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# Maximal monotonicity for the precomposition with a linear operator

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**Abstract.** We give the weakest constraint qualification known to us that assures the maximal monotonicity of the operator $A^* \circ T \circ A$ when $A$ is a linear continuous mapping between two reflexive Banach spaces and $T$ is a maximal monotone operator. As a special case we get the weakest constraint qualification that assures the maximal monotonicity of the sum of two maximal monotone operators on a reflexive Banach space. Then we give a weak constraint qualification assuring the Brézis-Haraux-type approximation of the range of the subdifferential of the precomposition to $A$ of a proper convex lower-semicontinuous function in non-reflexive Banach spaces, extending and correcting in a special case an older result due to Riahi.

**Keywords.** Maximal monotone operator, Fitzpatrick function, subdifferential, Brézis-Haraux-type approximation

## 1 Introduction

The literature on maximal monotone operators is quite rich especially in the recent years when their connections to convex analysis, underlined with the help of some functions (cf. [11], [14], [19], [21], [22]) were more and more intensively studied and used. One of the most interesting problems which involves both maximal monotone operators and convex analysis is the one of finding sufficient conditions that assure the maximal monotonicity of the operator $A^* \circ T \circ A$ when $A$ is a linear continuous mapping between two reflexive Banach spaces and $T$ is

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a maximal monotone operator. From the papers dealing with this problem we refer here to [1], [5], [12], [14] and [22], the latter unifying the results concerning this issue from the others and giving four equivalent constraint qualifications, the weakest in the literature known to us. Finding a weaker sufficient condition under which the sum of two maximal monotone operators on reflexive Banach spaces is maximal monotone has been an older challenge for many mathematicians, the problem having more than four decades behind. From Browder ([4]) and Rockafellar ([18]) in the 60's to the recent (yet unpublished) papers of Simons and Zălinescu ([21]), Borwein ([1]) or Jeyakumar and Wu ([10]), the conditions imposed on two maximal monotone operators in order to assure the maximal monotonicity of their sum became weaker and weaker, the latter paper containing the weakest constraint qualification that guarantees the mentioned result known to us so far. We mention here also Simons' book [19] where many sufficient conditions for the mentioned problem are recalled, compared and unified. This book and the lecture notes [15] due to Phelps are excellent references for anyone interested in maximal monotone operators. Within this paper we give a constraint qualification that guarantees the maximal monotonicity of $A^* \circ T \circ A$ and is satisfied also by some $A$ and $T$ that violate the other sufficient conditions known to us, already mentioned. This condition uses the so-called Fitzpatrick functions and has been developed from the one introduced by two of the authors in [2] for Fenchel duality. For a special choice of $A$ and $T$ we obtain a sufficient condition that guarantees the maximal monotonicity of the sum of two maximal monotone operators and we show that our constraint qualification is equivalent to the weakest one known to us.

Another result in maximal monotonicity for whose fulfilment we give a weaker sufficient condition is the one concerning the so-called Brézis-Haraux-type approximation of the range of $f \circ A$, where $f$ is a proper convex lower-semicontinuous function defined on the image space of $A$ with extended real values. Here we work in non-reflexive Banach spaces. Something similar has been done by Pennanen in [13] when the latter mentioned space is also reflexive. As a special case we recover and correct a result due to Riahi (cf. [16]) concerning the Brézis-Haraux-type approximation (cf. [19]) of the range of the sum of the subdifferentials of two lower-semicontinuous functions by the sum of the ranges of the two subdifferentials, for which we give a weaker constraint qualification than in the original paper.

The paper is structured as follows. The next section contains necessary preliminaries, notions and results used later, then we deal with the maximal monotonicity of $A^* \circ T \circ A$ and of the sum of two maximal monotone operators. Section 4 deals with the mentioned Brézis-Haraux-type approximations and it is followed by a short summary of the results proved within the paper. The list of references closes the paper.

## 2 Preliminaries

Within this section we introduce and recall notions and results in order to make the paper self-contained. Even if the main results in the paper are given in (reflexive) Banach spaces, some of the preliminaries are valid also for more general spaces, thus we begin by considering a non-trivial locally convex topological space $X$ and its continuous dual space $X^*$, endowed with the weak$^*$ topology $w(X^*, X)$. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset $C$ of $X$ we have the *indicator* function $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

and we denote by $\text{int}(C)$ and $\text{cl}(C)$ its *interior*, respectively its *closure* in the corresponding topology. For $C$ we define also the *linear hull* $\text{lin}(C)$ as the intersection of all the linear subspaces of $X$ containing $C$ and the *affine hull* $\text{aff}(C)$ which is the intersection of all the affine subsets of $X$ containing $C$. For $C \subseteq X$ convex we denote the *intrinsic relative algebraic interior* of $C$ by ${}^{ic}C$. One has $x \in^{ic} C$ if and only if $\cup_{\lambda>0} \lambda(C - x)$ is a closed linear subspace of $X$. We consider also the *first projection*, i.e. the function $\text{pr}_1 : X \times Y \to X$, for $Y$ some non-trivial locally convex space, defined as follows $\text{pr}_1(x, y) = x$ for any $(x, y) \in X \times Y$.

Given a function $f : X \to \overline{\mathbb{R}}$, we denote its *domain* by $\text{dom}(f) = \{x \in X : f(x) < +\infty\}$ and its *epigraph* by $\text{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the *subdifferential* of $f$ at $x$ by $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle\}$. We call $f$ *proper* if $f(x) > -\infty$ $\forall x \in X$ and $\text{dom}(f) \neq \emptyset$. The *conjugate* of the function $f$ is $f^* : X^* \to \overline{\mathbb{R}}$ introduced by

$$f^*(y) = \sup\big\{\langle y, x\rangle - f(x) : x \in X\big\}.$$

Between a function and its conjugate there is *Young's inequality*

$$f^*(y) + f(x) \geq \langle y, x\rangle \ \forall x \in X \ y \in X^*.$$

Consider also the *identity* function on $X$ defined as follows, $\text{id}_X : X \to X$, $\text{id}_X(x) = x$ $\forall x \in X$. When $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, we define the function $f \times g : X \times Y \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ through $f \times g(x, y) = (f(x), g(y))$, $(x, y) \in X \times Y$. When $f, g : X \to \overline{\mathbb{R}}$ are proper functions, we have the *infimal convolution* of $f$ and $g$ defined by

$$f \Box g : X \to \overline{\mathbb{R}}, \ f \Box g(a) = \inf\{f(x) + g(a - x) : x \in X\}.$$

Given a linear continuous mapping $A : X \to Y$, we have its *image-set* $\text{Im}(A) = AX = \{Ax : x \in X\} \subseteq Y$ and its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$. For the proper function $f : X \to \overline{\mathbb{R}}$ we define also the *marginal* function of $f$ through $A$ as $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf\big\{f(x) :$

$x \in X, Ax = y\}$, $y \in Y$. All along the present paper when an infimum or a supremum is attained we write min, respectively max instead of inf and sup.

Further we give some results concerning the composition of a function with a linear continuous operator.

**Lemma 1.** ([7]) *Let $X$ and $Y$ be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower-semicontinuous function such that $f \circ A$ is proper on $X$. Then*

$$\operatorname{epi}((f \circ A)^*) = \operatorname{cl}(\operatorname{epi}(A^* f^*)), \tag{1}$$

*where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$, for every locally convex topology $\tau$ on $X^*$ giving $X$ as dual.*

*Remark 1.* Significant choices for $\tau$ in the preceding lemma are the weak$^*$ topology $w(X^*, X)$ on $X^*$ or the norm topology of $X^*$ in case $X$ is a reflexive Banach space.

The following result was given in [2], but we give also its proof as part of it will be needed later.

