R}^n,\ g(x)\leq_{\mathbb{R}^m_+} u\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n}\Big\{Tv+\Lambda u-f(x+v)|\ g(x)\leq_{\mathbb{R}^m_+} u\Big\}.
\end{aligned}$$

Putting $\bar u:=u-g(x)$, one has

$$\begin{aligned}
\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n}\{Tv+\Lambda g(x)+\Lambda\bar u-f(x+v)|\ \bar u\in\mathbb{R}^m_+\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{x\in X}\bigcup_{v\in\mathbb{R}^n}\Big\{Tv+\Lambda g(x)-f(x+v)\\
&+ \{\Lambda\bar u|\ \bar u\in\mathbb{R}^m_+\}\Big\}\\
&= \max_{\mathbb{R}^p_+\setminus\{0\}}\bigcup_{x\in X}\Big\{\Lambda g(x)+\{\Lambda u|\ u\in\mathbb{R}^m_+\}\\
&+ \{Tv-f(x+v)|\ v\in\mathbb{R}^n\}\Big\}.
\end{aligned}$$

Setting $\bar{v} := x + v$, we obtain that

$$\begin{aligned}
\Phi_3^*(0,T,\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) + \{\Lambda u \mid u \in \mathbb{R}^m_+\} \\
&+ \{T\bar{v} - Tx - f(\bar{v}) \mid \bar{v} \in \mathbb{R}^n\}\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{\Lambda g(x) - Tx + \{\Lambda u \mid u \in \mathbb{R}^m_+\} \\
&+ \{T\bar{v} - f(\bar{v}) \mid \bar{v} \in \mathbb{R}^n\}\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{\Lambda u \mid u \in \mathbb{R}^m_+\} \\
&+ \{Tv - f(v) \mid v \in \mathbb{R}^n\} + \{\Lambda g(x) - Tx \mid x \in X\}\Big\}.
\end{aligned}$$

(ii) By Lemma 2.2, we can easy verify (ii). □

Consequently, we can formulate the following so-called Fenchel-Lagrange dual problem to (VO)

$$\begin{aligned}
(D^{VO}_{FL}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \Big[-\Phi_3^*(0,T,\Lambda)\Big] \\
= & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{f(v) - Tv \mid v \in \mathbb{R}^n\} \\
+ & \{-\Lambda u \mid u \in \mathbb{R}^m_+\} + \{Tx - \Lambda g(x) \mid x \in X\}\Big\}.
\end{aligned}$$

Proposition 3.12 *(weak duality)*

$$f(x) + \xi \not\leq_{\mathbb{R}^p_+\setminus\{0\}} 0,\ \forall x \in X,\ \forall \xi \in \Phi_3^*(0,T,\Lambda),$$

where $T \in \mathbb{R}^{p\times n}$ *and* $\Lambda \in \mathbb{R}^{p\times m}$.

Proposition 3.13 *(Strong duality)*

(i) (VO) *is stable with respect to* Φ_3 *if and only if for each solution* x^* *to* (VO) *there exists a solution* (T^*,Λ^*) *to* (D^{VO}_{FL}) *such that*

$$f(x^*) \in -\Phi_3^*(0,T^*,\Lambda^*). \tag{3.4}$$

(ii) Conversely, if $x^* \in X$ *and* $(T^*,\Lambda^*) \in \mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}$ *satisfy (3.4), then* x^* *is a solution to* (VO) *and* (T^*,Λ^*) *is a solution to* (D^{VO}_{FL}).

Similarly as for $(\widetilde{D}^{VO}_L)$, under the same restriction on Λ, we can introduce another dual problem. Indeed, let us suppose that $\Lambda \in L$. Then, according to Lemma 2.1(i) and (3.2), it holds

$$\begin{aligned}
\Phi_3^*(0,T,-\Lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{u\in\mathbb{R}^m_+} \{-\Lambda u\} + \bigcup_{v\in\mathbb{R}^n} \{Tv - f(v)\} \\
&+ \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\} \\
&\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{u\in\mathbb{R}^m_+} \{-\Lambda u\} \\
&+ \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\}.
\end{aligned}$$

Let us denote by $\widetilde{\Psi}(T,\Lambda) := \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup\limits_{v\in\mathbb{R}^n} \{Tv - f(v)\} + \bigcup\limits_{x\in X} \{-\Lambda g(x) - Tx\} \Big\}$. If the set $f^*(T)$ is externally stable, then $\widetilde{\Psi}(T,\Lambda)$ can be rewritten as

$$\widetilde{\Psi}(T,\Lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ f^*(T) + \bigcup_{x\in X} \{-\Lambda g(x) - Tx\} \Big\}.$$

The proposed map allows us to suggest the following dual problem

$$\begin{aligned}
(\widetilde{D}^{VO}_{FL}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L} \Big[-\widetilde{\Psi}(T,\Lambda) \Big] \\
= \; & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{f(v) - Tv\} + \bigcup_{x\in X} \{Tx + \Lambda g(x)\} \Big\}.
\end{aligned}$$

Proposition 3.14 *(weak duality)*

$$f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0, \; \forall x \in G, \; \forall \xi \in \widetilde{\Psi}(T,\Lambda),$$

where $T \in \mathbb{R}^{p\times n}$ *and* $\Lambda \in L$.

Proof: Let $(T,\Lambda) \in \mathbb{R}^{p\times n} \times L$ be fixed and $\xi \in \widetilde{\Psi}(T,\Lambda)$. In other words

$$\xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} Tv - f(v) + (-\Lambda g(x) - Tx), \; \forall v \in \mathbb{R}^n, \; \forall x \in X.$$

Choosing $v = x := \bar{x} \in G$, we obtain that

$$f(\bar{x}) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} -\Lambda g(\bar{x}).$$

On the other hand, since $\Lambda \in L$, $\bar{x} \in G$ it follows that $-\Lambda g(\bar{x}) \underset{\mathbb{R}^p_+}{\geq} 0$. Consequently, one has $f(x) + \xi \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} 0$. □

Proposition 3.15 *Let* $x^* \in G$, $(T^*,\Lambda^*) \in \mathbb{R}^{p\times n} \times L$ *such that* $f(x^*) \in -\widetilde{\Psi}(T^*,\Lambda^*)$. *Then* $f(x^*)$ *is simultaneously a minimal point to the primal problem* (VO) *and a maximal point to the dual problem* $(\widetilde{D}^{VO}_{FL})$.

Proof: Let $x^* \in G$, $(T^*, \Lambda^*) \in \mathbb{R}^{p\times n} \times L$ and $f(x^*) \in -\widetilde{\Psi}(T^*, \Lambda^*)$. The latter means

$$f(x^*) \in \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{v\in\mathbb{R}^n} \{f(v) - T^*v\} + \bigcup_{x\in X} \{T^*x + \Lambda^* g(x)\} \Big\}. \tag{3.5}$$

If $f(x^*)$ is not a minimal point to the primal problem (VO), then there exists $x \in G$ such that

$$f(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leq} f(x^*).$$

As mentioned before, since $\Lambda^* \in L$, $x \in G$ it holds $\Lambda^* g(x) \underset{\mathbb{R}^p_+}{\leq} 0$. Consequently, we have $f(x) + \Lambda^* g(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leq} f(x^*)$, or, equivalently,

$$f(x) - T^*x + T^*x + \Lambda^* g(x) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leq} f(x^*).$$

But

$$f(x) - T^*x + T^*x + \Lambda^* g(x) \in \bigcup_{v\in\mathbb{R}^n} \{f(v) - T^*v\} + \bigcup_{x\in X} \{T^*x + \Lambda^* g(x)\},$$

which is a contradiction to (3.5). Therefore $f(x^*)$ is a minimal point to the problem (VO). Moreover, if $f(x^*)$ is not a solution to $(\widetilde{D}^{VO}_{FL})$, then $\exists \widetilde{y} \in \bigcup\limits_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times L} \Big[-\widetilde{\Psi}(T, \Lambda)\Big]$ such that $f(x^*) \underset{\mathbb{R}^p_+\setminus\{0\}}{\leq} \widetilde{y}$. Let $(\widetilde{T}, \widetilde{\Lambda}) \in \mathbb{R}^{p\times n} \times L$ such that $\widetilde{y} \in -\widetilde{\Psi}(\widetilde{T}, \widetilde{\Lambda})$. From $\widetilde{\Lambda} g(x^*) \underset{\mathbb{R}^p_+}{\leq} 0$ follows

$$\widetilde{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\geq} f(x^*) + \widetilde{\Lambda} g(x^*) = f(x^*) - \widetilde{T}x^* + \widetilde{T}x^* + \widetilde{\Lambda} g(x^*),$$

which contradicts the fact that $\widetilde{y} \in -\widetilde{\Psi}(\widetilde{T}, \widetilde{\Lambda})$ in the same way as before. Accordingly, $f(x^*)$ is a solution to $(\widetilde{D}^{VO}_{FL})$. □

