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On Gap Functions for Equilibrium Problems

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Abstract: In this paper the construction of gap functions based on Fenchel duality is extended from finite-dimensional variational inequalities to equilibrium problems. Moreover, the proposed approach is applied to variational inequalities in a real Banach space.

Key words: equilibrium problem, Fenchel duality, weak regularity condition, gap functions, dual equilibrium problem, variational inequalities

AMS subject classification: 49N15, 58E35, 90C25

1 Introduction

Let X be a real topological vector space, $K \subseteq X$ be a nonempty closed and convex set. Assume that $f : K \times K \to \mathbb{R}$ is a function satisfying $f(x,x) = 0, \ \forall x \in K$. The equilibrium problem is to find $x \in K$ such that

$$(EP) \qquad f(x,y) \geq 0, \ \forall y \in K.$$

Since (EP) includes as special cases optimization problems, complementarity problems and variational inequalities (see [6]), some results for these problems have been extended to (EP) by several authors. In particular, the gap function approaches for solving variational inequalities (see for instance [2] and [16]) have been investigated for equilibrium problems in [5] and [14].
A function $\gamma : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be a gap function for (EP) [14, Defintion 2.1] if it satisfies the properties

(i) $\gamma(y) \geq 0, \ \forall y \in K$;

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(ii) $\gamma(x) = 0$ and $x \in K$ if and only if x is a solution for (EP).

Recently, in [1] the construction of gap functions for finite-dimensional variational inequalities has been related to the conjugate duality of an optimization problem. On the other hand, in [7] very weak sufficient conditions for Fenchel duality regarding convex optimization problems have been established in infinite dimensional spaces. The combination of both results allow us to propose new gap functions for (EP) based on Fenchel duality.
This paper is organized as follows. In section 2 we give some definitions and introduce the weak sufficient condition for the strong duality related to Fenchel duality. In the next section we propose some new functions by using Fenchel duality and we show that under certain assumptions they are gap functions for (EP). Section 4 summarizes early investigated gap functions for (EP). At the end the proposed approach is applied to variational inequalities in a real Banach space.

2 Mathematical preliminaries

Let X be a real locally convex space and X^* be its topological dual, the set of all continuous linear functionals over X. By $\langle x^*, x\rangle$ we denote the value of $x^* \in X^*$ at $x \in X$. For the nonempty set $C \subseteq X$, the indicator function $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

while the support function is $\sigma_C(x^*) = \sup\limits_{x\in C}\langle x^*, x\rangle$. Considering now a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\text{dom } f = \Big\{x \in X |\ f(x) < +\infty\Big\}$ its effective domain and by

$$\text{epi } f = \Big\{(x, r) \in \text{dom } f \times \mathbb{R} |\ f(x) \leq r\Big\}$$

its epigraph. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called proper if $\text{dom } f \neq \emptyset$. The (Fenchel-Moreau) conjugate function of f is $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(p) = \sup_{x\in X}[\langle p, x\rangle - f(x)].$$

Definition 2.1 Let the functions $f_i : X \to \mathbb{R} \cup \{+\infty\},\ i = 1, ..., m,$ be given. The function $f_1\square\cdots\square f_m : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f_1\square\cdots\square f_m(x) = \inf\Big\{\sum_{i=1}^{m} f_i(x_i) |\ \sum_{i=1}^{m} x_i = x\Big\}$$

is called the infimal convolution function of $f_1, ..., f_m$. The infimal convolution $f_1 \Box \cdots \Box f_m$ is called to be exact at $x \in X$ if there exist some $x_i \in X$, $i = 1, ..., m$, such that $\sum\limits_{i=1}^{m} x_i = x$ and

$$f_1 \Box \cdots \Box f_m(x) = f_1(x_1) + ... + f_m(x_m).$$

Furthermore, we say that $f_1 \Box \cdots \Box f_m$ is exact if it is exact at every $x \in X$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions such that dom $f \cap$ dom $g \neq \varnothing$. We consider the following optimization problem

$$(P) \qquad \inf_{x \in X} \Big\{ f(x) + g(x) \Big\}.$$

The Fenchel dual problem to (P) is

$$(D) \qquad \sup_{p \in X^*} \Big\{ -f^*(-p) - g^*(p) \Big\}.$$

In [7] a new weaker regularity condition has been introduced in a more general case in order to guarantee the existence of strong duality between a convex optimization problem and its Fenchel dual, namely that the optimal objective values of the primal and the dual are equal and the dual has an optimal solution. This regularity condition for (P) can be written as

