Partial Realization of Descriptor Systems*

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Abstract

We address the partial realization problem for linear descriptor systems. A general solution to this problem using the Markov parameters of the system defined via its Laurent series is provided. For proper descriptor systems, we also discuss a numerically feasible algorithm for computing a partial minimal realization based on the unsymmetric Lanczos process. Applications to model reduction for two examples from computational fluid dynamics and mechanical systems with holonomic constraints are also given.

Keywords. Partial realization, descriptor system, model reduction, Markov parameter, Krylov subspace method.

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1 Introduction

In recent years there has been a lot of interest in model reduction methods based on Krylov subspace methods, see, e.g., [Ant05, ASG01, Bai02, Fre00, Fre03]. One of the reasons here is that many large-scale models are described by linear differential equations with sparse coefficient matrices. Such systems arise, for example, from semidiscretization of partial differential equations by the finite-element or finite-difference methods or represent RCL circuits. Krylov subspace methods can often handle such problems very efficiently. The model reduction methods based on Krylov subspaces can be considered as methods of approximation at certain frequencies. If the frequencies of interest are finite the approximation problem is called *moment-matching* problem. In case of approximation at infinity one speaks of *partial realization*. The term partial realization originates from identifying linear systems from given input-output data, see, e.g., [GL83, Kal79] for the standard theory and [FH00, HMH01] for extensions involving descriptor systems. For model reduction purposes, partial realization is usually used in a somewhat different (though mathematically related) context.

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In order to formulate the partial realization problem considered here more precisely, consider a descriptor system

$$\begin{array}{rcl} E\,\dot{x}(t) & = & A\,x(t) \,+\, B\,u(t), \\ y(t) & = & C\,x(t), \end{array} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the descriptor variables, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ denote inputs and outputs, respectively, of the system, and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. We assume that the pencil $sE - A$ is regular, i.e., $\det(sE - A) \not\equiv 0$. Here, we will focus on the single-input, single-output case, i.e., $m = p = 1$. The system (1) describes the map $u \mapsto y$ between input and output in time-domain. In frequency-domain this map is given by $\mathbf{y}(s) = C(sE - A)^{-1}B\mathbf{u}(s)$, where $\mathbf{u}$, $\mathbf{y}$ are the Laplace transforms of the input and the output, respectively. The rational function $H(s) = C(sE-A)^{-1}B$ is called a transfer function of the system (1). The Laurent series of H at infinity is given by

$$H(s) = h_\nu s^\nu + \cdots + h_1 s + h_0 + h_{-1}s^{-1} + h_{-2}s^{-2} + \ldots,$$

see [Dai89]. The coefficients $h_\nu, \ldots, h_0, h_{-1}, h_{-2}, \ldots$ are called *Markov parameters* of the system (1). If $\nu = 0$, then H is called *proper*. The minimal partial realization problem can be stated as follows: For a given system (1) find a system

$$\begin{array}{rcl} E_r\,\dot{x}_r(t) & = & A_r\,x_r(t) \,+\, B_r\,u(t), \\ y_r(t) & = & C_r\,x_r(t), \end{array} \tag{2}$$

of order $r < n$ that matches the first $q = q(r)$ Markov parameters of the original system (1). Here $q(r)$ should be as large as possible. In Section 2 we show that it actually equals $q(r) = 2r + \nu + 1$.

The partial realization problem for standard systems with $E = I_n$ is treated in depth in the seminal paper [GL83]. For descriptor systems, it has been solved only for the very special case of descriptor systems with nonsingular matrix E, see [ASG01, Fre00, Gri97]. The case of a singular matrix E is considered to be an open problem [MS05]. Here, we will present a theoretical solution for the partial realization problem and we will also provide a numerical algorithm for computing a partial realization of a given proper descriptor system.

The rest of this paper is organized as follows. In the next section we prove the main result on partial realization for descriptor systems of arbitrary index. We also provide a numerically feasible method based on the unsymmetric Lanczos process to compute partial realizations for proper descriptor systems. In Section 3 we apply this algorithm to two examples and show how it can be used for model reduction.