**Lemma 2.** ([2]) *Let $X$ and $Y$ be non-trivial locally convex spaces, $\tau$ an arbitrary topology on $X^*$, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper function. Then*

$$\operatorname{cl}(\operatorname{epi}(A^* f^*)) = \operatorname{cl}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))), \tag{2}$$

*where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$.*

**Proof.** Take first $(x^*, r) \in A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$, so there is some $y^* \in Y^*$ such that $A^* y^* = x^*$ and $(y^*, r) \in \operatorname{epi}(f^*)$. This yields

$$A^* f^*(x^*) = \inf\{f^*(y^*) : A^* y^* = x^*\} \leq r,$$

therefore $(x^*, r) \in \operatorname{epi}(A^* f^*)$, followed immediately by $A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*)) \subseteq \operatorname{epi}(A^* f^*)$. Proving that $\operatorname{epi}(A^* f^*)$ is included into the closure of $A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$ will imply (2), so let $(x^*, r) \in \operatorname{epi}(A^* f^*)$, $\varepsilon > 0$ and take $\mathcal{V}(x^*)$ an open neighborhood of $x^*$ in $\tau$. Because

$$A^* f^*(x^*) = \inf\{f^*(y^*) : A^* y^* = x^*\} \leq r < r + \frac{\varepsilon}{2},$$

there exists an $y^* \in Y^*$ such that $A^* y^* = x^*$ and $f^*(y^*) \leq r + \varepsilon/2$. Thus $(x^*, r + \varepsilon/2) \in A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$ and, on the other hand, $(x^*, r + \varepsilon/2) \in \mathcal{V}(x^*) \times$

$(r-\varepsilon, r+\varepsilon)$. Because $\mathcal{V}(x^*)$ and $\varepsilon$ were arbitrarily chosen it follows $(x^*, r) \in \operatorname{cl}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*)))$. $\square$

Taking in (1) and (2) the closure in the product topology of $(X^*, \tau) \times \mathbb{R}$, with $\tau$ any locally convex topology on $X^*$ giving $X$ as dual, we get

$$\operatorname{epi}((f \circ A)^*) = \operatorname{cl}(\operatorname{epi}(A^* f^*)) = \operatorname{cl}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))). \tag{3}$$

**Definition 1.** A set $M \subseteq X$ is said to be *closed regarding the subspace* $Z \subseteq X$ if $M \cap Z = \operatorname{cl}(M) \cap Z$.

**Proposition 1.** *Let $X$, $Y$ and $U$ be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower-semicontinuous function such that $f \circ A$ is proper on $X$. Consider moreover the linear continuous mapping $M : U \to X^*$. Let $\tau$ be any locally convex topology on $X^*$ giving $X$ as dual. The following statements are equivalent.*

*(a)* $A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$ *is closed regarding the subspace* $\operatorname{Im}(M) \times \mathbb{R}$ *in the product topology of* $(X^*, \tau) \times \mathbb{R}$,

*(b)* $(f \circ A)^*(Mu) = \min\left\{f^*(y^*) : A^* y^* = Mu\right\}$ *for all* $u \in U$.

**Proof.** Because $f$ is proper, convex and lower-semicontinuous, $A$ linear and continuous and $f \circ A$ proper it follows that $(f \circ A)^*$ is proper, convex and lower-semicontinuous.

"$(a) \Rightarrow (b)$" Let $u \in U$. For any $y^* \in Y^*$ fulfilling $A^* y^* = Mu$ we have because of Young's inequality

$$f^*(y^*) \geq \langle y^*, Ax\rangle - f(Ax) = \langle A^* y^*, x\rangle - (f \circ A)(x) = \langle Mu, x\rangle - (f \circ A)(x) \ \forall x \in X,$$

and when taking the supremum subject to $x \in X$ in the right-hand side we get $f^*(y^*) \geq \sup_{x \in X}\{\langle Mu, x\rangle - (f \circ A)(x)\} = (f \circ A)^*(Mu)$. This holds for any $y^* \in Y^*$ satisfying $A^* y^* = Mu$, so we conclude

$$\inf\left\{f^*(y^*) : A^* y^* = Mu\right\} \geq (f \circ A)^*(Mu). \tag{4}$$

Let us prove now the reverse inequality. If $(f \circ A)^*(Mu) = +\infty$ then (4) yields $f^*(y^*) = +\infty = (f \circ A)^*(Mu)$ for any $y^* \in Y^*$ such that $A^* y^* = Mu$. Consider further $(f \circ A)^*(Mu) \in \mathbb{R}$. It follows $(Mu, (f \circ A)^*(Mu)) \in \operatorname{epi}((f \circ A)^*)$ and it is clear that it belongs also to $\operatorname{Im}(M) \times \mathbb{R}$. By (3), $(a)$ gives

$$\begin{aligned}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}) &= \operatorname{cl}(A^* \times \operatorname{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}) \\ &= \operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}),\end{aligned}$$

so $(Mu, (f \circ A)^*(Mu))$ belongs to the set in the left-hand side, too. This means that there is some $\bar{y}^* \in Y^*$ such that $A^*\bar{y}^* = Mu$ and $(\bar{y}^*, (f \circ A)^*(Mu)) \in \operatorname{epi}(f^*)$. The latter relation can be rewritten as $f^*(\bar{y}^*) \le (f \circ A)^*(Mu)$ and we get

$$\inf \{f^*(y^*) : A^* y^* = Mu\} \le f^*(\bar{y}^*) \le (f \circ A)^*(Mu). \tag{5}$$

Having (4) and (5) we are allowed to write

$$\inf \{f^*(y^*) : A^* y^* = Mu\} = (f \circ A)^*(Mu), \tag{6}$$

and the relations above regarding $\bar{y}^*$ show that the infimum in (6) is attained at $\bar{y}^*$, so $(b)$ is true as $u \in U$ has been taken arbitrarily.

"$(b) \Rightarrow (a)$" From (3) one gets $\operatorname{epi}((f \circ A)^*) \supseteq A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$, followed by

$$\operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}) \supseteq (A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}).$$

For any pair $(x^*, r) \in \operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R})$ there is some $u \in U$ such that $x^* = Mu$ and we have $(f \circ A)^*(x^*) = (f \circ A)^*(Mu) \le r$. The hypothesis $(b)$ grants the existence of an $\bar{y}^* \in Y^*$ satisfying both $A^*\bar{y}^* = Mu = x^*$ and $f^*(\bar{y}^*) = (f \circ A)^*(Mu) \le r$, i.e. $(\bar{y}^*, r) \in \operatorname{epi}(f^*)$. Thus $(x^*, r) = (A^*\bar{y}^*, r) \in A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$, and as it is in $\operatorname{Im}(M) \times \mathbb{R}$, too, and this pair has been arbitrarily chosen it follows

$$\operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}) \subseteq (A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}).$$

As the opposite inclusion stands, too, we get

$$\operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}) = (A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}),$$

which yields $(a)$ by (3) and Definition 1. □

**Corollary 1.** ([2]) *Let $X$ and $Y$ be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower-semicontinuous function such that $f \circ A$ is proper. Let $\tau$ be any locally convex topology on $X^*$ giving $X$ as dual. Then*

*(i)* $A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$ *is closed in the product topology of* $(X^*, \tau) \times \mathbb{R}$ *if and only if for any* $x^* \in X^*$ *one has*

$$(f \circ A)^*(x^*) = \min\{f^*(y^*) : A^* y^* = x^*\}.$$

*(ii) If* $A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$ *is closed in the product topology of* $(X^*, \tau) \times \mathbb{R}$*, then for any* $x \in \operatorname{dom}(f \circ A)$ *one has* $\partial(f \circ A)(x) = A^*\partial f(Ax)$.