(FRC) $\quad f^* \Box g^*$ is lower semicontinuous and

$$\text{epi } (f^* \Box g^*) \cap \Big(\{0\} \times \mathbb{R}\Big) = \Big(\text{epi}(f^*) + \text{epi}(g^*)\Big) \cap \Big(\{0\} \times \mathbb{R}\Big),$$

or, equivalently,

(FRC) $\quad f^* \Box g^*$ is a lower semicontinuous function and exact at 0.

Under this assumption, the following theorem states the existence of strong duality between (P) and (D).

Proposition 2.1 *Let (FRC) be fulfilled. Then $v(P) = v(D)$ and (D) has an optimal solution.*

Remark that considering the perturbation function $\Phi : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(x, z) = f(x) + g(x + z)$, one can obtain the Fenchel dual (D).

Indeed, the function Φ fulfills $\Phi(x,0) = f(x) + g(x)$, $\forall x \in X$ and choosing (D) as being

$$(D) \qquad \sup_{p \in X^*} \Big\{ -\Phi^*(0,p) \Big\}$$

(cf. [10]), this problem becomes actually the well-known Fenchel dual problem.

3 Gap functions based on Fenchel duality

In this section we consider the construction of gap functions for (EP) by using a similar approach that has been applied to gap functions for finite-dimensional variational inequalities (see [1]). Here, the Fenchel duality will play an important role. We assume that X is a real locally convex space. Let $x \in K$ be given. Then (EP) can be reduced to the optimization problem

$$(P^{EP}; x) \qquad \inf_{y \in K} f(x,y),$$

or, equivalently,

$$(P^{EP}; x) \qquad \inf_{y \in X} \widetilde{f}(x,y),$$

where

$$\widetilde{f}(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in K \times K; \\ +\infty, & \text{otherwise} \end{cases}$$

is the extension of f on $X \times X$. We mention that $x^* \in K$ is a solution of (EP) if and only if it is a solution of $(P^{EP}; x^*)$. Now let us reformulate $(P^{EP}; x)$ using the indicator function $\delta_K(y)$ as

$$(P^{EP}; x) \qquad \inf_{y \in X} \Big\{ \widetilde{f}(x,y) + \delta_K(y) \Big\}.$$

Then we can write the Fenchel dual to $(P^{EP}; x)$ as being

$$\begin{aligned} (D^{EP}; x) \qquad & \sup_{p \in X^*} \Big\{ -\sup_{y \in X} [\langle p, y\rangle - \widetilde{f}(x,y)] - \delta_K^*(-p) \Big\} \\ = \; & \sup_{p \in X^*} \Big\{ -\sup_{y \in K} [\langle p, y\rangle - f(x,y)] - \delta_K^*(-p) \Big\} \\ = \; & \sup_{p \in X^*} \Big\{ -f_{y,K}^*(x,p) - \delta_K^*(-p) \Big\}, \end{aligned}$$

where $f_{y,K}^*(x,p) := \sup_{y \in K} [\langle p, y\rangle - f(x,y)]$ is the partial conjugate of f with respect to the variable y and fixed x. Further, by $v(P)$ we denote the optimal objective value of the optimization problem (P). Let us introduce the following function for any $x \in K$

$$\gamma_F^{EP}(x) := -v(D^{EP};x) = -\sup_{p\in X^*}\Big\{-f^*_{y,K}(x,p) - \delta^*_K(-p)\Big\}$$
$$= \inf_{p\in X^*}\Big\{f^*_{y,K}(x,p) + \sigma_K(-p)\Big\}.$$

For $(P^{EP};x)$, the regularity condition (FRC) can be written as follows

$(FRC^{EP};x)$ $\quad f^*_{y,K}(x,\cdot)\Box\sigma_K$ is a lower semicontinuous function and exact at 0.