2 Main result

In case of a descriptor system with a nonsingular matrix E a partial realization can be obtained as

$$\begin{array}{rcl} \dot{x}_r(t) & = & W_r^\top A V_r\,x_r(t) \,+\, W_r^\top B\,u(t), \\ y_r(t) & = & C V_r\,x_r(t), \end{array} \tag{3}$$

where the columns of $V_r, W_r \in \mathbb{R}^{n\times r}$ represent biorthogonal (that is, $W_r^\top V_r = I_r$) bases of the two Krylov subspaces

$$\mathcal{V}_r \;=\; \mathrm{span}\{\, E^{-1}B, E^{-1}AE^{-1}B, \ldots, (E^{-1}A)^{r-1}E^{-1}B \,\}$$

and

$$\mathcal{W}_r = \operatorname{span}\{ C^\top, A^\top E^{-\top} C^\top, \ldots, (A^\top E^{-\top})^{r-1} C^\top \}.$$

It is well-known (see [Fre00, Gri97, JK97]) that the reduced-order model (3) is a partial realization of the system (1), matching the first $2r$ Markov parameters.

Next we show that even if the matrix E is singular a partial realization can be computed analogously. First, we construct transformation matrices W_r and V_r in a way that is suitable for theoretical purposes. After that we will show that they can be computed effectively using certain Krylov subspaces.

It is well known, see, e.g., [Dai89], that every regular pencil $sE - A$ can be transformed to the Weierstraß canonical form. That is, there exist nonsingular matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix}, \tag{4}$$

where N is nilpotent. Introducing a new state variable $[x_1^\top(t) \;\, x_2^\top(t)]^\top = Q^{-1}x(t)$ we can re-write the system (1) as

$$\dot{x}_1(t) = Jx_1(t) + B_1u(t), \tag{5}$$

$$N\dot{x}_2(t) = x_2(t) + B_2u(t), \tag{6}$$

$$y(t) = C_1x_1(t) + C_2x_2(t). \tag{7}$$

The systems $[I, J, B_1, C_1]$ and $[N, I, B_2, C_2]$ are called the slow and the fast subsystems of (1), respectively, see [Dai89]. It is clear that their orders are n_f and n_∞, respectively.

Remark 2.1. The transfer functions of the slow and fast subsystems of (1) are equal to

$$H_{slow}(s) = h_{-1}s^{-1} + h_{-2}s^{-2} + \ldots,$$
$$H_{fast}(s) = h_\nu s^\nu + \cdots + h_1 s + h_0,$$

respectively. In fact, using the Weirstraß canonical form we obtain

$$\begin{aligned} H(s) = C(sE - A)^{-1}B &= CQ \begin{bmatrix} (sI - J)^{-1} & 0 \\ 0 & (sN - I)^{-1} \end{bmatrix} PB \\ &= C_1(sI - J)^{-1}B_1 + C_2(sN - I)^{-1}B_2. \end{aligned}$$

Thus

$$H_{slow}(s) = C_1(sI - J)^{-1}B_1$$

represents the strictly proper part of H and

$$H_{fast}(s) = C_2(sN - I)^{-1}B_2$$

is the polynomial part of it. If H is proper, the transfer function of the fast subsystem is just a constant h_0.

The main idea of construction a partial realization for descriptor systems can be shortly expressed as follows: split the system into fast and slow subsystems and then reduce the order of the slow subsystem.

In the sequel we will need the following representation of the matrices P and Q in accordance to the partitioning in (4): $P = [\, P_f^\top \; P_\infty^\top \,]^\top$ and $Q = [\, Q_f \; Q_\infty \,]$, where $P_f^\top, Q_f \in \mathbb{R}^{n \times n_f}$.

Before we formulate the main result we have to do some preparatory work. First, let us introduce a projection-like matrix

$$\mathcal{P} = Q \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} P. \tag{8}$$

Second, using $\mathcal{P}$ we define matrices $\mathcal{O}_r \in \mathbb{R}^{r\times n}$ and $\mathcal{R}_r \in \mathbb{R}^{n\times r}$ by

$$\mathcal{O}_r = \begin{bmatrix} C \\ C\mathcal{P}A \\ C(\mathcal{P}A)^2 \\ \vdots \\ C(\mathcal{P}A)^{r-1} \end{bmatrix} \quad \text{and} \quad \mathcal{R}_r = \begin{bmatrix} \mathcal{P}B, \, \mathcal{P}A\mathcal{P}B, \, \ldots, \, (\mathcal{P}A)^{r-1}\mathcal{P}B \end{bmatrix}.$$