4 Special cases

This section aims to investigate some special cases of dual problems based on alternative definitions of the conjugate maps and the subgradient for a set-valued map having vector variables. In Definition 2.3, if we choose $U := [t, ..., t]^T \in \mathbb{R}^{p\times n}$ for $t \in \mathbb{R}^n$, as a variable of the conjugate maps, then this reduces to the definition consired in this section. Remark that duality results for vector optimization developed by Tanino and Sawaragi (see [15] and [17]) are essentially not distinguishable in both cases. The advantage of considering conjugate maps with vector variable consists in the fact that the corresponding dual problems have a more simple form than ones in Section 3 and they can be easily reduced to the duals for scalar optimization problems. Let us recall first the definitions of the conjugate maps with vector variables (cf. Definition 2.3).

Definition 4.1 *[10, Definition 7.2.3] (the type II Fenchel transform)*
Let $h : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ be a set-valued map.

(i) The set-valued map $h_p^*: \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ defined by

$$h_p^*(\lambda) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} \Big[(\lambda^T x)_p - h(x)\Big],\ \lambda \in \mathbb{R}^n$$

is called the (type II) conjugate map of h;

(ii) The conjugate map of h_p^*, h_p^{**} is called the biconjugate map of h, i.e.

$$h_p^{**}(x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda\in\mathbb{R}^n} \Big[(\lambda^T x)_p - h_p^*(\lambda)\Big],\ x \in \mathbb{R}^n;$$

(iii) $\lambda \in \mathbb{R}^n$ is said to be a subgradient of the set-valued map h at $(\bar{x};\bar{y})$ if $\bar{y} \in h(\bar{x})$ and

$$\bar{y} - (\lambda^T \bar{x})_p \in \min_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in\mathbb{R}^n} [h(x) - (\lambda^T x)_p],$$

where $(\lambda^T x)_p = (\lambda^T x, \ldots, \lambda^T x)^T \in \mathbb{R}^p$.

Like in Section 3, let $f: \mathbb{R}^n \to \mathbb{R}^p$, $g: \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued functions and $X \subseteq \mathbb{R}^n$. Based on the perturbation functions introduced in Section 3, let us suggest some dual problems having vector variables. For convenience, in this section we use the following notations.

$$\varphi_1: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},\ \ \varphi_1(x,u) = \begin{cases} f(x), & x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise}; \end{cases}$$

$$\varphi_2: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \cup \{+\infty\},\ \ \varphi_2(x,v) = \begin{cases} f(x+v), & x \in G, \\ +\infty, & \text{otherwise}; \end{cases}$$

$$\varphi_3: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \cup \{+\infty\},$$

$$\varphi_3(x,v,u) = \begin{cases} f(x+v), & x \in X,\ g(x) \leqq_{\mathbb{R}^m_+} u, \\ +\infty, & \text{otherwise}. \end{cases}$$

Let us notice that throughout this section instead of φ_{ip}^*, $i = 1,2,3$, we write φ_i^*, $i = 1,2,3$.

Lagrange duality. By using the dual objective map having a vector variable with respect to φ_1, the Lagrange dual problem to (VO) was introduced in [17]. Let us now explain how we obtain this dual.

Lemma 4.1 *Let $\lambda \in \mathbb{R}^m$. Then*

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T x)_p |\ x \in \mathbb{R}^m_+\} = \begin{cases} \{0\}, & \text{if } \lambda \geqq_{\mathbb{R}^m_+} 0; \\ \emptyset, & \text{otherwise}, \end{cases}$$

where $(\lambda^T x)_p = (\lambda^T x, \ldots, \lambda^T x)^T \in \mathbb{R}^p$.

Proof: Let $z \in \min_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T x)_p |\ x \in \mathbb{R}^m_+\}$. Then $\exists \bar{x} \in \mathbb{R}^m_+$ such that $z = (\lambda^T \bar{x})_p$ and it holds

$$(\lambda^T \bar{x})_p \not\geq_{\mathbb{R}^p_+\setminus\{0\}} (\lambda^T x)_p,\ \forall x \in \mathbb{R}^m_+,$$

or, equivalently,

$$\lambda^T \bar{x} \leq \lambda^T x, \ \forall x \in \mathbb{R}^m_+.$$

In other words, it holds $\lambda^T \bar{x} = \min_{x \in \mathbb{R}^n} \lambda^T x$. Since $\inf_{x \in \mathbb{R}^m_+} \lambda^T x = \begin{cases} 0, & \text{if } \lambda \underset{\mathbb{R}^m_+}{\geq} 0; \\ -\infty, & \text{otherwise,} \end{cases}$ we obtain the conclusion. □

Proposition 4.1 *Let* $\lambda \in \mathbb{R}^m$. *Then*

$$\varphi_1^*(0,\lambda) = \begin{cases} \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \left\{ (\lambda^T g(x))_p - f(x) |\ x \in X \right\}, & \textit{if } \lambda \underset{\mathbb{R}^m_+}{\leq} 0; \\ \varnothing, & \textit{otherwise.} \end{cases}$$

Proof: Let $\lambda \in \mathbb{R}^m$. Then by definition

$$\begin{aligned} \varphi_1^*(0,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ (\lambda^T u)_p - \varphi_1(x,u) |\ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \right\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ (\lambda^T u)_p - f(x) |\ x \in X, \ g(x) \underset{\mathbb{R}^m_+}{\leq} u \right\}. \end{aligned}$$

Setting $\bar{u} := u - g(x)$, we have

$$\begin{aligned} \varphi_1^*(0,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ (\lambda^T g(x))_p + (\lambda^T \bar{u})_p - f(x) |\ x \in X, \ \bar{u} \in \mathbb{R}^m_+ \right\} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \left\{ \{ (\lambda^T g(x))_p - f(x) |\ x \in X \} + \{ (\lambda^T \bar{u})_p |\ \bar{u} \in \mathbb{R}^m_+ \} \right\}. \end{aligned}$$

In view of Lemma 2.1(i) and Lemma 4.1, one has

$$\begin{aligned} \varphi_1^*(0,\lambda) &\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{ (\lambda^T g(x))_p - f(x) |\ x \in X \} + \max_{\mathbb{R}^p_+\setminus\{0\}} \{ (\lambda^T u)_p |\ u \in \mathbb{R}^m_+ \} \\ &= \max_{\mathbb{R}^p_+\setminus\{0\}} \{ (\lambda^T g(x))_p - f(x) |\ x \in X \} - \min_{\mathbb{R}^p_+\setminus\{0\}} \{ (-\lambda^T u)_p |\ u \in \mathbb{R}^m_+ \} \\ &= \begin{cases} \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{ (\lambda^T g(x))_p - f(x) |\ x \in X \}, & \text{if } \lambda \underset{\mathbb{R}^m_+}{\leq} 0; \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

For $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$ it remains to show that

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \{ (\lambda^T g(x))_p - f(x) |\ x \in X \} \subseteq \varphi_1^*(0,\lambda).$$

Let $\bar{y} \in \max\limits_{\mathbb{R}^p_+\setminus\{0\}} \{ (\lambda^T g(x))_p - f(x) |\ x \in X \}$. This means $\bar{y} \in \{ (\lambda^T g(x))_p - f(x) |\ x \in X \}$ and

$$\bar{y} \underset{\mathbb{R}^p_+\setminus\{0\}}{\not\leq} (\lambda^T g(x))_p - f(x), \ \forall x \in X. \tag{4.1}$$