Theorem 3.1 *Assume that $\forall x \in K$ the regularity condition $(FRC^{EP};x)$ is fulfilled. Let for each $x \in K$, $y \mapsto f(x,y)$ be convex and lower semicontinuous. Then γ_F^{EP} is a gap function for (EP).*

Proof:

(i) By weak duality it holds

$$v(D^{EP};x) \leq v(P^{EP};x) \leq 0.$$

Namely, one has $\gamma_F^{EP}(x) = -v(D^{EP};x) \geq 0,\ \forall x \in K$.

(ii) If $\bar{x}$ is a solution of (EP), then $v(P^{EP};\bar{x}) = 0$. On the other hand, by Proposition 2.1 the strong duality between $(P^{EP};x)$ and $(D^{EP};x)$ holds. In other words

$$v(D^{EP};\bar{x}) = v(P^{EP};\bar{x}) = 0.$$

That means $\gamma_F^{EP}(\bar{x}) = 0$. Conversely, let $\gamma_F^{EP}(\bar{x}) = 0$. Then

$$0 = v(D^{EP};\bar{x}) \leq v(P^{EP};\bar{x}) \leq 0.$$

Therefore $\bar{x}$ is a solution of (EP). □

It is well known that (EP) is closely related to the so-called dual equilibrium problem (cf. [12]), find $x \in K$ such that

$$(DEP) \qquad f(y,x) \leq 0,\ \forall y \in K,$$

or, equivalently,

$$(DEP) \qquad -f(y,x) \geq 0,\ \forall y \in K.$$

By K^{EP} and K^{DEP} we denote the solution sets of problems (EP) and (DEP), respectively. In order to suggest another gap function for (EP) we need some definitions and results.

Definition 3.1 [12, Definition 2.1]
The bifunction $f : K \times K \to \mathbb{R}$ is said to be

(i) monotone if, for each pair of points $x, y \in K$, we have

$$f(x, y) + f(y, x) \leq 0;$$

(ii) pseudomonotone if, for each pair of points $x, y \in K$, we have

$$f(x, y) \geq 0 \text{ implies } f(y, x) \leq 0.$$

Definition 3.2 [12, Definition 2.2]
A function $\varphi : K \to \mathbb{R}$ is said to be

(i) quasiconvex if, for each pair of points $x, y \in K$ and for all $\alpha \in [0, 1]$, we have

$$\varphi(\alpha x + (1 - \alpha)y) \leq \max\left\{\varphi(x), \varphi(y)\right\};$$

(ii) explicitly quasiconvex if it is quasiconvex and for each pair of points $x, y \in K$ such that $\varphi(x) \neq \varphi(y)$ and for all $\alpha \in (0, 1)$, we have

$$\varphi(\alpha x + (1 - \alpha)y) < \max\left\{\varphi(x), \varphi(y)\right\}.$$

A function $\varphi : K \to \mathbb{R}$ is said to be (explicitly) quasiconcave if $-\varphi$ is (explicitly) quasiconvex.

Definition 3.3 [12, Definition 2.3]
A function $\varphi : K \to \mathbb{R}$ is said to be u-hemicontinuous if, for all $x, y \in K$ and $\alpha \in [0, 1]$, the function $\tau(\alpha) = \varphi(\alpha x + (1 - \alpha)y)$ is upper semicontinuous at 0.

Proposition 3.1 [12, Proposition 2.1]

(i) *If f is pseudomonotone, then $K^{EP} \subseteq K^{DEP}$.*

(ii) *If $f(\cdot, y)$ is u-hemicontinuous, $\forall y \in K$ and $f(x, \cdot)$ is explicitly quasiconvex $\forall x \in K$, then $K^{DEP} \subseteq K^{EP}$.*

By using (DEP), in the same way as before, we introduce a new gap function for (EP). Let $x \in K$ be a solution of (DEP). This is equivalent to that x is a solution to the optimization problem

$$(P^{DEP}; x) \qquad \inf_{y \in K} [-f(y, x)],$$

which turns out to be

$$(P^{DEP}; x) \qquad \inf_{y \in X} \widehat{f}(x, y)$$

with the extended function

$$\widehat{f}(x, y) = \begin{cases} -f(y, x), & \text{if } (x, y) \in K \times K; \\ +\infty, & \text{otherwise.} \end{cases}$$