Further, define a Hankel matrix $\mathcal{H}_r \in \mathbb{R}^{r\times r}$ by

$$\mathcal{H}_k = \mathcal{O}_r\mathcal{R}_r = \begin{bmatrix} h_{-1} & h_{-2} & \ldots & h_{-r} \\ h_{-2} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \vdots \\ h_{-r} & \ldots & \ldots & h_{-2r+1} \end{bmatrix}. \tag{9}$$

We assume that all the principle minors of $\mathcal{H}_r$ are not equal zero. This implies existence of a factorization $\mathcal{H}_r = LU$, where L and U are nonsingular lower and upper triangular matrices, respectively.

Using the above notation we the define matrices

$$W_r = [(L^{-1}\mathcal{O}_r\mathcal{P})^\top \;\; P_\infty^\top], \quad V_r = [\mathcal{R}_r U^{-1} \;\; Q_\infty].$$

Consider a system $[E_r, A_r, B_r, C_r]$ constructed as

$$[E_r, A_r, B_r, C_r] = [W_r^\top E V_r, W_r^\top A V_r, W_r^\top B, C V_r]. \tag{10}$$

Theorem 2.2. *The system* (10) *has the same* $2r+\nu+1$ *Markov parameters* $h_\nu, h_{\nu-1}, \ldots, h_{-2r}$ *as the original system* (1)*.*

Proof. The Markov parameters $h_{-1}, h_{-2}, h_{-3}, \ldots$ of the system $[E, A, B, C]$ coincide with the Markov parameters of the standard system $[I, J, P_f B, CQ_f]$. This follows from the fact that the transfer functions of these two systems have the same strictly proper part, see Remark 2.1. Introduce matrices $\tilde{W}_r, \tilde{V}_r \in \mathbb{R}^{n_f\times r}$ such that $[I, \tilde{W}_r^\top J\tilde{V}_r, \tilde{W}_r^\top P_f B, CQ_f\tilde{V}_r]$ is a minimal partial realization of $[I, J, P_f B, CQ_f]$. These matrices can be constructed as follows, see, e.g., [Gug03]. Build the observability matrix $\tilde{\mathcal{O}}_r$ and the controllability matrix $\tilde{\mathcal{R}}_r$ of the system $[I, J, P_f B, CQ_f]$ as

$$\tilde{\mathcal{O}}_r = \begin{bmatrix} CQ_f \\ CQ_f J \\ CQ_f J^2 \\ \vdots \\ CQ_f J^{r-1} \end{bmatrix} \quad \text{and} \quad \tilde{\mathcal{R}}_r = \begin{bmatrix} P_f B, \, JP_f B, \, \ldots, \, J^{r-1}P_f B \end{bmatrix}.$$

Their product $\tilde{\mathcal{H}}_r = \tilde{\mathcal{O}}_r\tilde{\mathcal{R}}_r$ equals $\mathcal{H}_r = \mathcal{O}_r\mathcal{R}_r$ because $[I, J, P_f B, CQ_f]$ is just the slow subsystem of $[E, A, B, C]$. Hence $\tilde{\mathcal{H}}_r$ admits an LU-factorization $\tilde{\mathcal{H}}_r = LU$ with the same factors L and U as for $\mathcal{H}_r$. Thus the matrices $\tilde{V}_r$ and $\tilde{W}_r$ are given by $\tilde{V}_r = \tilde{\mathcal{R}}_r U^{-1}$ and $\tilde{W}_r^\top = L^{-1}\tilde{\mathcal{O}}_r$. There are the following relations between the pairs $\tilde{\mathcal{O}}_r, \tilde{\mathcal{R}}_r$ and $\mathcal{O}_r, \mathcal{R}_r$:

$$\mathcal{R}_r = Q\begin{bmatrix} \tilde{\mathcal{R}}_r \\ 0 \end{bmatrix}, \quad \mathcal{O}_r Q = [\tilde{\mathcal{O}}_r \;\; *],$$

where $*$ stands for some matrix of appropriate size. These relations are straightforward to verify. Now, consider the system $[E_r, A_r, B_r, C_r]$. For the matrix