**Proof.** $(i)$ follows from Proposition 1 when taking $U = X^*$ and $M = \mathrm{id}_{X^*}$, while for $(ii)$ we refer to [2] and [9]. $\square$

*Remark 2.* Let $\tau$ be any locally convex topology on $X^*$ giving $X$ as dual. We know that $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*)) \subseteq \mathrm{epi}(A^*f^*) \subseteq \mathrm{epi}((f \circ A)^*)$ (see Lemma 1 and the proof of Lemma 2). From (3) it follows that $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ is closed in the product topology of $(X^*, \tau) \times \mathbb{R}$ if and only if

$$A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*)) = \mathrm{epi}(A^*f^*) = \mathrm{epi}((f \circ A)^*).$$

The second part of this section in devoted to monotone operators and some of their properties. Consider further $X$ a Banach space equipped with the norm $\|\cdot\|$, while the norm on $X^*$ is $\|\cdot\|_*$.

**Definition 2.** ([18]) A mapping (generally multivalued) $T : X \to 2^{X^*}$ is called *monotone operator* provided that for any $x, y \in X$ one has

$$\langle y^* - x^*, y - x \rangle \geq 0 \ \text{ whenever } x^* \in T(x) \text{ and } y^* \in T(y).$$

**Definition 3.** ([18]) For any monotone operator $T : X \to 2^{X^*}$ we have

· its *effective domain* $D(T) = \{x \in X : T(x) \neq \emptyset\}$,

· its *range* $R(T) = \cup\{T(x) : x \in X\}$,

· its *graph* $G(T) = \{(x, x^*) : x \in X, x^* \in T(x)\}$.

**Definition 4.** ([18]) A monotone operator $T : X \to 2^{X^*}$ is called *maximal* when its graph is not properly included in the graph of any other monotone operator $T' : X \to 2^{X^*}$.

The subdifferential of a proper convex lower-semicontinuous function on $X$ is a typical example of a maximal monotone operator (cf. [17]). As we shall see in Section 4, it belongs to many other classes of operators, too. We introduce also the *duality map* $J : X \to 2^{X^*}$ defined as follows

$$J(x) = \partial \frac{1}{2}\|x\|^2 = \left\{x^* \in X^* : \|x\|^2 = \|x^*\|^2 = \langle x^*, x \rangle \right\} \ \forall x \in X,$$

because it gives the following criterion for the maximal monotonicity of a monotone operator $T : X \to 2^{X^*}$.

**Proposition 2.** ([1], [19]) *A monotone operator $T$ on a reflexive Banach space $X$ is maximal if and only if the mapping $T(x+\cdot)+J(\cdot)$ is surjective for all $x \in X$.*

As underlined by many authors (cf. [1], [10], [11], [14], [19], [21], [22]), there are strong connections between the maximal monotone operators and convex analysis. They are best noticeable by the *Fitzpatrick functions* associated to the monotone operators (cf. [6]). Rediscovered after some years, they proved to be crucial in treating the problem of maximal monotonicity of the sum of maximal monotone operators within the latest papers on the subject ([1], [10], [21], [22], [23]). To a monotone operator $T : X \to 2^{X^*}$ Fitzpatrick attached the function

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}},\ \varphi_T(x, x^*) = \sup \big\{ \langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in T(y) \big\}.$$

For any monotone operator $T$ it is quite clear that $\varphi_T$ is a convex lower - semicontinuous function as an affine supremum. An important result regarding the Fitzpatrick functions and their conjugates in reflexive Banach spaces follows.

**Proposition 3.** ([21]) *Let $T$ be a maximal monotone operator on a reflexive Banach space $X$. Then for any pair $(x, x^*) \in X \times X^*$ we have*

$$\varphi_T^*(x^*, x) \geq \varphi_T(x, x^*) \geq \langle x^*, x\rangle.$$

*Moreover, $\varphi_T^*(x^*, x) = \varphi_T(x, x^*) = \langle x^*, x\rangle$ if and only if $(x, x^*) \in G(T)$.*

**Lemma 3.** *Let $T : Y \to 2^{Y^*}$ be a monotone operator and $A : X \to Y$ a linear continuous mapping, with $X$ and $Y$ Banach spaces. Then*

$$(\mathrm{pr}_1(\mathrm{dom}(\varphi_T)) - \mathrm{Im}(A)) \times Y^* = \mathrm{dom}(\varphi_T) - \mathrm{Im}(A) \times Y^*.$$

**Proof.** "$\subseteq$" Take $(y, y^*) \in (\mathrm{pr}_1(\mathrm{dom}(\varphi_T)) - \mathrm{Im}(A)) \times Y^*$. Thus there are some $t \in \mathrm{pr}_1(\mathrm{dom}(\varphi_T))$ and $a \in \mathrm{Im}(A)$ such that $y = t - a$. The existence of $t$ implies that there is some $t^* \in Y^*$ such that $(t, t^*) \in \mathrm{dom}(\varphi_T)$. Denoting $a^* = t^* - y^* \in Y^*$, one has $(a, a^*) \in \mathrm{Im}(A) \times Y^*$. Therefore

$$(y, y^*) = (t, t^*) - (a, a^*) \in \mathrm{dom}(\varphi_T) - \mathrm{Im}(A) \times Y^*.$$

"$\supseteq$" Let $(t, t^*) \in \mathrm{dom}(\varphi_T)$ and $(a, a^*) \in \mathrm{Im}(A) \times Y^*$. Then $t \in \mathrm{pr}_1(\mathrm{dom}(\varphi_T))$, so $t - a \in \mathrm{pr}_1(\mathrm{dom}(\varphi_T)) - \mathrm{Im}(A)$, thus

$$(t - a, t^* - a^*) \in (\mathrm{pr}_1(\mathrm{dom}(\varphi_T)) - \mathrm{Im}(A)) \times Y^*. \qquad \square$$

# 3 Maximal monotonicity for the precomposition with a linear operator

Within this section $X$ and $Y$ will be reflexive Banach spaces. Given the maximal monotone operator $T$ on $Y$ and the linear continuous mapping $A : X \to Y$, such that $A\big(\mathrm{pr}_1(\mathrm{dom}(\varphi_T))\big) \neq \emptyset$, we introduce the operator $T_A : X \to 2^{X^*}$ defined by $T_A(x) = A^* \circ T \circ A(x)$, $x \in X$, which is monotone, but not always maximal monotone.

## 3.1 Maximal monotonicity for $T_A$

Various conditions which assure the maximal monotonicity of $T_A$ were given in many recent papers, among which we mention [1], [5], [12], [14] and [22]. We prove, using an idea due to Borwein ([1]), that $T_A$ is maximal monotone provided that the following constraint qualification is fulfilled,

$(CQ) A^* \times \mathrm{id}_Y \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(\varphi_T^*))$ is closed regarding the subspace $X^* \times \mathrm{Im}(A) \times \mathbb{R}$.