The corresponding Fenchel dual problem for $(P^{DEP}; x)$ is

$$\begin{aligned} (D^{DEP}; x) \qquad & \sup_{p \in X^*} \Big\{ -\sup_{y \in X} [\langle p, y \rangle - \widehat{f}(x, y)] - \delta_K^*(-p) \Big\} \\ = \ & \sup_{p \in X^*} \Big\{ -\sup_{y \in K} [\langle p, y \rangle + f(y, x)] - \delta_K^*(-p) \Big\}, \end{aligned}$$

if we again rewrite $(P^{DEP}; x)$ using δ_K similar as done above for $(P^{EP}; x)$. Let us define the function

$$\begin{aligned} \gamma_F^{DEP}(x) : &= -v(D^{DEP}; x) \\ &= -\sup_{p \in X^*} \Big\{ -\sup_{y \in K} [\langle p, y \rangle + f(y, x)] - \delta_K^*(-p) \Big\} \\ &= \inf_{p \in X^*} \Big\{ \sup_{y \in K} [\langle p, y \rangle + f(y, x)] + \sigma_K(-p) \Big\}. \end{aligned}$$

Proposition 3.2 *Let $f : K \times K \to \mathbb{R}$ be a monotone bifunction. Then it holds*

$$\gamma_F^{DEP}(x) \le \gamma_F^{EP}(x), \ \forall x \in K.$$

Proof: By the monotonicity of f, we have

$$f(x, y) + f(y, x) \le 0, \ \forall x, y \in K,$$

or, equivalently, $f(y, x) \le -f(x, y), \ \forall x, y \in K$. Let $p \in X^*$ be fixed. Adding $\langle p, y \rangle$ and taking the supremum in both sides over all $y \in K$ yields

$$\sup_{y \in K} [\langle p, y \rangle + f(y, x)] \le \sup_{y \in K} [\langle p, y \rangle - f(x, y)].$$

After adding $\sigma_K(-p)$ and taking the infimum in both sides over $p \in X^*$, we conclude that $\gamma_F^{DEP}(x) \leq \gamma_F^{EP}(x),\ \forall x \in K.$ □

Theorem 3.2 *Let the assumptions of Theorem* 3.1 *and Proposition* 3.1(*ii*) *be fulfilled. Assume that* $f : K \times K \to \mathbb{R}$ *is a monotone bifunction. Then* γ_F^{DEP} *is a gap function for* (EP).

Proof:

(i) By weak duality it holds

$$\gamma_F^{DEP}(x) = -v(D^{DEP}; x) \geq -v(P^{DEP}; x) \geq 0,\ \forall x \in K.$$

(ii) Let $\bar{x}$ be a solution of $(EP.)$ By Theorem 3.1, $\bar{x}$ is solution of (EP) if and only if $\gamma_F^{EP}(\bar{x}) = 0$. In view of (i) and Proposition 3.2, we get

$$0 \leq \gamma_F^{DEP}(\bar{x}) \leq \gamma_F^{EP}(\bar{x}) = 0.$$

Whence $\gamma_F^{DEP}(\bar{x}) = 0$. Let now $\gamma_F^{DEP}(\bar{x}) = 0$. By weak duality it holds

$$0 = v(D^{DEP}; \bar{x}) \leq v(P^{DEP}; \bar{x}) \leq 0.$$

Consequently $v(P^{DEP}; \bar{x}) = 0$. That means $\bar{x} \in K^{DEP}$. Hence, according to Proposition 3.1(ii), $\bar{x}$ is a solution of (EP). □

4 Regularized gap functions

The current section purposes to summarize early investigated gap functions for (EP) (cf. [5] and [14]) in the same way as in Section 3. Throughout this section we assume that X is a real reflexive Banach space and $h : K \times K \to \mathbb{R}$ is a bifunction such that for each $x \in K,\ y \mapsto h(x, y)$ is convex, differentiable and

(a) $h(x, y) \geq 0,\ \forall x, y \in K;$

(b) $h(x, x) = 0,\ \forall x \in K;$

(c) $h'_y(x, x) = 0,\ \forall x \in K$, where h'_y means the derivative of h in the sense of Gâteaux (cf. Definition 4.1) with respect to the second variable.