Choosing $\bar{u} = 0$, we have

$$\bar{y} = \bar{y} + (\lambda^T \bar{u})_p \in \left\{ \{ (\lambda^T g(x))_p - f(x) |\ x \in X \} + \{ (\lambda^T u)_p |\ u \in \mathbb{R}^m_+ \} \right\}.$$

On the other hand, since $(\lambda^T u)_p \leq_{\mathbb{R}^p_+} 0,\ \forall u \in \mathbb{R}^m_+$, one has $\bar{y} \geq_{\mathbb{R}^p_+} \bar{y} + (\lambda^T u)_p$ and by (4.1) it holds

$$\bar{y} + (\lambda^T u)_p \not\leq_{\mathbb{R}^p_+\setminus\{0\}} (\lambda^T g(x))_p - f(x) + (\lambda^T u)_p,\ \forall x \in X,\ \forall u \in \mathbb{R}^m_+.$$

Consequently, we obtain that

$$\bar{y} \not\leq_{\mathbb{R}^p_+\setminus\{0\}} (\lambda^T g(x))_p - f(x) + (\lambda^T u)_p,\ \forall x \in X,\ \forall u \in \mathbb{R}^m_+.$$

In other words $\bar{y} \in \varphi_1^*(0, \lambda)$. □

In this case the dual problem to (VO) can be written as

$$
\begin{aligned}
(\widehat{D}_L^{VO}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda \in \mathbb{R}^m} \Big[-\varphi_1^*(0, \lambda) \Big] \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda \leq_{\mathbb{R}^m_+} 0} \min_{\mathbb{R}^p_+\setminus\{0\}} \{ f(x) - (\lambda^T g(x))_p |\ x \in X \} \\
= \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\lambda \geq_{\mathbb{R}^m_+} 0} \min_{\mathbb{R}^p_+\setminus\{0\}} \{ f(x) + (\lambda^T g(x))_p |\ x \in X \}.
\end{aligned}
$$

Proposition 4.2 *[15, Theorem 6.1.4]*

(i) The problem (VO) is stable with respect to φ_1 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \geq_{\mathbb{R}^m_+} 0$ to the dual problem $(\widehat{D}_L^{VO})$ such that

$$f(\bar{x}) \in \min_{\mathbb{R}^p_+\setminus\{0\}} \{ f(x) + (\bar{\lambda}^T g(x))_p |\ x \in X \}$$

and $\bar{\lambda}^T g(\bar{x}) = 0$.

(ii) Conversely, if $\bar{x} \in G$ and $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \geq_{\mathbb{R}^m_+} 0$ satisfy the above conditions, then $\bar{x}$ and $\bar{\lambda}$ are solutions to (VO) and $(\widehat{D}_L^{VO})$, respectively.

Remark 4.1 Let $p = 1$ and the assumptions of Theorem 2.8 in [4] (see also [21]) be fulfilled. Then Proposition 4.2 coincides with the optimality conditions (cf. Theorem 2.9 in [4]) for the Lagrange dual problem in scalar optimization.

Example 4.1 Consider the vector optimization problem

$$(VO_1) \qquad \min_{\mathbb{R}^2_+\setminus\{0\}} \{ (x_1, x_2) |\ 0 \leq x_i \leq 1,\ x_i \in \mathbb{R},\ i = 1, 2 \}.$$

Let us construct the Lagrange dual problem to (VO_1). Before doing this, in view of $(\widehat{D}_L^{VO})$, for $\lambda \geq_{\mathbb{R}^m_+} 0$, one has to calculate

$$\min_{\mathbb{R}^p_+\setminus\{0\}} \{ f(x) + (\lambda^T g(x))_p |\ x \in X \}.$$

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T \in \mathbb{R}^4$ and the function $g : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $g(x) = (-x_1, x_1 - 1, -x_2, x_2 - 1)^T$. In other words, we have

$$\min_{\mathbb{R}^2_+\setminus\{0\}} \left\{ \begin{pmatrix} x_1 - \lambda_1 x_1 + \lambda_2(x_1 - 1) - \lambda_3 x_2 + \lambda_4(x_2 - 1) \\ x_2 - \lambda_1 x_1 + \lambda_2(x_1 - 1) - \lambda_3 x_2 + \lambda_4(x_2 - 1) \end{pmatrix} \Big| \; (x_1, x_2)^T \in \mathbb{R}^2 \right\},$$

or, equivalently,

$$\min_{\mathbb{R}^2_+\setminus\{0\}} \left\{ \begin{pmatrix} (\lambda_2 - \lambda_1 + 1)x_1 + (\lambda_4 - \lambda_3)x_2 \\ (\lambda_2 - \lambda_1)x_1 + (\lambda_4 - \lambda_3 + 1)x_2 \end{pmatrix} \Big| \; (x_1, x_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} \lambda_2 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix}.$$

Let

$$B_1 = \begin{pmatrix} \lambda_2 - \lambda_1 + 1 & \lambda_4 - \lambda_3 \\ \lambda_2 - \lambda_1 & \lambda_4 - \lambda_3 + 1 \end{pmatrix}.$$

Taking into account Theorem 11.20 in [12], if $\exists \mu \in \text{int } \mathbb{R}^2_+$ such that (cf. Lemma 5.1)

$$\mu^T B_1 = 0^T, \tag{4.2}$$

then $\min\limits_{\mathbb{R}^2_+\setminus\{0\}} \{B_1 x | \; x \in \mathbb{R}^2\} = \{B_1 x | \; x \in \mathbb{R}^2\}$. If (4.2) is not fulfilled, it follows that $\min\limits_{\mathbb{R}^2_+\setminus\{0\}} \{B_1 x | \; x \in \mathbb{R}^2\} = \varnothing$. From (4.2) follows $\begin{cases} (\lambda_2 - \lambda_1 + 1)\mu_1 + (\lambda_2 - \lambda_1)\mu_2 = 0 \\ (\lambda_4 - \lambda_3)\mu_1 + (\lambda_4 - \lambda_3 + 1)\mu_2 = 0. \end{cases}$
Consequently, we have

$$\lambda_1 = \lambda_2 + \frac{\mu_1}{\mu_1 + \mu_2}, \quad \lambda_3 = \lambda_4 + \frac{\mu_2}{\mu_1 + \mu_2}.$$

Let us define

$$L_1 := \left\{ \lambda \in \mathbb{R}^4 | \; \exists \mu \in \text{int } \mathbb{R}^2_+ \text{ such that } \lambda_1 = \lambda_2 + \frac{\mu_1}{\mu_1 + \mu_2}, \quad \lambda_3 = \lambda_4 + \frac{\mu_2}{\mu_1 + \mu_2} \right\}.$$

In conclusion, we obtain the Lagrange dual problem $(\widehat{D}_L^{VO_1})$ as follows

$$\max_{\mathbb{R}^2_+\setminus\{0\}} \bigcup_{\substack{\lambda \underset{\mathbb{R}^4_+}{\geqq} 0 \\ \lambda \in L_1}} \left\{ \begin{pmatrix} (\lambda_2 - \lambda_1 + 1)x_1 + (\lambda_4 - \lambda_3)x_2 \\ (\lambda_2 - \lambda_1)x_1 + (\lambda_4 - \lambda_3 + 1)x_2 \end{pmatrix} - \begin{pmatrix} \lambda_2 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \Big| \; (x_1, x_2)^T \in \mathbb{R}^2 \right\}.$$