$$E_r = W_r^\top E V_r = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

we have

$$\begin{aligned} E_{11} &= L^{-1}\mathcal{O}_r\mathcal{P}E\mathcal{R}_r U^{-1} = L^{-1}\mathcal{O}_r Q\begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} PEQ\begin{bmatrix} \tilde{\mathcal{R}}_r \\ 0 \end{bmatrix} U^{-1} \\ &= L^{-1}\mathcal{O}_r Q\begin{bmatrix} \tilde{\mathcal{R}}_r \\ 0 \end{bmatrix} U^{-1} = L^{-1}\tilde{\mathcal{O}}_r\tilde{\mathcal{R}}_r U^{-1} = I, \end{aligned}$$

and $E_{22} = P_\infty E Q_\infty = N$. The matrices E_{12} and E_{21} are zero matrices of appropriate sizes, as can be readily verified. Analogously for the matrix

$$A_r = W_r^\top A V_r = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

we obtain

$$\begin{aligned} A_{11} &= L^{-1}\mathcal{O}_r\mathcal{P}A\mathcal{R}_r U^{-1} = L^{-1}\mathcal{O}_r Q\begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} PAQ\begin{bmatrix} \tilde{\mathcal{R}}_r \\ 0 \end{bmatrix} U^{-1} \\ &= L^{-1}\mathcal{O}_r Q\begin{bmatrix} J\tilde{\mathcal{R}}_r \\ 0 \end{bmatrix} U^{-1} = L^{-1}\tilde{\mathcal{O}}_r J\tilde{\mathcal{R}}_r U^{-1} = \tilde{W}_r^\top J\tilde{V}_r, \end{aligned}$$

and $A_{22} = P_\infty A Q_\infty = I$. The matrices A_{12} and A_{21} are also zero matrices of appropriate sizes. Thus the slow subsystem of $[E_r, A_r, B_r, C_r]$ coincides with the system realized by $[I, \tilde{W}_r^\top A\tilde{V}_r, \tilde{W}_r^\top P_f B, CQ_f\tilde{V}_r]$. But this system has the same $2r$ Markov parameters $h_{-1}, h_{-2}, \ldots, h_{-2r}$ as $[E, A, B, C]$. The fast subsystem of $[E_r, A_r, B_r, C_r]$ is by construction exactly the fast subsystem of the original system. The later guarantees the matching of the first $\nu + 1$ Markov parameters $h_\nu, \ldots, h_1, h_0$. $\square$

Remark 2.3. From now on we shall consider only proper systems. We make this assumption because otherwise the minimal partial realization problem is numerically ill-posed. In fact, suppose that the system (1) is improper. This implies that the Laurent series of its transfer function has a nonconstant polynomial part

$$H(s) = h_\nu s^\nu + \cdots + h_1 s + h_0 + h_{-1}s^{-1} + h_{-2}s^{-2} + \ldots,$$

for some $\nu \geq 1$. Due to inevitable round-off errors it is impossible to match the first Markov parameters perfectly. I.e., the transfer function of a reduced system would have the form

$$H_r(s) = (h_\nu + \delta_\nu)s^\nu + \cdots + (h_1 + \delta_1)s + (h_0 + \delta_0) + (h_{-1} + \delta_{-1})s^{-1} + \ldots,$$

for small $\delta_\nu, \delta_{\nu-1}, \ldots, \delta_{-q}$. Hence

$$H_r(s) - H(s) = \delta_\nu s^\nu + \cdots + \delta_1 s + \delta_0 + \delta_{-1} s^{-1} + \ldots$$

is of order $O(s^\nu)$ that is large for $s \gg 1$ which makes a reasonable numerical approximation of the transfer function at infinity impossible.