**Theorem 1.** *If $(CQ)$ is fulfilled then $T_A$ is a maximal monotone operator.*

**Proof.** Let us fix first some $z \in X$ and $z^* \in X^*$ and consider the functions $f, g : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$f(x, x^*) = \inf\{\varphi_T(A(x+z), y^*) - \langle y^*, Az\rangle : A^*y^* = x^* + z^*\}$$

and $$g(x, x^*) = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle,\ (x, x^*) \in X \times X^*.$$
As $f$ and $g$ are convex and the latter is continuous we can apply Fenchel's duality theorem (cf. [23]) that guarantees the existence of some pair $(\bar{x}^*, \bar{x}) \in X^* \times X$ such that

$$\begin{aligned} \inf_{(x,x^*)\in X\times X^*} \{f(x, x^*) + g(x, x^*)\} &= \max_{(x^*,x)\in X^*\times X} \{-f^*(x^*, x) - g^*(-x^*, -x)\} \\ &= -f^*(\bar{x}^*, \bar{x}) - g^*(-\bar{x}^*, -\bar{x}). \end{aligned} \tag{7}$$

Let us calculate the conjugates of $f$ and $g$. Before this we introduce the linear continuous operator $B = A \times \mathrm{id}_{Y^*}$. For any $(w^*, w) \in X^* \times X$ we have

$$\begin{aligned} f^*(w^*, w) &= \sup_{\substack{x\in X,\\ x^*\in X^*}} \{\langle w^*, x\rangle + \langle x^*, w\rangle - \inf_{A^*y^*=x^*+z^*} \{\varphi_T(A(x+z), y^*) - \langle y^*, Az\rangle\}\} \\ &= \sup_{(x,x^*)\in X\times X^*} \{\langle w^*, x\rangle + \langle x^*, w\rangle + \sup_{A^*y^*=x^*+z^*} \{-\varphi_T(A(x+z), y^*) + \langle y^*, Az\rangle\}\} \\ &= \sup_{\substack{(x,x^*)\in X\times X^*, y^*\in Y^*,\\ A^*y^*=x^*+z^*}} \{\langle w^*, x\rangle + \langle x^*, w\rangle - \varphi_T(A(x+z), y^*) + \langle y^*, Az\rangle\} \\ &= \sup_{\substack{x\in X, y^*\in Y^*,\\ u=x+z\in X}} \{\langle w^*, u-z\rangle + \langle A^*y^* - z^*, w\rangle - \varphi_T(A(u), y^*) + \langle A^*y^*, z\rangle\} \\ &= \sup_{\substack{u\in X,\\ y^*\in Y^*}} \{\langle w^*, u\rangle + \langle A^*y^*, w+z\rangle - \varphi_T(A(u), y^*)\} - \langle w^*, z\rangle - \langle z^*, w\rangle \\ &= \sup_{\substack{u\in X,\\ y^*\in Y^*}} \{\langle w^*, u\rangle + \langle y^*, A(w+z)\rangle - (\varphi_T \circ B)(u, y^*)\} - \langle w^*, z\rangle - \langle z^*, w\rangle \\ &= (\varphi_T \circ B)^*(w^*, A(w+z)) - \langle w^*, z\rangle - \langle z^*, w\rangle \end{aligned}$$

and

$$
\begin{aligned}
g^*(w^*,w) &= \sup_{\substack{x\in X,\\ x^*\in X^*}} \left\{\langle w^*,x\rangle + \langle x^*,w\rangle - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x^*\|_*^2 + \langle z^*,x\rangle\right\}\\
&= \sup_{x\in X}\left\{\langle w^*+z^*,x\rangle - \frac{1}{2}\|x\|^2\right\} + \sup_{x^*\in X^*}\left\{\langle x^*,w\rangle - \frac{1}{2}\|x^*\|_*^2\right\}\\
&= \frac{1}{2}\|w^*+z^*\|_*^2 + \frac{1}{2}\|w\|^2.
\end{aligned}
$$

Proposition 1 assures that $(CQ)$ is equivalent to the fact that for any $(w^*,w)\in X^*\times X$ one has

$$(\varphi_T\circ B)^*(w^*,Aw) = \min_{(y^*,y)\in Y^*\times Y}\{\varphi_T^*(y^*,y) : B^*(y^*,y) = (w^*,Aw)\}.$$

For any $(x,x^*)\in X\times X^*$ and $y^*\in Y^*$ such that $A^*y^* = x^*+z^*$, by Proposition 3, we have

$$
\begin{aligned}
\varphi_T(A(x+z),y^*) &- \langle y^*,Az\rangle + g(x,x^*) \geq \langle y^*,A(x+z)\rangle - \langle y^*,Az\rangle\\
&+ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 - \langle z^*,x\rangle\\
&= \langle y^*,Ax\rangle - \langle z^*,x\rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2\\
&= \langle A^*y^*-z^*,x\rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2\\
&= \langle x^*+z^*-z^*,x\rangle + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2\\
&= \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 + \langle x^*,x\rangle \geq 0.
\end{aligned}
$$

Taking in the left-hand side the infimum subject to all $y^*\in Y^*$ fulfilling $A^*y^* = x^*+z^*$, we get $f(x,x^*)+g(x,x^*)\geq 0$. Thus $\inf_{(x,x^*)\in X\times X^*}\{f(x,x^*)+g(x,x^*)\}\geq 0$ and taking it into (7) one gets $f^*(\bar{x}^*,\bar{x}) + g^*(-\bar{x}^*,-\bar{x})\leq 0$, i.e.

$$(\varphi_T\circ B)^*(\bar{x}^*,A(\bar{x}+z)) - \langle\bar{x}^*,z\rangle - \langle z^*,\bar{x}\rangle + \frac{1}{2}\|-\bar{x}^*+z^*\|_*^2 + \frac{1}{2}\|-\bar{x}\|^2 \leq 0. \quad (8)$$

From Proposition 1 we have

$$(\varphi_T\circ B)^*(\bar{x}^*,A(\bar{x}+z)) = \min_{(y^*,y)\in Y^*\times Y}\{\varphi_T^*(y^*,y) : B^*(y^*,y) = (\bar{x}^*,A(\bar{x}+z))\},$$

with the minimum attained at some $(\bar{y}^*,\bar{y})\in Y^*\times Y$. As the adjoint operator of $B$ is $B^* : Y^*\times Y\to X^*\times Y$, $B^*(y^*,y) = (A^*y^*,y)$, it follows $B^*(\bar{y}^*,\bar{y}) =$

$(A^*\bar{y}^*, \bar{y}) = (\bar{x}^*, A(\bar{x}+z))$. Taking the last two relations into (8) we have

$$\begin{aligned}
0 &\geq \varphi_T^*(\bar{y}^*, \bar{y}) - \langle \bar{x}^*, z\rangle - \langle z^*, \bar{x}\rangle + \frac{1}{2}\|\bar{x}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2 \\
&= \varphi_T^*(\bar{y}^*, A(\bar{x}+z)) - \langle A^*\bar{y}^*, z\rangle - \langle z^*, \bar{x}\rangle + \frac{1}{2}\|A^*\bar{y}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2 \\
&= \varphi_T^*(\bar{y}^*, A(\bar{x}+z)) - \langle \bar{y}^*, Az\rangle - \langle \bar{y}^*, A\bar{x}\rangle + \langle \bar{y}^*, A\bar{x}\rangle - \langle z^*, \bar{x}\rangle + \frac{1}{2}\|\bar{x}\|^2 \\
&+ \frac{1}{2}\|A^*\bar{y}^* - z^*\|_*^2 = \big(\varphi_T^*(\bar{y}^*, A(\bar{x}+z)) - \langle \bar{y}^*, A(\bar{x}+z)\rangle\big) \\
&+ \big(\langle A^*\bar{y}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|A^*\bar{y}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2\big) \geq 0,
\end{aligned}$$

where the last inequality comes from Proposition 3. Thus the inequalities above must be fulfilled as equalities, so

$$\varphi_T^*(\bar{y}^*, A(\bar{x}+z)) - \langle \bar{y}^*, A(\bar{x}+z)\rangle = 0,$$

i.e., by Proposition 3, $\bar{y}^* \in T \circ A(\bar{x}+z)$ and

$$\langle A^*\bar{y}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|A^*\bar{y}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2 = 0,$$

i.e. $z^* - A^*\bar{y}^* \in \partial \frac{1}{2}\|\cdot\|^2(\bar{x})$. Further one has $A^*\bar{y}^* \in A^* \circ T \circ A(z+\bar{x}) = T_A(z+\bar{x})$ and $z^* - A^*\bar{y}^* \in J(\bar{x})$, so $z^* \in T_A(z+\bar{x}) + J(\bar{x})$. As $z$ and $z^*$ have been arbitrarily chosen, Proposition 2 yields the conclusion. □

