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Fenchel-Lagrange versus Geometric Duality in Convex Optimization*

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Abstract. We present a new duality theory in order to treat convex optimization problems and we prove that the geometric duality used by C.H. Scott and T.R. Jefferson in different papers during the last quarter of century is a special case of it. Moreover, weaker sufficient conditions in order to achieve strong duality are considered and optimality conditions are derived in each case. Next we apply our approach to some problems considered by Scott and Jefferson determining their duals. We give some weaker sufficient conditions in order to achieve strong duality and the corresponding optimality conditions. Finally, posynomial geometric programming is viewed also as a particular case of the duality approach we present.

Key Words. Geometric programming, convex optimization, perturbation theory, Lagrange and Fenchel duality, conjugate functions.

1. Introduction

A quarter of century after the inception of their series of papers on geometric programming (see Ref. 1, among other papers) Scott and Jefferson have published recently the article cited as Ref. 2 where they treat a class of optimization problems by means of geometric duality. This fact denotes that the approach often used by these authors enjoys a continuous attention. Our main purpose is to introduce a duality theory for convex optimization problems which has many advantages in competition to the classical generalized geometric duality established by Peterson (Ref. 3) and used later in a simplified version by Scott and Jefferson (sometimes together with S. Jorjani) in many papers (see Refs. 1-2, 4-14).

We have proved in Ref. 15 that Peterson's geometric dual problems can be obtained using the perturbation theory presented in Refs. 16-17. Here we show

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that the geometric dual problem used by Scott and Jefferson is actually the Fenchel-Lagrange dual of the primal geometric problem. Moreover we provide much weaker sufficient conditions in order to achieve strong duality than the ones considered by them. Later we review some problems treated by means of geometric duality by these authors and we prove that their dual problems can be determined easier with our approach, where artificial constructions of the implicit, explicit and cone constraints are not required. Unlike the cited authors, that do not bother about the sufficient conditions for strong duality, mentioning just that they come from Peterson's work Ref. 3, we resort to weaker conditions that the ones obtained there. Using them, we display the strong duality assertion and the optimality conditions for each of these problems.

The Fenchel-Lagrange dual problem has been developed by Boţ and Wanka in some papers (Refs. 15, 18-23) and it is constructed by means of the perturbation theory in Refs. 16-17. Its name, introduced for the first time in Ref. 19 reveals its origin, as it is a combination between the well-known dual problems due to W. Fenchel and J.L. Lagrange. So far it has been considered for problems with constraint functions defined over the whole space $\mathbb{R}^n$, while here they are defined over a subset of $\mathbb{R}^n$.

We consider a convex optimization problem (P), the so-called primal problem, that consists in minimizing a function defined over the space $\mathbb{R}^n$ whose values are not necessarily all finite when its variable is required to belong to a subset of $\mathbb{R}^n$ and to satisfy the non-positivity of some constraint real-valued functions defined over the same subset. To it we determine the Fenchel-Lagrange dual problem, describing its construction in detail. Then we introduce a constraint qualification whose fulfillment is sufficient in order to have strong duality for the two problems mentioned above. After proving the strong duality assertion we formulate and prove also the optimality conditions for these problems and this ends the second part of the present work.

Further we consider a special case of the general convex optimization problem (P) denoted (P_K) where the variable is forced to belong also to a closed convex cone in $\mathbb{R}^n$. The generalized geometric problem used by Scott and Jefferson turns out to be a special case of this problem for a suitable choice of the functions and sets involved. Strong duality and optimality conditions for (P_K) follow and from them are derived the ones regarding the mentioned geometric problem. The sufficient conditions required for strong duality are weaker than the ones used before by the cited authors, as we do not ask the functions and the sets involved to be also closed alongside their convexity.

The fourth part of the paper deals with some problems Scott and Jefferson have treated by means of geometric programming duality. We prove that their artificial constructions in order to bring the problems into the form of the primal geometric problem are not necessary, as the same duals can be obtained simpler by applying our Fenchel-Lagrange duality. A large variety of problems is presented: minmax program (Ref. 11), entropy constrained program (Ref. 14), facility location problem (Ref. 12), quadratic concave fractional program (Ref. 9), problem of minimizing a sum of convex ratios (Ref. 10) and quasiconcave multiplicative program (Ref. 2). Strong duality, obtained under weaker conditions than in the original papers, and optimality conditions assertions are delivered for each of those problems. We also show that the posynomial geometric programming (Ref. 24) can be viewed as a special case of the Fenchel-Lagrange duality.