If the system (1) is proper, then the transfer function of its fast subsystem is just a constant $H_{fast}(s) \equiv h_0$. This allows us to construct the minimal partial realization of $[E, A, B, C]$ as

$$\begin{array}{rcl} \dot{x}_r(t) & = & W_r^\top A V_r \, x_r(t) \, + \, W_r^\top B \, u(t), \\ y_r(t) & = & C V_r \, x_r(t) + h_0 u(t), \end{array} \tag{11}$$

where $W_r^\top = L^{-1}\mathcal{O}_r\mathcal{P}$ and $V_r = \mathcal{R}_r U^{-1}$. As for the first Markov parameter h_0 it can be computed as

$$\begin{aligned} h_0 &= C(sE - A)^{-1}B - C(sI - \mathcal{P}A)^{-1}\mathcal{P}B \\ &= (h_0 + h_{-1}s^{-1} + h_{-2}s^{-2} + \ldots) - (h_{-1}s^{-1} + h_{-2}s^{-2} + \ldots), \end{aligned}$$

for an arbitrary s. The method for constructing the transformation matrices V_r and W_r described above is not suitable for implementation. There are some reasons for that. One of them is the necessity to compute the Markov parameters $h_{-i} = C(\mathcal{P}A)^{i-1}\mathcal{P}B$ explicitly. If $\mathcal{P}A$ has a dominating eigenvalue, then information corresponding to the rest of the spectrum can be lost during such computation, see [Gri97]. Another reason is that the Hankel matrix (9) is usually ill-conditioned. This implies loss of accuracy in its LU-factorization and hence in the transformation matrices V_r, W_r, see [Gri97]. Fortunately these transformation matrices can be constructed in a stable manner as biorthogonal bases of the Krylov subspaces

$$\begin{aligned} \mathcal{V}_r &= \mathrm{span}\{\, \mathcal{P}B, \mathcal{P}A\mathcal{P}B, (\mathcal{P}A)^2\mathcal{P}B, \ldots, (\mathcal{P}A)^{r-1}\mathcal{P}B \,\}, \\ \mathcal{W}_r &= \mathrm{span}\{\, C^\top, A^\top\mathcal{P}^\top C^\top, \ldots, (A^\top\mathcal{P}^\top)^{r-1}C^\top \,\}, \end{aligned}$$

using the biorthogonal Lanczos method. To prove this recall that the system $[I, \mathcal{P}A, \mathcal{P}B, C]$ has the same Markov parameters $h_{-1}, h_{-2}, h_{-3}, \ldots$ as the original system $[E, A, B, C]$. Moreover, due to properties of the biorthogonal Lanczos process the first $2r$ Markov parameters of $[I, \mathcal{P}A, \mathcal{P}B, C]$ and $[I, W_r^\top\mathcal{P}AV_r, W_r^\top\mathcal{P}B, CV_r]$ coincide, see, e.g., [Fre00]. This implies that the minimal partial realization (11) of $[E, A, B, C]$ can be obtained alternatively as

$$\begin{array}{rcl} \dot{x}_r(t) & = & W_r^\top \mathcal{P} A V_r \, x_r(t) \, + \, W_r^\top \mathcal{P} B \, u(t), \\ y_r(t) & = & C V_r \, x_r(t) + h_0 u(t). \end{array} \tag{12}$$

An algorithmic description of the procedure is provided in Algorithm 1.

An implementation of the biorthogonal Lanczos process becomes efficient as soon as we have a possibility to compute the matrix-vector product $\mathcal{P}Ax$ fast. In general this is not possible, even if the matrix A is sparse. But for some classes of descriptor systems such a product can be computed efficiently. Some examples of such systems are presented in the next section.

Algorithm 1 Computation of a minimal partial realization for descriptor system

Input: A descriptor system $[E, A, B, C]$; the desired order r of the reduced-order system.

Output: A minimal partial realization $[I, A_r, B_r, C_r, D_r]$ of $[E, A, B, C]$.

1: Compute the matrix A_r using the unsymmetric Lanczos process applied to the matrix $\mathcal{P}A$ and the starting vectors $\mathcal{P}B$, C.

2: Set $B_r = C^\top \mathcal{P} B e_1$ and $C_r = e_1^\top$, where $e_1 = [1, 0, \ldots, 0] \in \mathbb{R}^r$.

3: Set $D_r = C(sE - A)^{-1}B - C(sI - \mathcal{P}A)^{-1}\mathcal{P}B$, for some $s \in \mathbb{R}$.