*Remark 3.* We compare in the following the constraint qualification $(CQ)$ to some generalized interior-point regularity conditions given in the literature in order to assure the maximality of the monotone operator $T_A$. Under the condition in [5] one gets the fulfillment of the ones considered in [1] and [14], which imply the ones in [12] and [22], that are actually equivalent (according to [22]) to

$$(CQ_Z) \qquad \bigcup_{\lambda>0} \lambda(D(T) - \mathrm{Im}(A)) \text{ is a closed linear subspace.}$$

Assume $(CQ_Z)$ fulfilled. Then $\mathrm{cl}(\mathrm{lin}(D(T) - \mathrm{Im}(A))) \subseteq \bigcup_{\lambda>0} \lambda(D(T) - \mathrm{Im}(A))$ from the way the linear hull is defined. Lemma 5.3$(a)$ in [21] states $D(T) \subseteq \mathrm{pr}_1(\mathrm{dom}(\varphi_T))$, which yields

$$\bigcup_{\lambda>0} \lambda(D(T) - \mathrm{Im}(A)) \subseteq \bigcup_{\lambda>0} \lambda(\mathrm{pr}_1(\mathrm{dom}(\varphi_T)) - \mathrm{Im}(A)),$$

so one gets

$$\mathrm{cl}(\mathrm{lin}(D(T) - \mathrm{Im}(A))) \subseteq \bigcup_{\lambda>0} \lambda(\mathrm{pr}_1(\mathrm{dom}(\varphi_T)) - \mathrm{Im}(A)). \tag{9}$$

On the other hand, for any $y \in \operatorname{Im}(A)$ we have $D(T) - y \subseteq \operatorname{cl}(\operatorname{lin}(D(T) - \operatorname{Im}(A)))$, which yields, by Lemma 5.3($b$) in [21], $\operatorname{pr}_1(\operatorname{dom}(\varphi_T)) \subseteq \operatorname{cl}(\operatorname{lin}(D(T) - \operatorname{Im}(A))) + y$. As $y$ has been arbitrarily chosen we get $\operatorname{pr}_1(\operatorname{dom}(\varphi_T)) - \operatorname{Im}(A) \subseteq \operatorname{cl}(\operatorname{lin}(D(T) - \operatorname{Im}(A)))$, which implies

$$\bigcup_{\lambda>0} \lambda(\operatorname{pr}_1(\operatorname{dom}(\varphi_T) - \operatorname{Im}(A))) \subseteq \operatorname{cl}(\operatorname{lin}(D(T) - \operatorname{Im}(A))).$$

This and (9) give

$$\bigcup_{\lambda>0} \lambda(\operatorname{pr}_1(\operatorname{dom}(\varphi_T) - \operatorname{Im}(A))) = \operatorname{cl}(\operatorname{lin}(D(T) - \operatorname{Im}(A))).$$

By Lemma 3 we get that $\bigcup_{\lambda>0} \lambda\big(\operatorname{dom}(\varphi_T) - \operatorname{Im}(A) \times Y^*\big)$ is a closed linear subspace, so, taking into account that $B = A \times \mathrm{id}_{Y^*}$, $0 \in^{ic} (\operatorname{dom}(\varphi_T) - \operatorname{Im}(B))$. This yields, by Theorem 2.3.8($vii$) in [23],

$$(\varphi_T \circ B)^*(w^*, Aw) = \min_{(y^*,y) \in Y^* \times Y} \{\varphi_T^*(y^*, y) : B^*(y^*, y) = (w^*, Aw)\},$$

which is equivalent to $(CQ)$. Therefore $(CQ_Z) \Rightarrow (CQ)$. A counterexample to show that it is possible to have $(CQ)$ satisfied and $(CQ_Z)$ violated will be given later.

*Remark 4.* The maximal monotonicity of $T_A$ is valid also when imposing the constraint qualification

$$(\widetilde{CQ}) \qquad A^* \times \mathrm{id}_Y \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(\varphi_T^*)) \text{ is closed.}$$

The only difference in the proof is that we use Corollary 1($i$) instead of Proposition 1. One may notice that we have $(CQ_Z) \Rightarrow (\widetilde{CQ}) \Rightarrow (CQ)$, i.e. $(\widetilde{CQ})$ is still weaker than $(CQ_Z)$.

The remaining part of the section is dedicated to the proof of the fact that $(CQ)$ is indeed weaker than $(CQ_Z)$.

*Example 1.* Let $X = \mathbb{R}$ and $Y = \mathbb{R} \times \mathbb{R}$. Then $X^* = \mathbb{R}$ and $Y^* = \mathbb{R} \times \mathbb{R}$. Consider the operator $T : \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R} \times \mathbb{R}}$ defined by

$$T(x,y) = \begin{cases} \{0\} \times \mathbb{R}, & \text{if } x > 0, y = 0, \\ (-\infty, 0] \times \mathbb{R}, & \text{if } x = y = 0, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad \forall (x,y) \in \mathbb{R} \times \mathbb{R}.$$

Is is not difficult to notice that, considering the functions $f, g : \mathbb{R} \to \overline{\mathbb{R}}$, $f = \delta_{[0,+\infty)}$ and $g = \delta_{\{0\}}$, which are proper, convex and lower-semicontinuous, for any $(x,y) \in \mathbb{R} \times \mathbb{R}$ we have $T(x,y) = \big(\partial f(x), \partial g(y)\big)$, thus $T$ is a maximal monotone operator. Taking $A : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $Ax = (x,x)$, one gets, for any $x \in \mathbb{R}$,

$$T_A(x) = A^* \circ T \circ A(x) = \partial f(x) + \partial g(x) = \partial g(x),$$

which yields that $T_A$ is a maximal monotone operator, too.

Let us calculate the conjugate of $\varphi_T$ to see if $(CQ)$ is fulfilled. We have for all $(x, y, x^*, y^*) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$\varphi_T(x, y, x^*, y^*) = \begin{cases} 0, & \text{if } x \geq 0, x^* \leq 0, y = 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and}$$

$$\varphi_T^*(x^*, y^*, x, y) = \begin{cases} 0, & \text{if } x^* \leq 0, x \geq 0, y = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

Thus the epigraph of the conjugate is

$$\text{epi}(\varphi_T^*) = (-\infty, 0] \times \mathbb{R} \times [0, +\infty) \times \{0\} \times [0, +\infty),$$

so

$$A^* \times \text{id}_{\mathbb{R}\times\mathbb{R}} \times \text{id}_{\mathbb{R}}(\text{epi}(\varphi_T^*)) = \mathbb{R} \times [0, +\infty) \times \{0\} \times [0, +\infty),$$

which is closed, i.e. $(\widetilde{CQ})$ is valid. Thus it is closed regarding the subspace $\mathbb{R} \times \text{Im}(A) \times \mathbb{R} = \mathbb{R} \times \Delta_{\mathbb{R}} \times \mathbb{R}$, too, i.e. $(CQ)$ is satisfied for the chosen $T$ and $A$. Here we used the notation $\Delta_X = \{(x, x) : x \in X\}$, in case $X = \mathbb{R}$.

Let us calculate now $\underset{\lambda>0}{\cup} \lambda(D(T) - \text{Im}(A))$ in order to check the validity of $(CQ_Z)$. It is clear that $D(T) = \big((0, +\infty) \times \{0\}\big) \cup \{(0, 0)\} = [0, +\infty) \times \{0\}$ and $\text{Im}(A) = \Delta_{\mathbb{R}}$. We have $D(T) - \text{Im}(A) = \{[x, +\infty) \times \{x\} : x \in \mathbb{R}\}$, so

$$\underset{\lambda>0}{\cup} \lambda(D(T) - \text{Im}(A)) = \{[x, +\infty) \times \{x\} : x \in \mathbb{R}\} = \{(x, y) \in \mathbb{R} : x \geq y\},$$

which is not a subspace, thus $(CQ_Z)$ is violated. Therefore, even if $(CQ_Z)$ implies $(CQ)$, the reverse implication does not always hold, i.e. $(CQ)$ is indeed weaker than $(CQ_Z)$.