2. The Fenchel-Lagrange dual problem

Let us consider a convex subset X of $\mathbb{R}^n$ and a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $g = (g_1, , , , g_k)^T : X \to \mathbb{R}^k$, with $g_i, i = 1, ..., k$, convex functions. Using them we introduce the following convex optimization problem, further called primal problem,

$$(P) \qquad \inf_{\substack{x \in X, \\ g(x) \leqq 0}} f(x).$$

As usual, $g(x) \leqq 0$ means $g_i(x) \leq 0$ for all $i = 1, ..., k$. We denote by $\operatorname{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ the effective domain of the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.

A short observation is necessary: in the previous works of two of the authors (Refs. 18-23) the constraint functions g_i, $i = 1, ..., k$, are defined over the whole space, unlike here. By defining the constraint functions over a subset of $\mathbb{R}^n$ we are able to treat a larger class of convex optimization problems. In order to determine the so-called Fenchel-Lagrange dual problem of (P) we need to introduce the perturbation function (cf. Refs. 18-23) $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\Phi(x, y, z) = \begin{cases} f(x+y), & \text{if } x \in X, g(x) \leqq z, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables y and z. Following the path of the perturbation method described in Refs. 16-17 the next step is to calculate the conjugate function of Φ. Within this paper two types of conjugate functions are used. For a function defined over the whole space, $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the conjugate function is

$$f^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad f^*(x^*) = \sup_{x \in \mathbb{R}^n} \big\{\langle x^*, x\rangle - f(x)\big\}.$$

When a function is defined over a subset of X of $\mathbb{R}^n$, let it be $k : X \to \mathbb{R}$, we define for it the so-called conjugate relative to the set X

$$k_X^* : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad k_X^*(x^*) = \sup_{x \in X} \big\{\langle x^*, x\rangle - k(x)\big\}.$$

For a function $k : X \to \mathbb{R}$ let us consider its extension to the whole space

$$h : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad h(x) = \begin{cases} k(x), & \text{if } x \in X, \\ +\infty, & \text{otherwise.} \end{cases}$$

One may notice that the conjugate of k relative to the set X is identical to the conjugate of the function h.

Let us proceed with the definition for $\Phi^* : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\begin{aligned}\Phi^*(x^*, p, q) &= \sup_{\substack{x,y\in\mathbb{R}^n,\\ z\in R^k}} \Big\{ \langle x^*, x\rangle + \langle p, y\rangle + \langle q, z\rangle - \Phi(x,y,z) \Big\} \\ &= \sup_{\substack{x\in X, y\in\mathbb{R}^n,\\ g(x)\leqq z}} \Big\{ \langle x^*, x\rangle + \langle p, y\rangle + \langle q, z\rangle - f(x+y) \Big\}\end{aligned}$$

We have used the denotations $\langle x^*, x\rangle := x^{*T}x$, $\langle p, y\rangle := p^T y$ and $\langle q, z\rangle := q^T z$ for the Euclidean scalar products in the corresponding spaces $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. In order to calculate Φ^* let us introduce new variables r instead of y and s instead of z by

$$r := x + y, \quad s := z - g(x).$$

As the supremum above is computed with respect to three independent variables, x, r and s, it can be separated into a sum of three suprema

$$\begin{aligned}\Phi^*(x^*, p, q) &= \sup_{s\in\mathbb{R}^k_+} \langle q, s\rangle + \sup_{r\in\mathbb{R}^n} \Big\{ \langle p, r\rangle - f(r) \Big\} \\ &+ \sup_{x\in X} \Big\{ \langle x^* - p, x\rangle + \langle q, g(x)\rangle \Big\} \\ &= \begin{cases} f^*(p) - \inf\limits_{x\in X} \Big\{ \langle p - x^*, x\rangle - \langle q, g(x)\rangle \Big\}, & \text{if } q \in -\mathbb{R}^k_+, \\ +\infty, & \text{otherwise,}\end{cases}\end{aligned}$$

where $\mathbb{R}^k_+ = \{z : z \in \mathbb{R}^k, 0 \leqq z\}$.

According to Ref. 16 the dual problem to the problem (P) is

$$(D) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}^k}} \Big\{ -\Phi^*(0, p, q) \Big\},$$

that becomes in our case after changing the sign of the variable q

$$(D) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}^k_+}} \Big\{ -f^*(p) + \inf_{x\in X} [\langle p, x\rangle + \langle q, g(x)\rangle] \Big\}.$$