3 Examples

The representation of the projection-like matrix (8) is not suitable for computing the matrix-vector product $\mathcal{P}Ax$ due to the following reason. Even if it were possible to compute the matrices Q and P accurately, $\mathcal{P}$ would be dense, making the Lanczos procedure inefficient. Instead we will use the following representation:

$$\mathcal{P} = P_r(EP_r + A(I - P_r))^{-1}, \tag{13}$$

where

$$P_r = Q \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

is the spectral projection onto the right deflating subspace of the pencil $sE - A$ corresponding to finite eigenvalues, see [Sty02]. The represention (13) can be easily verified by substitution. Here we consider two different models: a semidescretized Stokes equation and a constrained mass-spring system. Both examples have been taken from [MS05].

3.1 Semidiscretized Stokes Equation

Consider the instationary Stokes equation describing the flow of incompressible fluid

$$\begin{array}{rcll} \dfrac{\partial v}{\partial t} & = & \Delta v - \nabla \rho + f, & (\xi, t) \in \Omega \times (0, t_e), \\ 0 & = & \operatorname{div} v, & (\xi, t) \in \Omega \times (0, t_e) \end{array} \tag{14}$$

with appropriate initial and boundary conditions. Here $v(\xi, t) \in \mathbb{R}^d$ is the velocity vector ($d = 2$ or 3 is the dimension of the spatial domain), $\rho(\xi, t) \in \mathbb{R}$ is the pressure, $f(\xi, t) \in \mathbb{R}^d$ is the vector of external forces, $\Omega \subset \mathbb{R}^d$ is a bounded open domain and $t_e > 0$ is the endpoint of the time interval. The spatial discretization of the Stokes equation (14) by the finite difference method on a uniform staggered grid leads to a descriptor system

$$\begin{array}{rcl} \dot{\mathbf{v}}_h(t) & = & A_{11}\mathbf{v}_h(t) + A_{12}\boldsymbol{\rho}_h(t) + B_1 u(t), \\ 0 & = & A_{12}^\top \mathbf{v}_h(t) \qquad\qquad\quad + B_2 u(t), \\ y(t) & = & C_1 \mathbf{v}_h(t) + C_2 \boldsymbol{\rho}_h(t), \end{array} \tag{15}$$

where $\mathbf{v}_h(t) \in \mathbb{R}^{n_\mathbf{v}}$ and $\boldsymbol{\rho}_h(t) \in \mathbb{R}^{n_\rho}$ are the semidiscretized vectors of velocities and pressures, respectively, see [Ber90]. The matrix $A_{11} \in \mathbb{R}^{n_\mathbf{v} \times n_\mathbf{v}}$ is the discrete Laplace operator, while $-A_{12} \in \mathbb{R}^{n_\mathbf{v} \times n_\rho}$ and $-A_{12}^\top \in \mathbb{R}^{n_\rho \times n_\mathbf{v}}$ are, respectively, the discrete gradient and divergence operators. The matrices $B_1 \in \mathbb{R}^{n_\mathbf{v}}$, $B_2 \in \mathbb{R}^{n_\rho}$ and the control input $u(t) \in \mathbb{R}$ are designed here for experimental purposes and may result either from boundary conditions or external forces or both, $y(t)$ is an appropriately chosen output of the system. The order $n = n_{\mathbf{v}} + n_{\boldsymbol{\rho}}$ of system (15) depends on the level of refinement of the discretization and is usually very large. Note that the matrix coefficients in (15) given by

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & 0 \end{bmatrix}$$

are sparse and have a special block structure. Using this structure, the projection P_r onto the right deflating subspace of the pencil $sE - A$ can be computed as

$$P_r = \begin{bmatrix} \Pi & 0 \\ -(A_{12}^\top A_{12})^{-1} A_{12}^\top A_{11} \Pi & 0 \end{bmatrix},$$

where $\Pi = I - A_{12}(A_{12}^\top A_{12})^{-1} A_{12}^\top$ is the orthogonal projection onto $\mathrm{Ker}\,(A_{12}^\top)$ along $\mathrm{Im}\,(A_{12})$, see [Sty05]. The product $\mathcal{P}A$ in this case is given by

$$\mathcal{P}A = \begin{bmatrix} \Pi A_{11} \Pi & 0 \\ -(A_{12}^\top A_{12})^{-1} A_{12}^\top A_{11} \Pi A_{11} \Pi & 0 \end{bmatrix}. \tag{16}$$