## 3.2 Maximal monotonicity for the sum of two maximal monotone operators

An important special case of the problem treated in Theorem 1, i.e. the maximal monotonicity of $T_A$ is the situation when the sum of two maximal monotone operators is maximal monotone. This case is obtained from the general one by taking $Y = X \times X$, $A(x) = (x, x)$ for any $x \in X$ and $T : X \times X \to X^* \times X^*$, $T(x, y) = (T_1(x), T_2(y))$ when $(x, y) \in X \times X$, where $T_1$ and $T_2$ are maximal monotone operators on $X$. It is a simple verification to show that $T$ is maximal monotone. Having these choices, for any $x \in X$ we have $T_A(x) = T_1(x) + T_2(x)$. Moreover, the condition on the domain of $\varphi_T$ becomes $\text{pr}_1(\text{dom}(\varphi_{T_1})) \cap \text{pr}_1(\text{dom}(\varphi_{T_2})) \neq \emptyset$.

The literature concerning the maximal monotonicity of $T_1 + T_2$ is richer than the one in the more general case. Let us mention here, alongside the papers already cited above, also [4], [10], [18] and [21]. A comprehensive study on this problem is available in [19], where many sufficient conditions for the maximal monotonicity of the sum of two maximal monotone operators are compared and classified. The weakest such condition available in the literature known to us is the one given in [10], namely

$$(CQ_{JW}) \qquad (\mathrm{epi}(\varphi_{T_1}^*))_x + (\mathrm{epi}(\varphi_{T_2}^*))_x \text{ is closed for each } x \in X,$$

where, for any $x \in X$, $(\mathrm{epi}(\varphi_{T_i}^*))_x = \{(x^*, r) : (x^*, x, r) \in \mathrm{epi}(\varphi_{T_i}^*)\}$, $i = 1, 2$. Our constraint qualification $(CQ)$ becomes in this special case

$(CQ^s)$ $\quad \{(x^* + y^*, x, y, r) : \varphi_{T_1}^*(x^*, x) + \varphi_{T_2}^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$.

We prove now that $(CQ^s)$ is equivalent to $(CQ_{JW})$, i.e. we have also the weakest constraint qualification that guarantees the maximal monotonicity of the sum of two maximal monotone operators, too.

Let us proceed. We know from Proposition 1 that $(CQ^s)$ is equivalent to

$$(\varphi_T \circ B)^*(w^*, Aw) = \min_{(z^*,z) \in Y^* \times Y} \{\varphi_T^*(z^*, z) : B^*(z^*, z) = (w^*, Aw)\}$$

$\forall (w^*, w) \in X^* \times X$, with the minimum attained at some $(\bar{y}^*, \bar{y}) \in Y^* \times Y$. We write this for the present choice of $Y$, $A$ and $T$. First, $B = A \times \mathrm{id}_{X^* \times X^*} : X \times X^* \times X^* \to X \times X \times X^* \times X^*$ is in this case defined by $B(x, x^*, y^*) = (x, x, x^*, y^*)$ for $(x, x^*, y^*) \in X \times X^* \times X^*$, so $B^*(x^*, y^*, x, y) = (x^* + y^*, x, y)$. We also have $\varphi_T(x, y, x^*, y^*) = \varphi_{T_1}(x, x^*) + \varphi_{T_2}(y, y^*)$ and $\varphi_T^*(x^*, y^*, x, y) = \varphi_{T_1}^*(x^*, x) + \varphi_{T_2}^*(y^*, y)$ $\forall (x, y) \in X \times X$ $\forall (x^*, y^*) \in X^* \times X^*$. Introducing the function (cf. [10], [21])

$$\rho : X \times X^* \to \overline{\mathbb{R}}, \ \rho(v, v^*) = \inf_{x^*, y^* \in X^*} \{\varphi_{T_1}(v, x^*) + \varphi_{T_2}(v, y^*) : x^* + y^* = v^*\},$$

its conjugate is $\rho^* : X^* \times X \to \overline{\mathbb{R}}$,

$$\begin{aligned} \rho^*(w^*, w) &= \sup_{(v,v^*) \in X \times X^*} \Big\{\langle w^*, v\rangle + \langle v^*, w\rangle - \inf_{\substack{x^*, y^* \in X^*, \\ x^* + y^* = v^*}} \{\varphi_{T_1}(v, x^*) + \varphi_{T_2}(v, y^*)\}\Big\} \\ &= \sup_{\substack{x^*, y^* \in X^*, \\ v \in X}} \{\langle w^*, v\rangle + \langle x^* + y^*, w\rangle - \varphi_{T_1}(v, x^*) - \varphi_{T_2}(v, y^*)\}. \end{aligned}$$

The conjugate of $\varphi_T \circ B$ is now for any $w^* \in X^*$ and $w \in X$

$$(\varphi_T \circ B)^*(w^*, w, w) = \sup_{\substack{x^*, y^* \in X^*, \\ x \in X}} \{\langle w^*, x\rangle + \langle x^* + y^*, w\rangle - \varphi_{T_1}(x, x^*) - \varphi_{T_2}(x, y^*)\}$$

and it is easy to notice that $(\varphi_T \circ B)^*(w^*, w, w) = \rho^*(w^*, w)$. Using these, $(CQ^s)$ is equivalent to

$$\begin{aligned} \rho^*(w^*, w) &= \min_{\substack{x,y \in X, \\ x^*, y^* \in X^*}} \{\varphi_{T_1}^*(x^*, x) + \varphi_{T_2}^*(y^*, y) : x^* + y^* = w^*, x = y = w\} \\ &= \min_{\substack{x^*, y^* \in X^*, \\ x^* + y^* = w^*}} \{\varphi_{T_1}^*(x^*, w) + \varphi_{T_2}^*(y^*, w)\}. \end{aligned}$$

Theorem 3.1 in [10] yields that this is equivalent to $(CQ_{JW})$. Thus our constraint qualification is equivalent to the weakest condition that assures that the sum of two maximal monotone operators on a reflexive Banach space is maximal monotone. We have the following statement.

**Theorem 2.** *Let $T_1$ and $T_2$ be maximal monotone operators on $X$ such that $\mathrm{pr}_1(\mathrm{dom}(\varphi_{T_1})) \cap \mathrm{pr}_1(\mathrm{dom}(\varphi_{T_2})) \neq \emptyset$. If $(CQ^s)$ (or, equivalently, $(CQ_{JW})$) is fulfilled, then $T_1 + T_2$ is a maximal monotone operator on $X$.*

*Remark 5.* The other constraint qualification we gave, $(\widetilde{CQ})$, becomes in this case

$$(\widetilde{CQ}^s) \qquad \{(x^* + y^*, x, y, r) : \varphi_{T_1}^*(x^*, x) + \varphi_{T_2}^*(y^*, y) \leq r\} \text{ is closed.}$$

One may prove that $(\widetilde{CQ}^s)$ is weaker than the other constraint qualifications mentioned within this subsection, except $(CQ^s)$ and $(CQ_{JW})$ which are implied by it.