Let $\bar{x} = (0,0)^T \in \mathbb{R}^2$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)^T \in L_1$ be vectors such that $\bar{\lambda} \underset{\mathbb{R}^4_+}{\geqq} 0$ and $\bar{\lambda}^T g(\bar{x}) = 0$. Then, from $\bar{\lambda}^T g(\bar{x}) = 0$ follows $\bar{\lambda}_2 + \bar{\lambda}_4 = 0$. As $\bar{\lambda}_2, \bar{\lambda}_4 \geq 0$, this implies that $\bar{\lambda}_2 = \bar{\lambda}_4 = 0$. Moreover, as $\bar{\lambda} \in L_1$, it holds $\bar{\lambda}_1 = \frac{\mu_1}{\mu_1 + \mu_2}$, $\bar{\lambda}_3 = \frac{\mu_2}{\mu_1 + \mu_2}$. In other words, $\bar{\lambda}_1 = \alpha := \frac{\mu_1}{\mu_1 + \mu_2}$, $\bar{\lambda}_3 = 1 - \alpha$, $0 < \alpha < 1$. On the other hand, it is clear that

$$\begin{aligned} f(\bar{x}) = (0,0)^T \;&\in\; \min_{\mathbb{R}^2_+\setminus\{0\}} \{f(x) + (\bar{\lambda}\bar{\lambda}^T g(x))_2 | \; x \in \mathbb{R}^2\} \\ &= \left\{ \begin{pmatrix} \frac{\mu_2}{\mu_1 + \mu_2}(x_1 - x_2) \\ \frac{\mu_1}{\mu_1 + \mu_2}(x_2 - x_1) \end{pmatrix} \Big| \; (x_1, x_2)^T \in \mathbb{R}^2 \right\} \\ &= \left\{ \begin{pmatrix} (\alpha - 1)y \\ \alpha y \end{pmatrix} \Big| \; y \in \mathbb{R} \right\}, \; 0 < \alpha < 1. \end{aligned}$$

According to Proposition 4.2(ii), $\bar{x} = (0,0)^T$ and $\bar{\lambda} = (\alpha, 0, 1 - \alpha, 0)^T$, $0 < \alpha < 1$ are solutions to (VO_1) and $(\widehat{D}_L^{VO_1})$, respectively.

Fenchel duality. Before considering the next dual problem, we need the following assertion.

Lemma 4.2 *Let $t \in \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$. If the set* $\max\limits_{\mathbb{R}^p_+\backslash\{0\}} \{(t^T x)_p |\ x \in Y\} \neq \varnothing$, *then*

$$\max_{\mathbb{R}^p_+\backslash\{0\}} \{(t^T x)_p |\ x \in Y\} = \{(\max_{x\in Y} t^T x)_p\}.$$

Proof: Let $t \in \mathbb{R}^n$. By assumption, there exists $\bar{x} \in Y$ such that

$$(t^T \bar{x})_p \underset{\mathbb{R}^p_+\backslash\{0\}}{\not\leq} (t^T x)_p,\ \forall x \in Y,$$

or, equivalently,

$$t^T \bar{x} \geq t^T x,\ \forall x \in Y.$$

Therefore $t^T \bar{x} = \max\limits_{x\in Y} t^T x$. □

Proposition 4.3 *Let $t \in \mathbb{R}^n$. Then*

$$\varphi_2^*(0,t) = \begin{cases} f_p^*(t) - (\min\limits_{x\in G} t^T x)_p, & \textit{if} \max\limits_{\mathbb{R}^p_+\backslash\{0\}} \{(-t^T x)_p |\ x \in G\} \neq \varnothing, \\ \varnothing, & \textit{otherwise.} \end{cases}$$

Proof: Let $t \in \mathbb{R}^n$. By definition

$$\begin{aligned} \varphi_2^*(0,t) &= \max_{\mathbb{R}^p_+\backslash\{0\}} \Big\{(t^T v)_p - \varphi_2(x,v) |\ x \in \mathbb{R}^n,\ v \in \mathbb{R}^n\Big\} \\ &= \max_{\mathbb{R}^p_+\backslash\{0\}} \Big\{(t^T v)_p - f(x+v) |\ x \in G,\ v \in \mathbb{R}^n\Big\}. \end{aligned}$$

Substituting $\bar{v} := x + v$, we get

$$\begin{aligned} \varphi_2^*(0,t) &= \max_{\mathbb{R}^p_+\backslash\{0\}} \Big\{(t^T \bar{v})_p - (t^T x)_p - f(\bar{v}) |\ x \in G,\ \bar{v} \in \mathbb{R}^n\Big\} \\ &= \max_{\mathbb{R}^p_+\backslash\{0\}} \Big\{\{(t^T \bar{v})_p - f(\bar{v}) |\ \bar{v} \in \mathbb{R}^n\} + \{(-t^T x)_p |\ x \in G\}\Big\}. \end{aligned}$$

According to Lemma 2.1(i), it follows that

$$\varphi_2^*(0,t) \subseteq \max_{\mathbb{R}^p_+\backslash\{0\}} \{(t^T v)_p - f(v) |\ v \in \mathbb{R}^n\} + \max_{\mathbb{R}^p_+\backslash\{0\}} \{(-t^T x)_p |\ x \in G\}.$$

It is clear that unless $\max\limits_{\mathbb{R}^p_+\backslash\{0\}} \{(-t^T x)_p |\ x \in G\} \neq \varnothing$, $\varphi_2^*(0,t) = \varnothing$.
Since $\max\limits_{\mathbb{R}^p_+\backslash\{0\}} \{(-t^T x)_p |\ x \in G\} \neq \varnothing$, by Lemma 4.2 it holds

$$\max_{\mathbb{R}^p_+\backslash\{0\}} \{(-t^T x)_p |\ x \in G\} = \{(-\min_{x\in G} t^T x)_p\}.$$

In other words

$$\begin{aligned} \varphi_2^*(0,t) &\subseteq \max_{\mathbb{R}^p_+\backslash\{0\}} \{(t^T v)_p - f(v) |\ v \in \mathbb{R}^n\} - (\min_{x\in G} t^T x)_p \\ &= f_p^*(t) - (\min_{x\in G} t^T x)_p. \end{aligned}$$

Let now $\bar{y} \in f_p^*(t) - (\min_{x \in G} t^T x)_p$. Then

$$\bar{y} \in \max_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ \{(t^T v)_p - f(v) |\ v \in \mathbb{R}^n\} - (\min_{x \in G} t^T x)_p \Big\}.$$

This means that

$$\bar{y} \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\leq} (t^T v)_p - f(v) - (\min_{x \in G} t^T x)_p, \ \forall v \in \mathbb{R}^n.$$

Moreover, from

$$(t^T v)_p - f(v) - (\min_{x \in G} t^T x)_p \underset{\mathbb{R}^p_+}{\geq} (t^T v)_p - f(v) - (t^T x)_p, \ \forall x \in G, \ \forall v \in \mathbb{R}^n$$

follows

$$(t^T v)_p - f(v) - (t^T x)_p \underset{\mathbb{R}^p_+ \setminus \{0\}}{\not\geq} \bar{y}, \ \forall x \in G, \ \forall v \in \mathbb{R}^n.$$

Whence $\bar{y} \in \varphi_2^*(0, t)$. □

The Fenchel dual problem can be stated now as follows

$$(\widehat{D}_F^{VO}) \qquad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in \mathbb{R}^n} \Big[- \varphi_2^*(0, t) \Big]$$
$$= \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in \mathbb{R}^n} \Big[- f_p^*(t) + (\min_{x \in G} t^T x)_p \Big]$$

From Theorem 2.2 and Proposition 4.3 follows the following assertion.

Proposition 4.4

(i) The problem (VO) is stable with respect to φ_2 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{t} \in \mathbb{R}^n$ to the dual problem $(\widehat{D}_F^{VO})$ such that

$$f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x \in G} \bar{t}^T x)_p \tag{4.3}$$

and $\bar{t}^T \bar{x} = \min_{x \in G} \bar{t}^T x$.

(ii) Conversely, if $\bar{x} \in G$ and $\bar{t} \in \mathbb{R}^n$ satisfy the above conditions, then $\bar{x}$ and $\bar{t}$ are solutions to (VO) and $(\widehat{D}_F^{VO})$, respectively.

Remark 4.2 Let $p = 1$ and the assumptions of Theorem 2.8 in [4] be fulfilled. Then Proposition 4.4 is nothing else than the result which provides the optimality conditions (cf. Theorem 2.10 in [4]) for the Fenchel dual problem in scalar optimization.

Fenchel-Lagrange duality. The last dual problem in this section deals with the perturbation function φ_3.