This representation has been obtained from (13) and the fact that the solution of

$$(EP_r + A(I - P_r))\, x = Ab, \quad \text{or} \quad \begin{bmatrix} A_{11} + \Pi - \Pi A_{11} \Pi & A_{12} \\ A_{12}^\top & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \Pi(A_{11} - I)\Pi b_1 + b_1 \\ -(A_{12}^\top A_{12})^{-1} A_{12}^\top A_{11} \Pi (A_{11} - I)\Pi b_1 + b_2 \end{bmatrix}.$$

The latter identity can be verified by substitution. The spatial discretization of the Stokes equation (14) on the square domain $\Omega = [0,1] \times [0,1]$ by the finite difference method on a uniform staggered 30×30 grid leads to a problem of order $n = 2820$. In our experiments $B = [\,B_1^T,\ B_2^T\,]^T \in \mathbb{R}^n$ is chosen at random and we are interested in the first velocity component, i.e., $C = [\,1,\ 0, \ldots, 0\,] \in \mathbb{R}^n$. We approximate the semidiscretized Stokes system (14) by a model of order 10. The approximation error is shown in Figure 1. The figure shows that the approximation quality is very good for a wide range of frequencies although we only aim at matching Markov parameters, i.e., coefficients of the transfer function's Laurent series.

3.2 Constrained damped mass-spring system

Consider the holonomically constrained damped mass-spring system illustrated in Figure 2.

The ith mass m_i is connected to the $(i+1)$st mass by a spring and a damper with constants k_i and d_i, respectively, and also to the ground by a spring and a damper with constants κ_i and δ_i, respectively. Additionally, the first mass is connected to the last one by a rigid bar and it is influenced by the control $u(t)$. The vibration of this system is described by a descriptor system

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \mathbf{v}(t), \\ M\dot{\mathbf{v}}(t) &= -Q\,\mathbf{p}(t) - R\mathbf{v}(t) + G^T \boldsymbol{\lambda}(t) + B_2 u(t), \\ 0 &= G\,\mathbf{p}(t), \\ y(t) &= C_1 \mathbf{p}(t), \end{aligned} \tag{17}$$

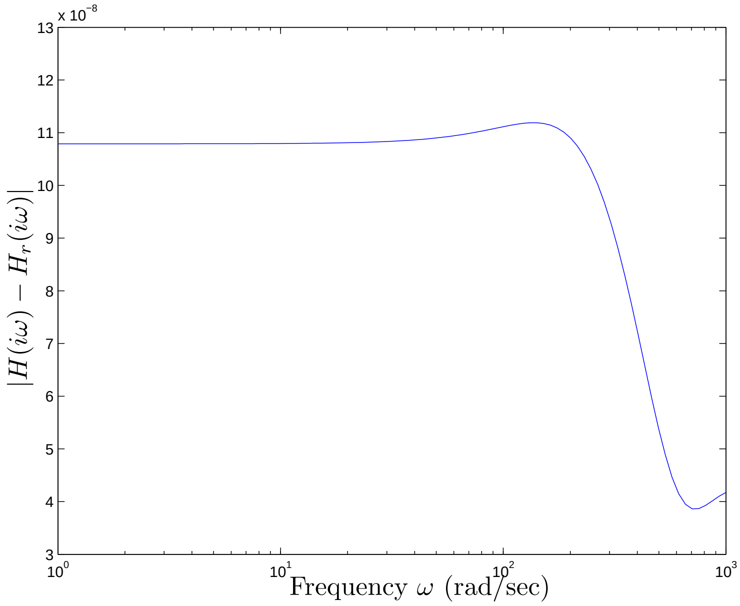

Figure 1: Absolute error plots for the semidiscretized Stokes equation.