Definition 4.1 [13] A functional $g : X \to \mathbb{R}$ is said to be differentiable (in the sense of Gâteaux) at the point $x \in X$ if there exists $g'(x) \in X^*$ such that

$$\lim_{t \to 0} \frac{g(x + th) - g(x)}{t} = \langle g'(x), h \rangle$$

is finite.

Proposition 4.1 [13, cf. Proposition 2.1]
Let $f(x, y)$ be a convex, differentiable bifunction with respect to y and $h(x, y)$ be a function fulfilling the conditions $(a) - (c)$. Then $\bar{x}$ is a solution of (EP) if and only if it is a solution of the auxiliary equilibrium problem, find $\bar{x} \in K$ such that

$$(EP_h) \qquad f(\bar{x}, y) + h(\bar{x}, y) \geq 0, \ \forall y \in K.$$

Proof: Since in [13] has been used the alternative formulation, namely the variables were exchanged in (EP), let us show how the proof looks at our case. Indeed, it is clear that $\bar{x}$ is a solution of (EP), then it is also a solution of (EP_h). Let $\bar{x}$ be a solution of (EP_h). Then $\bar{x}$ is a solution of the optimization problem

$$\inf_{y \in K} [f(\bar{x}, y) + h(\bar{x}, y)]. \tag{4.1}$$

Since K is convex, $\bar{x}$ is a solution of (4.1) if and only if

$$\langle f'_y(\bar{x}, \bar{x}) + h'_y(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0, \ \forall y \in K,$$

or, equivalently,

$$\langle f'_y(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0, \ \forall y \in K.$$

In view of the convexity of $f(\bar{x}, \cdot)$ we obtain

$$f(\bar{x}, y) - f(\bar{x}, \bar{x}) \geq \langle f'_y(\bar{x}, \bar{x}), y - \bar{x} \rangle \geq 0, \ \forall y \in K.$$

That means $f(\bar{x}, y) \geq 0, \ \forall y \in K$. □

Corollary 4.1 *Let $f(x, y)$ be a concave, differentiable bifunction with respect to x. Then $\bar{x}$ is a solution of (DEP) if and only if it is a solution of the dual auxiliary equilibrium problem, find $\bar{x} \in K$ such that*

$$(DEP_h) \qquad - f(y, \bar{x}) + h(\bar{x}, y) \geq 0, \ \forall y \in K.$$

Proof: Since $-f(x, y)$ is convex and differentiable with respect to x, choosing $-f(y, x)$ instead of $f(x, y)$, we can apply Proposition 4.1. □

In [5], the authors proposed the following gap function for (EP)

$$\gamma_h^{EP}(x) := \sup_{y \in K} [-f(x, y) - h(x, y)],$$

while instead of (c) was taken the condition

($\bar{c}$) $h(x,(1-\lambda)x+\lambda y)=o(\lambda),\ \lambda\in[0,1]$.

Gap functions of such type have been investigated also for finite-dimensional variational inequalities (see, for instance [2],[8] and [16]), whose important property under certain assumptions is the differentiability. Recently, in a finite-dimensional space, the differentiability of such type of a gap function for (EP) has been considered in [14].

Theorem 4.1 *Let the assumptions of Proposition 4.1 be fulfilled. Then γ_h^{EP} is a gap function for (EP).*

Proof:

(i) $\gamma_h^{EP}(x)=\sup\limits_{y\in K}[-f(x,y)-h(x,y)]\geq -f(x,x)-h(x,x)=0.$

(ii) If $\bar{x}$ is a solution of (EP), then by (a) we have

$$f(\bar{x},y)+h(\bar{x},y)\geq 0,\ \forall y\in K.$$

Whence $\gamma_h^{EP}(\bar{x})=\sup\limits_{y\in K}[-f(\bar{x},y)-h(\bar{x},y)]\leq 0$. Therefore, by (i), we obtain $\gamma_h^{EP}(\bar{x})=0$. Let now $\gamma_h^{EP}(\bar{x})=0$. Consequently

$$f(\bar{x},y)+h(\bar{x},y)\geq 0,\ \forall y\in K.$$

By Proposition 4.1, this is true if and only if $f(\bar{x},y)\geq 0,\ \forall y\in K$. □