Proposition 4.5 *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$. Assume that* $\max_{\mathbb{R}^p_+\setminus\{0\}}\{(\lambda^T g(x) - t^T x)_p |\ x \in X\} \neq \varnothing$, *then*

$$\varphi_3^*(0,t,\lambda) = \begin{cases} f_p^*(t) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p, & \text{if } \lambda \leq_{\mathbb{R}^m_+} 0; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof: Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$. By definition

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{(t^T v)_p + (\lambda^T u)_p - \varphi_3(x,v,u) |\ x \in \mathbb{R}^n, v \in \mathbb{R}^n, u \in \mathbb{R}^m\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{(t^T v)_p + (\lambda^T u)_p - f(x+v) |\ x \in X, v \in \mathbb{R}^n, g(x) \leq_{\mathbb{R}^m_+} u\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \bigcup_{v\in\mathbb{R}^n} \Big\{(t^T v)_p + (\lambda^T u)_p - f(x+v) |\ g(x) \leq_{\mathbb{R}^m_+} u\Big\}.
\end{aligned}$$

Taking $\bar{u} := u - g(x)$, one has

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \bigcup_{v\in\mathbb{R}^n} \Big\{(t^T v)_p + (\lambda^T g(x))_p + (\lambda^T \bar{u})_p - f(x+v) |\ \bar{u} \in \mathbb{R}^m_+\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \bigcup_{v\in\mathbb{R}^n} \Big\{(t^T v)_p + (\lambda^T g(x))_p - f(x+v) + \{(\lambda^T \bar{u})_p |\ \bar{u} \in \mathbb{R}^m_+\}\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{(\lambda^T g(x))_p + \{(\lambda^T \bar{u})_p |\ \bar{u} \in \mathbb{R}^m_+\} \\
&+ \{(t^T v)_p - f(x+v) |\ v \in \mathbb{R}^n\}\Big\}.
\end{aligned}$$

Setting now $\bar{v} := x + v$, it follows that

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{(\lambda^T g(x))_p + \{(\lambda^T \bar{u})_p |\ \bar{u} \in \mathbb{R}^m_+\} \\
&+ \{(t^T \bar{v})_p - (t^T x)_p - f(\bar{v}) |\ \bar{v} \in \mathbb{R}^n\}\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{x\in X} \Big\{(\lambda^T g(x))_p - (t^T x)_p + \{(\lambda^T \bar{u})_p |\ \bar{u} \in \mathbb{R}^m_+\} \\
&+ \{(t^T \bar{v})_p - f(\bar{v}) |\ \bar{v} \in \mathbb{R}^n\}\Big\} \\
&= \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{(\lambda^T u)_p |\ u \in \mathbb{R}^m_+\} + \{(t^T v)_p - f(v) |\ v \in \mathbb{R}^n\} \\
&+ \{(\lambda^T g(x))_p - (t^T x)_p |\ x \in X\}\Big\}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\varphi_3^*(0,t,\lambda) &\subseteq \max_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T u)_p |\ u \in \mathbb{R}^m_+\} \\
&+ \max_{\mathbb{R}^p_+\setminus\{0\}} \{(t^T v)_p - f(\bar{v}) |\ v \in \mathbb{R}^n\} \\
&+ \max_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T g(x))_p - (t^T x)_p |\ x \in X\}.
\end{aligned}$$

Moreover, we can easy verify that

$$\max_{\mathbb{R}^p_+\setminus\{0\}} \{(\lambda^T g(x) - t^T x)_p |\ x \in X\} = \{(\max_{x\in X}[\lambda^T g(x) - t^T x])_p\}.$$

By Lemma 4.1 we conclude that

$$\varphi_3^*(0,t,\lambda) \subseteq \begin{cases} f_p^*(t) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p, & \text{if } \lambda \leq_{\mathbb{R}^m_+} 0; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Let us show now the converse inclusion. Let $t \in \mathbb{R}^n$, $\lambda \leq_{\mathbb{R}^m_+} 0$ and $\bar{y} \in f_p^*(t) + (\max_{x\in\mathbb{R}^n}[\lambda^T g(x) - t^T x])_p$. Then it holds

$$\bar{y} \in \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{\{(t^T v)_p - f(v) |\ v \in \mathbb{R}^n\} + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p\Big\}.$$

In other words

$$\bar{y} \not\leq_{\mathbb{R}^p_+\setminus\{0\}} (t^T v)_p - f(v) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p,\ \forall v \in \mathbb{R}^n.$$

Since

$$(t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p \leq_{\mathbb{R}^p_+} (t^T v)_p - f(v) + (\max_{x\in X}[\lambda^T g(x) - t^T x])_p,\ \forall x \in X,$$

we conclude that

$$\bar{y} \not\leq_{\mathbb{R}^p_+\setminus\{0\}} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p,\ \forall x \in X,\ \forall v \in \mathbb{R}^n,$$

or, equivalently,

$$\bar{y} + (\lambda^T u)_p \not\leq_{\mathbb{R}^p_+\setminus\{0\}} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p + (\lambda^T u)_p,\ \forall x \in X,\ \forall v \in \mathbb{R}^n,\ \forall u \in \mathbb{R}^m_+.$$

On the other hand, because of $(\lambda^T u)_p \leq_{\mathbb{R}^p_+} 0$, $\forall u \in \mathbb{R}^m_+$ it holds $\bar{y} \geq_{\mathbb{R}^p_+} \bar{y} + (\lambda^T u)_p$, $u \in \mathbb{R}^m_+$.

Whence, we obtain that

$$\bar{y} \not\leq_{\mathbb{R}^p_+\setminus\{0\}} (t^T v)_p - f(v) + (\lambda^T g(x) - t^T x)_p + (\lambda^T u)_p,\ \forall x \in X,\ \forall v \in \mathbb{R}^n,\ \forall u \in \mathbb{R}^m_+.$$

Therefore $\bar{y} \in \varphi_3^*(0,t,\lambda)$. □

As a consequence, we can suggest the following dual problem to (VO)

$$\begin{aligned} (\widehat{D}^{VO}_{FL}) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(t,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m} \Big[-\varphi_3^*(0,t,\lambda)\Big] \\ = \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\substack{t\in\mathbb{R}^n \\ \lambda \leq_{\mathbb{R}^m_+} 0}} \Big[-f_p^*(t) + (\min_{x\in X}[t^T x - \lambda^T g(x)])_p\Big] \\ = \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\substack{t\in\mathbb{R}^n \\ \lambda \geq_{\mathbb{R}^m_+} 0}} \Big[-f_p^*(t) + (\min_{x\in X}[t^T x + \lambda^T g(x)])_p\Big]. \end{aligned}$$

According to Theorem 2.2 and Proposition 4.5 one can give the following result.

Proposition 4.6

(i) The problem (VO) is stable with respect to φ_3 if and only if for each solution $\bar{x}$ to (VO), there exists a solution $\bar{t} \in \mathbb{R}^n$, $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geq} 0$ to the dual problem $(\widehat{D}^{VO}_{FL})$ such that

$$f(\bar{x}) \in -f_p^*(\bar{t}) + (\min_{x\in X}[\bar{t}^T x + \bar{\lambda}^T g(x)])_p. \tag{4.4}$$

Moreover it holds

$$\bar{t}^T \bar{x} + \bar{\lambda}^T g(\bar{x}) = \min_{x\in X}[\bar{t}^T x + \bar{\lambda}^T g(x)] \quad and \quad \bar{\lambda}^T g(\bar{x}) = 0. \tag{4.5}$$

(ii) Conversely, if $\bar{x} \in G$ and $\bar{t} \in \mathbb{R}^n$, $\bar{\lambda} \in \mathbb{R}^m$ with $\bar{\lambda} \underset{\mathbb{R}^m_+}{\geq} 0$ satisfy (4.4)–(4.5), then $\bar{x}$ and $(\bar{t}, \bar{\lambda})$ are solutions to (VO) and $(\widehat{D}^{VO}_{FL})$, respectively.

Remark 4.3 In the scalar case Proposition 4.6 is nothing else than the assertion dealing with the optimality conditions for the Fenchel-Lagrange duality (cf. Theorem 2.11 in [4]).

Further we show some relations between the dual objective maps investigated in this section.