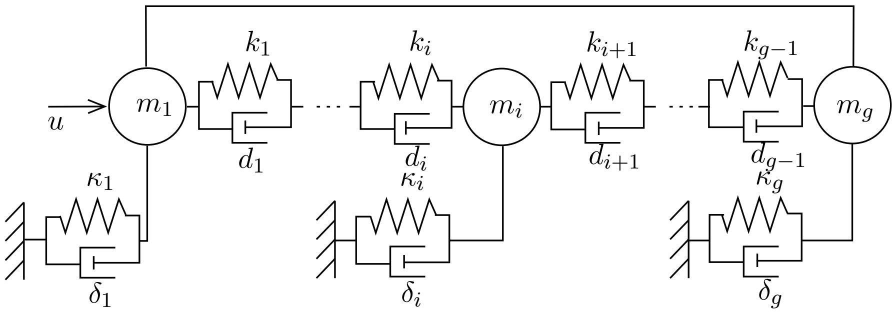

Figure 2: A damped mass-spring system with a holonomic constraint.

where $\mathbf{p}(t) \in \mathbb{R}^g$ is the position vector, $\mathbf{v}(t) \in \mathbb{R}^g$ is the velocity vector, $\boldsymbol{\lambda}(t) \in \mathbb{R}$ is the Lagrange multiplier, $M = \operatorname{diag}(m_1, \ldots, m_g)$ is the mass matrix, D and K are the tridiagonal damping and stiffness matrices. The projection-like matrix $\mathcal{P}$ is given by

$$\mathcal{P} = \begin{bmatrix} I - V^\top F & 0 & M^{-1}WRV^\top \\ M^{-1}WRV^\top F & M^{-1}W & M^{-1}W(Q - RM^{-1}WR)V^\top \\ VQ(I - V^\top F) & VRM^{-1}W & VQ(M^{-1}WRV^\top + M^{-1}W(Q - RM^{-1}WR)V^\top) \end{bmatrix},$$

where $V = (FM^{-1}F^\top)^{-1}FM$ and $W = I - F^\top V$. This representation has been obtained from (8) using the explicit representation of the transformation matrices P and Q provided in [Sch95].

In our experiments we take $m_1 = \ldots = m_g = 100$ and

$$\begin{aligned} k_1 = \ldots = k_{g-1} = \kappa_2 = \ldots = \kappa_{g-1} = 2, &\quad \kappa_1 = \kappa_g = 4, \\ d_1 = \ldots = d_{g-1} = \delta_2 = \ldots = \delta_{g-1} = 5, &\quad \delta_1 = \delta_g = 10. \end{aligned}$$

For $g = 100$, we obtain a descriptor system of order $n = 201$ with $m = 1$ input and $p = 1$

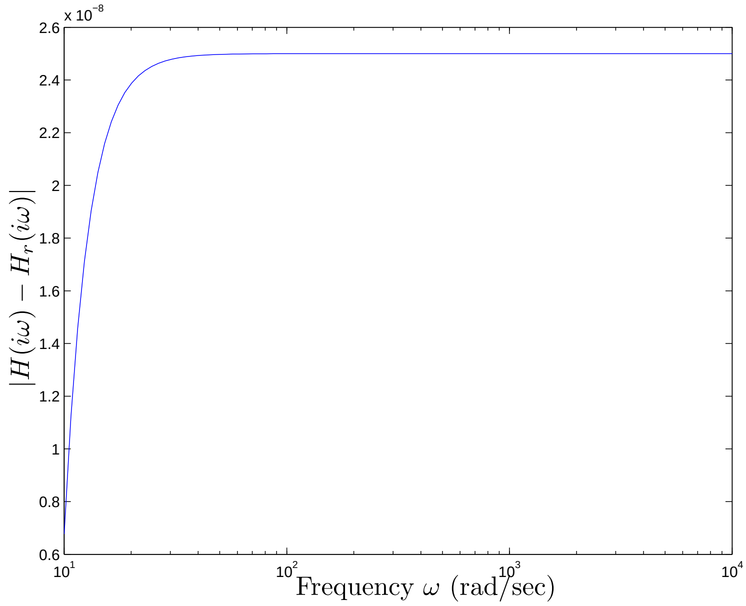

Figure 3: Absolute error plot and error bound for the damped mass-spring system.

output. We approximate this system by a system of order 10. The approximation result is presented in Figure 3.

Again we observe a very good match of the transfer function of the original system and the reduced-order system.

4 Conclusion

In this paper we have presented a solution of the partial realization problem for state-space systems in descriptor form. For a particular case of proper descriptor systems a numerical algorithm based on the unsymmetric Lanczos method has been proposed. This algorithm has been succesfully tested on examples from computational fluid dynamics and multibody mechanical systems. Future work will focus on extending these results to the multi-input, multi-output case.

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