On the other hand, γ_h^{EP} is closely related to another function $\gamma_h^{DEP}:K\to\mathbb{R}\cup\{+\infty\}$ defined by (see [5])

$$\gamma_h^{DEP}(x):=\sup_{y\in K}[f(y,x)-h(x,y)].$$

Proposition 4.2 *Let $f:K\times K\to\mathbb{R}$ be a monotone bifunction. Then it holds*

$$\gamma_h^{DEP}(x)\leq\gamma_h^{EP}(x),\ \forall x\in K. \tag{4.2}$$

Proof: By the monotonicity of f, we have

$$f(x,y)+f(y,x)\leq 0,\ \forall x,y\in K,$$

or, equivalently,

$$f(y,x)\leq -f(x,y),\ \forall x,y\in K.$$

After adding $-h(x,y)$ and taking the infimum in both sides over $y\in K$, we conclude that $\gamma_h^{DEP}(x)\leq\gamma_h^{EP}(x),\ \forall x\in K$. □

Theorem 4.2 *Let $f : K \times K \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ be concave with respect to x and convex with respect to y. Assume that f is a monotone differentiable bifunction and the assumptions of Proposition 3.1(ii) are fulfilled. Then γ_h^{DEP} is a gap function for (EP).*

Proof:

(i) $\gamma_h^{DEP}(x) = \sup_{y \in K}[f(y,x) - h(x,y)] \geq f(x,x) - h(x,x) = 0.$

(ii) By Theorem 4.1 $\bar{x}$ is a solution of (EP) if and only if $\gamma_h^{EP}(\bar{x}) = 0$. According to (4.2) it holds

$$0 \leq \gamma_h^{DEP}(\bar{x}) \leq \gamma_h^{EP}(\bar{x}) = 0.$$

In other words $\gamma_h^{DEP}(\bar{x}) = 0$. Let now $\gamma_h^{DEP}(\bar{x}) = 0$. Then

$$-f(y,\bar{x}) + h(\bar{x},y) \geq 0, \ \forall y \in K.$$

Taking into account Corollary 4.1 and Proposition 3.1(ii) we conclude that $f(\bar{x},y) \geq 0, \ \forall y \in K$. □

5 Applications to variational inequalities

In this section we apply the approach proposed in Section 3 to variational inequalities in a real Banach space. Let us notice that the approach based on the conjugate duality including Fenchel one, has been first considered for finite-dimensional variational inequalities (cf. [1]). We assume that X is a real Banach space. Taking $f(x,y) := \langle F(x), y - x \rangle$, (EP) reduces to the variational inequality problem of finding $x \in K$ such that

$$(VI) \qquad \langle F(x), y - x \rangle \geq 0, \ \forall y \in K,$$

where $F : K \to X^*$ is a given mapping and $K \subseteq X$ is a closed and convex set. For $x \in K$, (VI) can be rewritten as the optimization problem

$$(P^{VI}; x) \qquad \inf_{y \in X} \Big\{ \langle F(x), y - x \rangle + \delta_K(y) \Big\},$$

in the sense that x is a solution of (VI) if and only if it is a solution of $(P^{VI}; x)$. In view of γ_F^{EP}, we introduce the function based on Fenchel duality for (VI) by

$$\begin{aligned}\gamma_F^{VI}(x) &= \inf_{p\in X^*}\left\{\sup_{y\in X}[\langle p,y\rangle - \langle F(x), y-x\rangle] + \sigma_K(-p)\right\}\\ &= \inf_{p\in X^*}\left\{\sup_{y\in X}\langle p-F(x),y\rangle + \sigma_K(-p)\right\} + \langle F(x),x\rangle.\end{aligned}$$

From

$$\sup_{y\in X}\langle p-F(x),y\rangle = \begin{cases} 0, & \text{if } p=F(x),\\ +\infty, & \text{otherwise,}\end{cases}$$

follows that

$$\gamma_F^{VI}(x) = \inf_{p=F(x)}\sup_{y\in K}\langle -p,y\rangle + \langle F(x),x\rangle = \sup_{y\in K}\langle F(x), x-y\rangle,$$

which turns out to be the so-called Auslender's gap function (see [1] and [3]).