Proposition 4.7 *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ with $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$. If $\max\limits_{\mathbb{R}^p_+\setminus\{0\}}\{(-t^T x)_p |\ x \in G\} \neq \varnothing$ and $\max\limits_{\mathbb{R}^p_+\setminus\{0\}}\{(\lambda^T g(x) - t^T x)_p |\ x \in X\} \neq \varnothing$, then*

$$\varphi_2^*(0,t) \subseteq \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+.$$

Proof: Let $t \in \mathbb{R}^n$ and $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$. Assume that $z \in \varphi_2^*(0,t) = f_p^*(t) - (\min\limits_{x\in G} t^T x)_p$. Since $g(x) \underset{\mathbb{R}^m_+}{\leq} 0$, for $x \in G$ one has $-\lambda^T g(x) \leq 0,\ \forall x \in G$. After adding $t^T x$ in both sides we have

$$\min_{x\in X}[t^T x - \lambda^T g(x)] \leq \min_{x\in G}[t^T x - \lambda^T g(x)] \leq \min_{x\in G} t^T x,$$

or, equivalently,

$$-(\min_{x\in G} t^T x)_p \underset{\mathbb{R}^p_+}{\leq} -(\min_{x\in X}[t^T x - \lambda^T g(x)])_p.$$

This means that

$$-(\min_{x\in G} t^T x)_p \in -(\min_{x\in X}[t^T x - \lambda^T g(x)])_p - \mathbb{R}^p_+.$$

Therefore

$$z \in f_p^*(t) - (\min_{x\in X}[t^T x - \lambda^T g(x)])_p - \mathbb{R}^p_+.$$

In other words $z \in \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+$. □

Proposition 4.8 *Let $t \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ with $\lambda \underset{\mathbb{R}^m_+}{\leq} 0$. If the set $f_p^*(t)$ is external stable and $\max\limits_{\mathbb{R}^p_+\setminus\{0\}}\{(\lambda^T g(x) - t^T x)_p |\ x \in X\} \neq \varnothing$, then*

$$\varphi_1^*(0,\lambda) \subseteq \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+.$$

Proof: Let $t \in \mathbb{R}^n$ and $\lambda \leq_{\mathbb{R}^m_+} 0$ be fixed. Then it is clear that

$$\begin{aligned}
\varphi_1^*(0,\lambda) &= \max_{\mathbb{R}^p_+ \setminus \{0\}} \{(\lambda^T g(x))_p - f(x) |\ x \in X\} \\
&\subseteq \{(\lambda^T g(x))_p - f(x) |\ x \in X\} \\
&\subseteq \{(t^T x)_p - f(x) |\ x \in \mathbb{R}^n\} + \{-(t^T x - \lambda^T g(x))_p |\ x \in X\}.
\end{aligned}$$

On the other hand, in view of the relation

$$-\{(p^T x - \lambda^T g(x))_p |\ x \in X\} \subseteq -\min_{x \in X} (p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+$$

and by the external stability of $f_p^*(t)$, we have

$$\begin{aligned}
\varphi_1^*(0,\lambda) &\subseteq f_p^*(t) - \mathbb{R}^p_+ - \min_{x \in X} (p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+ \\
&= f_p^*(t) - \min_{x \in X} (p^T x - \lambda^T g(x))_p - \mathbb{R}^p_+ \\
&= \varphi_3^*(0,t,\lambda) - \mathbb{R}^p_+.
\end{aligned}$$

□

5 Applications to the vector variational inequality

5.1 Gap functions for the vector variational inequality

Let $F : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ be a matrix-valued function and $K \subseteq \mathbb{R}^n$. The vector variational inequality problem consists in finding $x \in K$ such that

$$(VVI) \qquad F(x)^T(y - x) \not\leq_{\mathbb{R}^p_+ \setminus \{0\}} 0, \ \forall y \in K.$$

Definition 5.1 *(cf. [6] and [10]) A set-valued map $\gamma : K \rightrightarrows \mathbb{R}^p$ is said to be a gap function for (VVI) if it satisfies the following conditions:*

(i) $0 \in \gamma(x)$ if and only if $x \in K$ solves the problem (VVI);

(ii) $0 \not\geq_{\mathbb{R}^p_+ \setminus \{0\}} \gamma(y), \ \forall y \in K$.

For (VVI) the following gap function has been investigated (see [6])

$$\gamma_A^{VVI}(x) = \max_{\mathbb{R}^p_+ \setminus \{0\}} \left\{ F(x)^T(x - y) |\ y \in K \right\}.$$

Recall that γ_A^{VVI} is a generalization of Auslender's gap function for the scalar variational inequality problem (cf. [3]).
On the other hand, the dual problems and duality results investigated in Section 3 allow us to introduce some new gap functions for (VVI). Let us mention that such a similar approach has been proposed for scalar variational inequalities in [2]. We remark that $x \in K$ is a solution to the problem (VVI) if and only if 0 is a minimal

point of the set $\Big\{F(x)^T(y-x)|\ y\in K\Big\}$. This means that x is a solution to the following vector optimization problem

$$(P^{VVI};x) \qquad \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{F(x)^T(y-x)|\ y\in K\Big\}.$$

Let $x\in K$ be fixed. Setting $\widetilde{f}_x(y):=F(x)^T(y-x)$ instead of f in (D_F^{VO}), the Fenchel dual problem to $(P^{VVI};x)$ turns out to be

$$(D_F^{VVI};x) \qquad \max_{\mathbb{R}^p_+\backslash\{0\}} \bigcup_{T\in\mathbb{R}^{p\times n}} \min_{\mathbb{R}^p_+\backslash\{0\}} \Big\{\bigcup_{y\in\mathbb{R}^n}\{(F(x)^T-T)y\}-F(x)^Tx+\bigcup_{y\in K}\{Ty\}\Big\}.$$

We define the following map for any $x\in K$

$$\gamma_F^{VVI}(x):=\bigcup_{T\in\mathbb{R}^{p\times n}}\widetilde{\Phi}_2^*(0,T;x),$$

where $\widetilde{\Phi}_2^*(0,T;x)$ is defined by

$$\widetilde{\Phi}_2^*(0,T;x)=\max_{\mathbb{R}^p_+\backslash\{0\}}\Big\{\bigcup_{y\in\mathbb{R}^n}\{(T-F(x)^T)y\}+F(x)^Tx+\bigcup_{y\in K}\{-Ty\}\Big\}.$$

Theorem 5.1 *Let for any $x\in K$ the problem $(P^{VVI};x)$ be stable with respect to $\widetilde{\Phi}_2(0,\cdot;x)$. Then γ_F^{VVI} is a gap function for (VVI).*

Proof:

(i) Let $x\in K$ be a solution to the problem (VVI). As the problem $(P^{VVI};x)$ is stable, by Proposition 3.10(i), there exists a solution $T_x\in\mathbb{R}^{p\times n}$ to $(D_F^{VVI};x)$ such that

$$\widetilde{f}_x(x)=0\in-\widetilde{\Phi}_2^*(0,T_x;x). \tag{5.1}$$

In other words, $0\in\widetilde{\Phi}_2^*(0,T_x;x)$ and this implies that

$$0\in\bigcup_{T\in\mathbb{R}^{p\times n}}\widetilde{\Phi}_2^*(0,T;x)=\gamma_F^{VVI}(x).$$

Conversely, let $x\in K$ and $0\in\gamma_F^{VVI}(x)$. Hence, there exists $T_x\in\mathbb{R}^{p\times n}$ such that

$$0\in\widetilde{\Phi}_2^*(0,T_x;x)\text{ or, equivalently, }0=F(x)^T(x-x)\in-\widetilde{\Phi}_2^*(0,T_x;x).$$

According to Proposition 3.10(ii), x is a solution to $(P^{VVI};x)$ and also to the problem (VVI).

(ii) Let $y\in K$ be fixed. Then, in view of Proposition 3.9, for any $T\in\mathbb{R}^{p\times n}$, one has

$$f_y(z)+\xi\not\leq_{\mathbb{R}^p_+\backslash\{0\}}0,\ \forall z\in K,\ \forall\xi\in\widetilde{\Phi}_2^*(0,T;y),$$

or, equivalently,

$$F(y)^T(z-y)+\xi\not\leq_{\mathbb{R}^p_+\backslash\{0\}}0,\ \forall z\in K,\ \forall\xi\in\bigcup_{T\in\mathbb{R}^{p\times n}}\widetilde{\Phi}_2^*(0,T;y)=\gamma_F^{VVI}(y).$$

Setting $z=y$, we get

$$\xi\not\leq_{\mathbb{R}^p_+\backslash\{0\}}0,\ \forall\xi\in\gamma_F^{VVI}(y).$$

□

According to Proposition 3.2, we can give the following result relative to the stability with respect to $\widetilde{\Phi}_2(0,\cdot;x),\ x \in K$.