It is obvious from the construction of the dual that the weak duality assertion between (P) and (D), i. e. the value of the primal objective function at any primal feasible point is greater than or equal to the value of the dual objective function at any dual feasible point, always stands. We will not mention further for each pair of dual problems we treat that the weak duality is true, as it is valid in the most general case without any supplementary assumption. By strong duality we understand the situation in which the optimal objective values of the primal and dual are equal and the dual has an optimal solution. Unlike weak duality, strong duality can fail in the general case. To avoid any undesired event of this kind, we introduce a constraint qualification that guarantees the validity of strong duality in case it is fulfilled. First let us divide the index set {1, ..., k} into two subsets,

$$L := \left\{ i \in \{1, ..., k\} \,\middle|\, \begin{array}{l} g_i : X \to \mathbb{R} \text{ is the restriction to } X \text{ of an} \\ \text{affine function } \tilde{g}_i : \mathbb{R}^n \to \mathbb{R} \end{array} \right\},$$

and $N := \{1, ..., k\} \backslash L$. The constraint qualification follows

$$(CQ) \qquad \exists x' \in \mathrm{ri}(X) \cap \mathrm{ri}(\mathrm{dom}(f)) : \begin{cases} g_i(x') \leq 0, & i \in L, \\ g_i(x') < 0, & i \in N, \end{cases}$$

where $\mathrm{ri}(X)$ denotes the relative interior of the set X. We are ready now to formulate the strong duality assertion. Before that let us denote by $\inf(P)$ and $\sup(D)$ the optimal objective values of the primal and dual problem, respectively.

Theorem 2.1. Provided that the constraint qualification (CQ) is fulfilled, there is strong duality between problems (P) and (D), i. e. their optimal objective values are equal and the dual problem has an optimal solution.

Proof. We can write the problem (P) equivalently

$$(P) \qquad \inf_{\substack{x \in X \cap \mathrm{dom}(f), \\ g(x) \leqq 0}} f(x).$$

By Theorem 6.5 in Ref. 17, (CQ) yields

$$x' \in \mathrm{ri}\big(X \cap \mathrm{dom}(f)\big) = \mathrm{ri}(X) \cap \mathrm{ri}\big(\mathrm{dom}(f)\big).$$

Theorem 5.7 in Ref. 25 states under the present hypotheses the existence of a $\bar{q} \geqq 0$ such that

$$\inf(P) = \max_{q \geqq 0} \inf_{x \in X \cap \mathrm{dom}(f)} [f(x) + \langle q, g(x) \rangle] = \inf_{x \in X \cap \mathrm{dom}(f)} [f(x) + \langle \bar{q}, g(x) \rangle].$$

Defining

$$h : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad h(x) = \begin{cases} \langle \bar{q}, g(x) \rangle, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X, \end{cases}$$

we can rewrite the right-hand side term by

$$\inf(P) = \inf_{x \in \mathbb{R}^n} \big[f(x) + h(x)\big].$$

Because $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(h)) = \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(X) \neq \emptyset$, by Theorem 31.1 (Fenchel's Duality Theorem) in Ref. 17, there exists a $\bar{p} \in \mathbb{R}^n$ such that this infimum is equal to

$$\begin{aligned} \inf(P) &= \max_{p \in \mathbb{R}^n} \big[- f^*(p) - h^*(-p) \big] = -f^*(\bar{p}) - h^*(-\bar{p}) \\ &= -f^*(\bar{p}) - \sup_{x \in \mathbb{R}^n} \{ \langle -\bar{p}, x \rangle - h(x) \} \\ &= -f^*(\bar{p}) - \sup_{x \in X} \{ \langle -\bar{p}, x \rangle - \langle \bar{q}, g(x) \rangle \} \\ &= -f^*(\bar{p}) + \inf_{x \in X} \{ \langle \bar{p}, x \rangle + \langle \bar{q}, g(x) \rangle \}. \end{aligned} \tag{1}$$

In the right-hand term of (1) one may recognize the objective function of the dual problem (D) at $(\bar{p}, \bar{q})$. From weak duality it follows that the supremum of (D) is attained, becoming maximum, at $(\bar{p}, \bar{q})$, which turns out to be an optimal solution of the dual problem. $\square$

Remark 2.1. Let us notice that in the proof above we have first proved that under the fulfillment of (CQ) there holds strong duality between the primal problem and its Lagrange dual problem. Then we proved strong duality between the Lagrange dual and its Fenchel dual problem, the last one proving to be exactly the Fenchel-Lagrange dual problem we introduced earlier.

Next we derive necessary and sufficient optimality conditions regarding the problems (P) and (D).

Theorem 2.2.

(a) If the constraint qualification (CQ) is fulfilled and the primal problem (P) has an optimal solution $\bar{x}$, then the dual problem has an optimal solution $(\bar{p}, \bar{q})$ and the following optimality conditions are satisfied

  (i) $f(\bar{x}) + f^*(\bar{p}) = \langle \bar{p}, \bar{x} \rangle$,

  (ii) $\inf_{x \in X} \Big[\langle \bar{p}, x \rangle + \langle \bar{q}, g(x) \rangle \Big] = \langle \bar{p}, \bar{x} \rangle$,

  (iii) $\langle \bar{q}, g(\bar{x}) \rangle = 0$.