# 4 Brézis - Haraux - type approximation of the range of the subdifferential of the precomposition with a linear operator

Within this part $X$ and $Y$ are considered Banach spaces, unless otherwise specified. Let us mention that, unlike the previous section, here we do not ask these spaces to be moreover reflexive. We rectify, weaken and generalize a statement due to Riahi ([16]) concerning the so-called Brézis-Haraux-type approximation of the range of the sum of the subdifferentials of two proper convex lower-semicontinuous functions, giving it for the operator $T_A$ introduced in the previous section. Riahi's statement is recovered as special case, under a weaker sufficient condition than in the original paper.

## 4.1 Some preliminaries

We need to introduce some notions and to recall some results which are dealt with only within this part. First we introduce the so-called monotone operators of type $(D)$, originally introduced by Gossez in [8] as operators of dense type and known in the literature also as densely maximal, of type $3^*$, also known as star monotone and of the type $(BH)$, and operators of the type $(NI)$. Let us stress once again that we work in non-reflexive Banach spaces.

Before this we need to introduce the operator $\overline{T} : Y^{**} \to 2^{Y^*}$ as follows

$$G(\overline{T}) = \{(x^{**}, x^*) \in Y^{**} \times Y^* : \langle x^{**} - \hat{y}, x^* - y^* \rangle \geq 0 \ \forall (y, y^*) \in G(T),$$

where $\hat{y}$ denotes the canonical image of $y$ in $Y^{**}$. The elements in $G(\overline{T})$ are called in the literature *monotonically related* to $T$.

**Definition 5.** ([15]) A monotone operator $T : Y \to 2^{Y^*}$ is called of *type* $(D)$ provided that for any $(y^{**}, y^*) \in G(\overline{T})$ there is a net $(y_\alpha, y^*_\alpha)_\alpha \subseteq G(T)$ such that $y_\alpha \to y^{**}$ in the $\sigma(Y^{**}, Y^*)$-topology (cf. [8], [15], [16]), $(y_\alpha)_\alpha$ is bounded and $y^*_\alpha \to y^*$ in the norm-topology.

**Definition 6.** ([13], [18]) A monotone operator $T : Y \to 2^{Y^*}$ is called $3^*$-*monotone* if for all $x^* \in R(T)$ and $x \in D(T)$ there is some $\beta(x^*, x) \in \mathbb{R}$ such that $\inf_{y^* \in T(y)} \langle x^* - y^*, x - y \rangle \geq \beta(x^*, x)$.

**Definition 7.** ([13], [18]) An operator $T : Y \to 2^{Y^*}$ is called of *type* $(NI)$ if for all $(x^{**}, x^*) \in Y^{**} \times Y^*$ one has $\inf_{y^* \in T(y)} \langle \hat{y} - x^{**}, y^* - x^* \rangle \leq 0$.

Some necessary results follow.

**Lemma 4.** ([16]) *Given the type* $(D)$ *operator* $T : Y \to 2^{Y^*}$ *and the non-empty subset* $E \subseteq Y^*$ *such that for any* $x^* \in E$ *there is some* $x \in Y$ *fulfilling* $\inf_{y^* \in T(y)} \langle y^* - x^*, y - x \rangle > -\infty$, *one has* $E \subseteq \operatorname{cl}(R(T))$ *and* $\operatorname{int}(E) \subseteq R(\overline{T})$.

**Proposition 4.** *If* $T : Y \to 2^{Y^*}$ *is* $3^*$*-monotone and* $A : X \to Y$ *is a linear continuous mapping such that* $T_A$ *is of type* $(D)$, *then*

*(i)* $A^*(R(T)) \subseteq \operatorname{cl}(R(T_A))$,

*(ii)* $\operatorname{int}(A^*(R(T))) \subseteq R(\overline{T_A})$.

**Proof.** As $T$ is $3^*$-monotone, we have for any $s \in D(T)$ and any $s^* \in R(T)$ there is some $\beta(s^*, s) \in \mathbb{R}$ such that $\beta(s^*, s) \leq \inf_{x^* \in T(x)} \langle s^* - x^*, s - x \rangle$.

To apply Lemma 4 for $E = A^*(R(T))$ and $T_A$, we need to verify if they satisfy its hypothesis. Take some $u^* \in A^*(R(T))$, thus there is an $v^* \in R(T)$ such that

$u^* = A^*v^*$. We have for any $u \in X$

$$\inf_{x^*\in T_A(x)}\langle x^* - u^*, x - u\rangle = \inf_{t^*\in T\circ A(x)}\langle A^*t^* - A^*v^*, x - u\rangle = \inf_{t^*\in T\circ A(x)}\langle t^*$$

$$-v^*, A(x-u)\rangle \geq \inf_{t^*\in T(t)}\langle t^* - v^*, t - Au)\rangle \geq \beta(v^*, Au) > -\infty.$$

Having this fulfilled for any $x$, we can apply Lemma 4 which yields $(i)$ and $(ii)$. □

**Proposition 5.** ([15]) *In reflexive Banach spaces the maximal monotone operators coincide with the maximal monotone operators of type* $(D)$.

The last result we give here carries the $3^*$-monotonicity from $T$ to $T_A$.

**Proposition 6.** *If* $T : Y \to 2^{Y^*}$ *is* $3^*$*-monotone and* $A : X \to Y$ *is a linear continuous mapping, then* $T_A$ *is* $3^*$*-monotone, too.*

**Proof.** Take $x^* \in R(T_A)$, i.e. there is some $z \in X$ such that $x^* \in A^* \circ T \circ A(z)$. Thus there exists a $z^* \in T \circ A(z)$ satisfying $x^* = A^*z^*$. Clearly, $z^* \in R(T)$. Consider also an $x \in D(T_A)$ and denote $u = Ax \in D(T)$. When $y^* \in T_A(y)$ there is some $t^* \in T \circ A(y)$ such that $y^* = A^*t^*$. We have

$$\begin{aligned}
\inf_{y^*\in T_A(y)}\langle x^* - y^*, x - y\rangle &= \inf_{t^*\in T\circ A(y)}\langle A^*z^* - A^*t^*, x - y\rangle \\
&= \inf_{t^*\in T\circ A(y)}\langle z^* - t^*, A(x-y)\rangle \\
&\geq \inf_{t^*\in T(v)}\langle z^* - t^*, u - v\rangle \geq \beta(z^*, u) \in \mathbb{R},
\end{aligned}$$

as $T$ is $3^*$-monotone. Therefore $T_A$ is $3^*$-monotone, too. □

## 4.2 Rectifying and extending Riahi's results

We give here the main results in this section concerning the so-called Brézis-Haraux-type approximation (cf. [19]) of the range of the operator $T_A$, respectively of the subdifferential of the precomposition of a linear continuous mapping with a proper convex lower-semicontinuous function. Some results related to them were obtained by Pennanen in [13], but in reflexive spaces.

**Theorem 3.** *If* $T : Y \to 2^{Y^*}$ *is* $3^*$*-monotone and* $A : X \to Y$ *is a linear continuous mapping such that* $T_A$ *is of type* $(D)$*, then*

*(i)* $\mathrm{cl}(A^*(R(T))) = \mathrm{cl}(R(T_A))$,

*(ii)* $\mathrm{int}(R(T_A)) \subseteq \mathrm{int}(A^*(R(T))) \subseteq \mathrm{int}(R(\overline{T_A}))$.

**Proof.** By Proposition 4($i$) we have also $\mathrm{cl}(A^*(R(T))) \subseteq \mathrm{cl}(R(T_A))$ and $\mathrm{int}(A^*(R(T))) \subseteq \mathrm{int}(R(\overline{T_A}))$. Take some $x^* \in R(T_A)$. Then there are some $x \in X$ and $y^* \in T \circ A(x) \subseteq R(T)$ such that $x^* = A^* y^*$. Thus $x^* \in A^*(R(T))$, so $R(T_A) \subseteq A^*(R(T))$, so the same inclusion stands also between the closures, respectively the interiors, of these sets. Relations ($i$) and ($ii$) follow immediately by Proposition 4. □

*Remark 6.* The previous statement generalizes Theorem 1 in [16], which can be obtained for $Y = X \times X$, $Ax = (x, x)$ and $T = (T_1, T_2)$. The next consequence extends Corollary 1 in [16] which arises for the same choice of $Y$, $A$ and $T$.