The problem (VI) can be associated to the following variational inequality introduced by Minty, find $x\in K$ such that

$$(MVI) \qquad \langle F(y), y-x\rangle \geq 0,\ \forall y\in K.$$

Recall that setting $f(y,x) := \langle F(y), x-y\rangle$ in (DEP) we can obtain (MVI). As done in section 3, before we introduce another gap function for (VI), let us consider some definitions and assertions.

Definition 5.1 [11, 15] A mapping $F: K\to X^*$ is said to be

(i) monotone if, for each pair of points $x,y\in K$, we have

$$\langle F(y)-F(x), y-x\rangle \geq 0;$$

(ii) pseudo-monotone if, for each pair of points $x,y\in K$, we have

$$\langle F(x), y-x\rangle \geq 0 \text{ implies } \langle F(y), y-x\rangle \geq 0;$$

(iii) continuous on finite-dimensional subspaces if for any finite-dimensional subspace M of X with $K\cap M\neq\varnothing$ the restricted mapping $F: K\cap M\to X^*$ is continuous from the norm topology of $K\cap M$ to the weak* topology of X^*.

Proposition 5.1 [15, Lemma 3.1] *Let $F: K\to X^*$ be a pseudo-monotone mapping which is continuous on finite-dimensional subspaces. Then $x\in K$ is a solution of (VI) if and only if it is a solution of (MVI).*

As mentioned before, for given $x \in K$, (MVI) is equivalent to the optimization problem

$$(P^{MVI}; x) \qquad \inf_{y \in X} \widehat{f}_F(x, y),$$

where

$$\widehat{f}_F(x,y) = \begin{cases} \langle F(y), y - x \rangle, & \text{if } (x,y) \in K \times K; \\ +\infty, & \text{otherwise.} \end{cases}$$

Setting $f(y,x) := \langle F(y), x - y \rangle$ in γ_F^{DEP}, we have

$$\gamma_F^{MVI}(x) := \inf_{p \in X^*} \Big\{ \sup_{y \in K} [\langle p, y \rangle - \langle F(y), y - x \rangle] + \sigma_K(-p) \Big\}.$$

We remark that the monotonicity (pseudo-monotonicity) of F in the sense of Definition 5.1 implies that monotonicity (pseudo-monotonicity) of $f(x,y) = \langle F(x), y - x \rangle$ in the sense of Definition 3.1. Then by Proposition 3.2 we get the following assertion.

Proposition 5.2 *Let $F : K \to X^*$ be a monotone mapping. Then it holds*

$$\gamma_F^{MVI}(x) \le \gamma_F^{VI}(x), \ \forall x \in K.$$

Theorem 5.1 *Let $F : K \to X^*$ be a monotone mapping which is continuous on finite-dimensional subspaces. Then γ_F^{MVI} is a gap function for (VI).*

Proof:

(i) $\gamma_F^{DEP}(x) \ge 0$ implies that $\gamma_F^{MVI}(x) \ge 0, \ \forall x \in K$, as this is a special case.

(ii) By definition of a gap function, $\bar{x} \in K$ is a solution of (VI) if and only if $\gamma_F^{VI}(\bar{x}) = 0$. Taking into account (i) and Proposition 5.2, one has

$$0 \le \gamma_F^{MVI}(\bar{x}) \le \gamma_F^{VI}(\bar{x}) = 0.$$

In other words, $\gamma_F^{MVI}(\bar{x}) = 0$. Let now $\gamma_F^{MVI}(\bar{x}) = 0$. Then by weak duality, we can easily see that $\bar{x} \in K$ is a solution of (MVI). This follows using an analogous argumentation as in the proof of Theorem 3.2. Whence, according to Proposition 5.1, $\bar{x}$ solves (VI). □

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