Proposition 5.1 *Let for any $x \in K$ the set $\min_{\mathbb{R}^p_+\setminus\{0\}} \{F(x)^T y|\ y \in K\}$ be externally stable. Then the problem $(P^{VVI};x)$ is stable with respect to $\widetilde{\Phi}_2(0,\cdot;x)$.*

In connection with the Fenchel dual problem we call γ_F^{VVI} as the Fenchel gap function for the problem (VVI). Let now the ground set K be given by

$$K = \Big\{x \in \mathbb{R}^n|\ g(x) \leqq_{\mathbb{R}^m_+} 0\Big\},$$

where $g(x) = (g_1(x),...,g_m(x))^T,\ g_i : \mathbb{R}^n \to \mathbb{R}$. Before introducing two other gap functions, let us state the Lagrange and Fenchel-Lagrange dual problems for $(P^{VVI};x)$. Taking $\widetilde{f}_x$ instead of f in $\Phi_1^*(0,\Lambda)$ and $\Phi_3^*(0,T,\Lambda)$, respectively, we have

$$\begin{aligned}(D_L^{VVI};x) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{\Lambda \in \mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{-\Lambda q\} - F(x)^T x \\ & + \bigcup_{y\in\mathbb{R}^n} \{F(x)^T y - \Lambda g(y)\}\Big\}\end{aligned}$$

and

$$\begin{aligned}(D_{FL}^{VVI};x) \qquad & \max_{\mathbb{R}^p_+\setminus\{0\}} \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \min_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{-\Lambda q\} - F(x)^T x \\ & + \bigcup_{y\in\mathbb{R}^n} \{(F(x)^T - T)y\} + \bigcup_{y\in\mathbb{R}^n} \{Ty - \Lambda g(y)\}\Big\}.\end{aligned}$$

We introduce the following maps, for any $x \in K$, as follows

$$\gamma_L^{VVI}(x) := \bigcup_{\Lambda\in\mathbb{R}^{p\times m}} \widetilde{\Phi}_1^*(0,\Lambda;x),$$

where we define

$$\widetilde{\Phi}_1^*(0,\Lambda;x) = \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{\Lambda q\} + F(x)^T x + \bigcup_{y\in\mathbb{R}^n} \{\Lambda g(y) - F(x)^T y\}\Big\}$$

and

$$\gamma_{FL}^{VVI}(x) := \bigcup_{(T,\Lambda)\in\mathbb{R}^{p\times n}\times\mathbb{R}^{p\times m}} \widetilde{\Phi}_3^*(0,T,\Lambda;x),$$

defining

$$\begin{aligned}\widetilde{\Phi}_3^*(0,T,\Lambda;x) \ = \ & \max_{\mathbb{R}^p_+\setminus\{0\}} \Big\{ \bigcup_{q\in\mathbb{R}^m_+} \{\Lambda q\} + F(x)^T x + \bigcup_{y\in\mathbb{R}^n} \{(T - F(x)^T)y\} \\ & + \bigcup_{y\in\mathbb{R}^n} \{\Lambda g(y) - Ty\}\Big\}.\end{aligned}$$

In analogy to the proof of Theorem 5.1, by applying the duality assertions in Section 3, for (D_L^{VO}) and (D_{FL}^{VO}), respectively, the following theorem can be verified.

Theorem 5.2 *Let for any $x \in K$ the problem $(P^{VVI};x)$ be stable with respect to $\widetilde{\Phi}_1(0,\cdot;x)$ and $\widetilde{\Phi}_3(0,\cdot;x)$, respectively. Then γ_L^{VVI} and γ_{FL}^{VVI} are gap functions for (VVI).*

The origin of these new gap functions for (VVI) justifies to call them as Lagrange gap function γ_L^{VVI} and Fenchel-Lagrange gap function γ_{FL}^{VVI}, respectively.

5.2 Gap functions via Fenchel duality

According to the results in Section 4, we can suggest a further class of gap functions for (VVI). In this subsection, we restrict the construction of a gap function to the case of Fenchel duality. As mentioned before, for a fixed $x \in K$ we consider the following vector optimization problem relative to (VVI)

$$(P^{VVI}; x) \qquad \min_{\mathbb{R}^p_+ \setminus \{0\}} \Big\{ F(x)^T (y - x) \,|\; y \in K \Big\}.$$

For a fixed $x \in K$, taking $F(x)^T(y - x)$ as the objective function, $(\widehat{D}^{VO}_F)$ becomes

$$(\widehat{D}^{VVI}_F; x) \qquad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in \mathbb{R}^n} \Big\{ \min_{\mathbb{R}^p_+ \setminus \{0\}} [(F(x)^T(y - x) - (t^T y)_p |\; y \in \mathbb{R}^n] + (\min_{y \in K} t^T y)_p \Big\}.$$

We need the following auxiliary result.

Lemma 5.1 *Let $M \in \mathbb{R}^{p \times n}$. Then*

$$\min_{\mathbb{R}^p_+ \setminus \{0\}} \{My |\; y \in \mathbb{R}^n\} = \begin{cases} \{My |\; y \in \mathbb{R}^n\}, & \text{if } \exists \mu \in \text{int } \mathbb{R}^p_+ \text{ such that } \mu^T M = 0^T, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof: Let $M \in \mathbb{R}^{p \times n}$ be fixed and $\bar{y} \in \mathbb{R}^n$. According to Theorem 11.20 in [12], $M\bar{y} \in \min_{\mathbb{R}^p_+ \setminus \{0\}} \{My |\; y \in \mathbb{R}^n\}$ if and only if $\exists \mu \in$ int $\mathbb{R}^p_+$ such that

$$\mu^T M \bar{y} \leq \mu^T M y, \; \forall y \in \mathbb{R}^n. \tag{5.2}$$

As

$$\inf_{y \in \mathbb{R}^n} \mu^T M y = \begin{cases} 0, & \mu^T M = 0^T, \\ -\infty, & \text{otherwise,} \end{cases}$$

$M\bar{y} \in \min_{\mathbb{R}^p_+ \setminus \{0\}} \{My |\; y \in \mathbb{R}^n\}$ if and only if $\exists \mu \in$ int $\mathbb{R}^p_+$ such that

$$\mu^T M = 0^T. \tag{5.3}$$

This means that under the above assumption each $\bar{y} \in \mathbb{R}^n$ is a solution to (5.2). □

Let $C := [t, ..., t] \in \mathbb{R}^{n \times p}$ and for a fixed $x \in K$ the set $N(x)$ be defined by

$$N(x) := \{t \in \mathbb{R}^n |\; \exists \mu \in \text{int } \mathbb{R}^p_+ \text{ such that } (F(x) - C)\mu = 0\}.$$

In view of Lemma 5.1, one has

$$(\widehat{D}^{VVI}_F; x) \qquad \max_{\mathbb{R}^p_+ \setminus \{0\}} \bigcup_{t \in N(x)} \Big\{ -F(x)^T x + \{(F(x) - C)^T y |\; y \in \mathbb{R}^n\} + (\min_{y \in K} t^T y)_p \Big\}.$$

Let us introduce for $x \in K$ the following map

$$\widetilde{\gamma}^{VVI}_F(x) := F(x)^T x + \bigcup_{t \in N(x)} \Big[\{(C - F(x))^T y |\; y \in \mathbb{R}^n\} - (\min_{y \in K} t^T y)_p \Big].$$

Theorem 5.3 *Let for any $x \in K$ the set $\min_{\mathbb{R}^p_+ \setminus \{0\}} \{F(x)^T y |\; y \in K\}$ be externally stable. Then $\widetilde{\gamma}^{VVI}_F$ is a gap function for (VVI).*

Proof:

(i) Let $x \in K$ be fixed. As the set $\min_{\mathbb{R}^p_+\backslash\{0\}} \{F(x)^T y|\ y \in K\}$ is externally stable, by Proposition 5.1, the problem $(P^{VVI}; x)$ is stable. Taking $F(x)^T(y-x)$ instead of $f(y)$ in $f_p^*(t)$, by Lemma 5.1, we have

$$\begin{aligned} f_p^*(t) &= \max_{\mathbb{R}^p_+\backslash\{0\}} \{(t^T y)_p - F(x)^T(y-x)|\ y \in \mathbb{R}^n\} \\ &= F(x)^T x - \min_{\mathbb{R}^p_+\backslash\{0\}} \{(F(x)-C)^T y|\ y \in \mathbb{R}^n\} \\ &= F(x)^T x - \{(F(x)-C)^T y|\ y \in \mathbb{R}^n\}, \end{aligned}$$

where $C = [t, ..., t] \in \mathbb{R}^{n\times p}$ and $t \in N(x)$. Then (4.3) is equivalent to

$$0 \in -F(x)^T x + \{(F(x)-C)^T y|\ y \in \mathbb{R}^n\} + (\min_{y\in K} t^T y)_p. \tag{5.4}$$

Let $\bar{x} \in K$ be a solution to (VVI). By Proposition 4.4(i) and (5.4) it follows that $0 \in \widetilde{\gamma}_F^{VVI}(\bar{x})$. Let $\bar{x} \in K$ and $0 \in \widetilde{\gamma}_F^{VVI}(\bar{x})$. Then $\exists \bar{t} \in N(\bar{x})$ such that

$$0 \in F(\bar{x})^T \bar{x} + \{(\bar{C} - F(\bar{x}))^T y|\ y \in \mathbb{R}^n\} - (\min_{y\in K} \bar{t}^T y)_p,$$

where $\bar{C} = [\bar{t}, ..., \bar{t}] \in \mathbb{R}^{n\times p}$. Taking into account Proposition 4.4(ii) and (5.4), $\bar{x}$ is a solution to $(P^{VVI}; \bar{x})$. Consequently, $\bar{x}$ solves the problem (VVI).

(ii) Let $y \in K$. Choosing as $T := [t, ..., t]^T \in \mathbb{R}^{p\times n}$, by Proposition 3.9 and Proposition 4.3, it holds

$$F(y)^T(z-y) + \xi \not\leq_{\mathbb{R}^p_+\backslash\{0\}} 0,\ \forall z \in K,\ \forall \xi \in f_p^*(t) - (\min_{y\in K} t^T y)_p,\ t \in N(y),$$

or, equivalently,

$$F(y)^T(z-y) + \xi \not\leq_{\mathbb{R}^p_+\backslash\{0\}} 0,\ \forall z \in K,\ \forall \xi \in \widetilde{\gamma}_F^{VVI}(y).$$

Setting $z = y$, one has

$$\xi \not\leq_{\mathbb{R}^p_+\backslash\{0\}} 0,\ \forall \xi \in \widetilde{\gamma}_F^{VVI}(y).$$

□

Remark 5.1 In the case $p = 1$, the problem (VVI) reduces to the scalar variational inequality problem of finding $x \in K$ such that

$$(VI) \qquad F(x)^T(x-y) \geq 0,\ y \in K,$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector-valued function. Let $x \in K$ be fixed. By the definition of the set $N(x)$, there exists $\mu > 0$ such that $(F(x) - t)\mu = 0$. Therefore it holds $F(x) = t$. Consequently, the gap function for the variational inequality becomes

$$\begin{aligned} \gamma_F^{VI}(x) &= F(x)^T x + \max_{y\in K}(-F(x)^T y) \\ &= \max_{y\in K} F(x)^T(x-y), \end{aligned}$$

which coincides with Auslender's gap function (see [2] and [3]).

Example 5.1 Let $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be a constant matrix and

$$K = \{(x_1, x_2)^T \in \mathbb{R}^2 |\ 0 \le x_i \le 1,\ x_i \in \mathbb{R},\ i = 1, 2\}.$$

We consider the vector variational inequality problem of finding $x \in K$ such that

$$(VVI_1) \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (y - x) \underset{\mathbb{R}^2_+ \setminus \{0\}}{\not\leq} 0,\ \forall y \in K.$$

Let us describe $\widetilde{\gamma}_F^{VVI}$ for (VVI_1). Let $x = (x_1, x_2)^T \in \mathbb{R}^2$ be fixed. First we consider the set-valued map $W : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by (see $(\widehat{D}_F^{VVI}; x)$)

$$W(x_1, x_2) = \min_{\mathbb{R}^2_+ \setminus \{0\}} \{F(x)^T (y - x) - (t^T y)_2 |\ y \in \mathbb{R}^2\}.$$

Then

$$\begin{aligned} W(x_1, x_2) &= \min_{\mathbb{R}^2_+ \setminus \{0\}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} - \begin{pmatrix} t_1 y_1 + t_2 y_2 \\ t_1 y_1 + t_2 y_2 \end{pmatrix} \Big|\ (y_1, y_2)^T \in \mathbb{R}^2 \right\} \\ &= \min_{\mathbb{R}^2_+ \setminus \{0\}} \left\{ \begin{pmatrix} y_1 - x_1 - t_1 y_1 - t_2 y_2 \\ y_2 - x_2 - t_1 y_1 - t_2 y_2 \end{pmatrix} \Big|\ (y_1, y_2)^T \in \mathbb{R}^2 \right\} \\ &= \min_{\mathbb{R}^2_+ \setminus \{0\}} \left\{ \begin{pmatrix} (1 - t_1) y_1 - t_2 y_2 \\ -t_1 y_1 + (1 - t_2) y_2 \end{pmatrix} \Big|\ (y_1, y_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

If $\exists \mu = (\mu_1, \mu_2)^T \in \text{int } \mathbb{R}^2_+$ such that $(\mu_1, \mu_2) \begin{pmatrix} 1 - t_1 & -t_2 \\ -t_1 & 1 - t_2 \end{pmatrix} = 0$, or, equivalently, $\begin{cases} (1 - t_1)\mu_1 - t_1 \mu_2 & = 0 \\ -t_2 \mu_1 + (1 - t_2)\mu_2 & = 0. \end{cases}$ Then, by Lemma 5.1, it holds

$$W(x_1, x_2) = \left\{ \begin{pmatrix} (1 - t_1) y_1 - t_2 y_2 \\ -t_1 y_1 + (1 - t_2) y_2 \end{pmatrix} \Big|\ (y_1, y_2)^T \in \mathbb{R}^2 \right\} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

As $(\mu_1, \mu_2) \in \text{int } \mathbb{R}^2_+$, it must to be $\begin{vmatrix} (1 - t_1) & -t_1 \\ -t_2 & (1 - t_2) \end{vmatrix} = 0$. As a consequence, one has

$$t_1 + t_2 = 1 \text{ and } t_2 \mu_1 = t_1 \mu_2.$$

Whence

$$\begin{aligned} \widetilde{\gamma}_F^{VVI_1}(x) \ &= \ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \bigcup_{t \in N_1} \left[\left\{ \begin{pmatrix} t(y_2 - y_1) \\ (1 - t)(y_1 - y_2) \end{pmatrix} \Big|\ (y_1, y_2)^T \in \mathbb{R}^2 \right\} \right. \\ &\left. - \begin{pmatrix} \min\limits_{0 \le y_1 \le 1} (1 - t) y_1 + \min\limits_{0 \le y_2 \le 1} t y_2 \\ \min\limits_{0 \le y_1 \le 1} (1 - t) y_1 + \min\limits_{0 \le y_2 \le 1} t y_2 \end{pmatrix} \right], \end{aligned}$$

where the set N_1 is defined by

$$N_1 := \{t \in \mathbb{R} |\ \exists \mu \in \text{int } \mathbb{R}^2_+ \text{ such that } (1 - t)\mu_1 = t \mu_2\}.$$

Moreover, as $N_1 = (0, 1)$, we conclude that

$$\widetilde{\gamma}_F^{VVI_1}(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \bigcup_{t \in (0,1)} \left\{ \begin{pmatrix} ty \\ (t - 1)y \end{pmatrix} \Big|\ y \in \mathbb{R} \right\}.$$

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