**Corollary 2.** *Assume $X$ moreover reflexive and let $T : Y \to 2^{Y^*}$ be a $3^*$-monotone and $A : X \to Y$ a linear continuous mapping such that $T_A$ is maximal monotone. Then one has $\mathrm{cl}(A^*(R(T))) = \mathrm{cl}(R(T_A))$ and $\mathrm{int}(R(T_A)) = \mathrm{int}(A^*(R(T)))$.*

**Proof.** As $X$ is reflexive, Proposition 5 yields that $T_A$ is maximal monotone of type ($D$) and according to [15] we have that $\overline{T_A}$ and $T_A$ coincide. We apply Theorem 3 which yields the conclusion. □

The next statement generalizes Corollary 2 in [16], providing moreover a weaker constraint qualification under which one can assert the Brézis-Haraux-type approximation of the range of the sum of the subdifferentials of two proper convex lower-semicontinuous functions. First we give the constraint qualification that guarantees our more general result,

($\overline{CQ}$) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

**Theorem 4.** *Let the proper convex lower-semicontinuous function $f : Y \to \overline{\mathbb{R}}$ and the linear continuous operator $A : X \to Y$ such that $f \circ A$ is proper, and assume ($\overline{CQ}$) valid. Then one has*

*(i)* $\mathrm{cl}(A^*(R(\partial f))) = \mathrm{cl}(R(\partial(f \circ A)))$,

*(ii)* $\mathrm{int}(R(\partial(f \circ A))) \subseteq \mathrm{int}(A^*(R(\partial f))) \subseteq \mathrm{int}(D(\partial(A^* f^*)))$.

**Proof.** By Corollary 1($ii$) we know that ($\overline{CQ}$) implies $A^* \circ \partial f \circ A = \partial(f \circ A)$. Again, $f \circ A$ is proper, convex and lower-semicontinuous, so by Théoréme 3.1 in [8] we know that $\partial(f \circ A)$ is an operator of type ($D$), while according to Theorem $B$ in [17] (see also [13], [16]) $\partial f$ is $3^*$-monotone. Applying Theorem 3 for $T = \partial f$ we get $\mathrm{cl}(A^*(R(\partial f))) = \mathrm{cl}(R(A^* \circ \partial f \circ A))$, which with ($\overline{CQ}$) yields ($i$), and $\mathrm{int}(R(A^* \circ \partial f \circ A)) \subseteq \mathrm{int}(A^*(R(\partial f))) \subseteq \mathrm{int}(R(\overline{A^* \circ \partial f \circ A}))$. Using ($\overline{CQ}$) the latter becomes

$$\mathrm{int}(R(\partial(f \circ A))) \subseteq \mathrm{int}(A^*(R(\partial f))) \subseteq \mathrm{int}(R(\overline{\partial(f \circ A)})). \tag{10}$$

As from Corollary 1$(i)$ one may deduce that under $(\overline{CQ})$ $A^*f^* = (f \circ A)^*$, by Lemma 35.2 in [19] we get $R(\overline{\partial(f \circ A)}) = D(\partial(f \circ A)^*) = D(\partial(A^*f^*))$. Taking this into (10) we get $(ii)$. $\square$

When one takes $Y = X \times X$, $Ax = (x, x)$ and $f(x, y) = g(x) + h(y)$, where $x, y \in X$, the constraint qualification $(\overline{CQ})$ becomes (cf. [2])

$(\overline{CQ}^s)$ $\mathrm{epi}(g^*) + \mathrm{epi}(h^*)$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

**Corollary 3.** *Let $g$ and $h$ be two proper convex lower-semicontinuous functions on the Banach space $X$ with extended real values. Assume $(\overline{CQ}^s)$ satisfied. Then one has*

*(i)* $\mathrm{cl}(R(\partial g) + R(\partial h)) = \mathrm{cl}(R(\partial(g + h)))$,

*(ii)* $\mathrm{int}(R(\partial(g + h))) \subseteq \mathrm{int}(R(\partial g) + R(\partial h)) \subseteq \mathrm{int}(D(\partial(g^* \Box h^*)))$.

A similar result has been obtained by Riahi in Corollary 2 in [16]. There he said that under the constraint qualification

$$(\overline{CQ}_R) \qquad \bigcup_{t>0} t(\mathrm{dom}(g) - \mathrm{dom}(h)) \text{ is a closed linear subspace of } X,$$

one gets $\mathrm{cl}(R(\partial g) + R(\partial h)) = \mathrm{cl}(R(\partial(g + h)))$ and $\mathrm{int}(R(\partial g) + R(\partial h)) = \mathrm{int}(D(\partial (g^* \Box h^*)))$.

We prove that the latter is not always true when $(\overline{CQ}_R)$ stands. For a proper, convex and lower-semicontinuous function $g : X \to \overline{\mathbb{R}}$ Riahi's relation would become $\mathrm{int}(R(\partial g)) = \mathrm{int}(D(\partial g^*))$, which is equivalent, by Lemma 35.2 in [19] to

$$\mathrm{int}(R(\partial g)) = \mathrm{int}(R(\overline{\partial g})). \tag{11}$$

From Théoréme 3.1 in [8] we have that $\partial g$ is a monotone operator of type $(D)$ and it is also known that it is maximal monotone, too. According to Simons ([20]) $\partial g$ is also of type $(NI)$. Finally, by Theorem 20 in the same paper, we get that $\mathrm{int}(R(\overline{\partial g}))$ is convex, so (11) yields $\mathrm{int}(R(\partial g))$ convex. Unfortunately this is not always true, as Example 2.21 in [15], originally given by Fitzpatrick, shows. Take $X = c_0$, which is a Banach space with the usual supremum norm, and $g(x) = \|x\| + \|x - (1, 0, 0, ...)\|$, a proper, convex and continuous function on $c_0$. Skipping the calculatory details, it follows that $\mathrm{int}(R(\partial g))$ is not convex, unlike $\mathrm{int}\, R(\overline{\partial g})$. Thus (11) is false and the same happens to Riahi's allegation.

*Remark 7.* As proven in Proposition 3.1 in [3] (see also [2]), $(\overline{CQ}_R)$ implies $(\overline{CQ}^s)$, but the converse is not true, as shown by Example 3.1 in the same paper. Therefore our Corollary 3 extends, by weakening the constraint qualification, and corrects Corollary 2 in [16].

## 5 Conclusions

Given a maximal monotone operator $T$ on the reflexive Banach space $Y$ and the linear continuous operator $A : X \to Y$, where $X$ is a reflexive Banach space, too, we give a sufficient condition for the maximal monotonicity of $A^* \circ T \circ A$ weaker than the generalized interior-point regularity conditions known to us from the literature. Moreover, when $Y$, $A$ and $T$ are chosen such that the assertion turns into the maximal monotonicity of the sum of two maximal monotone operators on a reflexive Banach space $X$, we prove that our constraint qualification is actually equivalent to the weakest condition guaranteeing the mentioned result known to us from the literature. In the second part of the paper, where we work in non-reflexive Banach spaces, we rectify and extend a result due to Riahi, giving a weak constraint qualification in order to assure the so-called Brézis-Haraux-type approximation of the range of $\partial(f \circ A)$, where $f : Y \to \overline{\mathbb{R}}$ is a proper convex lower-semicontinuous function. For a special choice of functions we prove that the corrected version of Riahi's results holds under a weaker constraint qualification than required by him.

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