(b) If $\bar{x}$ is a feasible point to the primal problem (P) and $(\bar{p}, \bar{q})$ is feasible to the dual problem (D) fulfilling the optimality conditions (i)-(iii), then there is strong duality between (P) and (D) and the mentioned feasible points turn out to be optimal solutions.

Proof.

(a) Theorem 2.1 guarantees strong duality between (P) and (D). So the dual problem has an optimal solution. Let us denote it by $(\bar{p}, \bar{q})$. The equality of the optimal objective values of (P) and (D) implies

$$f(\bar{x}) + f^*(\bar{p}) - \inf_{x \in X} \Big[\langle \bar{p}, x\rangle + \langle \bar{q}, g(x)\rangle\Big] = 0. \tag{2}$$

It is obvious that

$$\inf_{x \in X} \Big[\langle \bar{p}, x\rangle + \langle \bar{q}, g(x)\rangle\Big] \leq \langle \bar{p}, \bar{x}\rangle + \langle \bar{q}, g(\bar{x})\rangle,$$

while Young's inequality states

$$f(\bar{x}) + f^*(\bar{p}) \geq \langle \bar{p}, \bar{x}\rangle.$$

Combining (2) with these relations leads to

$$0 \geq \langle \bar{p}, \bar{x}\rangle - \langle \bar{p}, \bar{x}\rangle - \langle \bar{q}, g(\bar{x})\rangle = -\langle \bar{q}, g(\bar{x})\rangle,$$

but $\langle \bar{q}, g(\bar{x})\rangle \leq 0$ because of the feasibility of $\bar{x}$ to (P) and $\bar{q}$ to (D), respectively. Therefore (iii) is true. Adding and subtracting $\langle \bar{p}, \bar{x}\rangle$ to (2) yields

$$\big[f(\bar{x}) + f^*(\bar{p}) - \langle \bar{p}, \bar{x}\rangle\big] + \big[\langle \bar{p}, \bar{x}\rangle - \inf_{x \in X}[\langle \bar{p}, x\rangle + \langle \bar{q}, g(x)\rangle]\big] = 0.$$

This gives immediately (i) and (ii).

(b) All the calculations presented above can be carried out in reverse order, so the assertion holds. □

Remark 2.2. We need to mention that *(b)* applies without any convexity assumption as well as constraint qualification. So the sufficiency of the optimality conditions (i)-(iii) is true in the most general case.

3. A particular case: the geometric programming duality

Our main stimulus in writing this paper was to prove that the simplified generalized geometric duality is nothing but a particular case of Fenchel-Lagrange duality. So far we have presented the basic facts and results regarding the Fenchel-Lagrange duality, but in order to reach our goal we consider first a special case of the primal problem (P) which is still more general than the geometric primal problem. So we consider the problem

$$(P_K) \qquad \inf_{\substack{x \in X, g(x) \leqq 0, \\ x \in K}} f(x),$$

where K is a closed convex cone in $\mathbb{R}^n$. Its Fenchel-Lagrange dual problem is

$$(D_K) \qquad \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}^k_+}} \Big\{ -f^*(p) + \inf_{x\in X\cap K}[\langle p,x\rangle + \langle q, g(x)\rangle]\Big\}.$$

The constraint qualification that is sufficient for the existence of strong duality in this case is, with the notations introduced before,

$$(CQ_K) \qquad \exists x' \in \mathrm{ri}(X)\cap \mathrm{ri}(\mathrm{dom}(f)) : \begin{cases} g_i(x') \leq 0, & i \in L, \\ g_i(x') < 0, & i \in N, \\ x' \in \mathrm{ri}(K). \end{cases}$$

Consequently we have the strong duality assertion regarding the problem (P_K) and an equivalent form of its dual problem.

Theorem 3.1. When the constraint qualification (CQ_K) is satisfied, there is strong duality between the primal problem (P_K) and the equivalent formulation of its dual,

$$(D'_K) \qquad \sup_{\substack{p\in\mathbb{R}^n, q\in\mathbb{R}^k_+,\\ t\in K^*}} \Big\{ -f^*(p) + \inf_{x\in X}[\langle p-t,x\rangle + \langle q, g(x)\rangle]\Big\}.$$

Proof. (P_K) being a special case of the problem (P), like (D_K) of its dual (D) and because $\mathrm{ri}(X\cap K) = \mathrm{ri}(X)\cap\mathrm{ri}(K)$, strong duality is valid for (P_K) and (D_K). But the presence of the cone K in the formula of (D_K) is not so desired, so we need to find an alternative formulation to this dual problem. Let us rewrite the infimum contained in its formulation in the following way

$$\inf_{x\in X\cap K}[\langle p,x\rangle + \langle q, g(x)\rangle] = \inf_{x\in\mathbb{R}^n}[\langle p,x\rangle + h(x) + \delta_K(x)],$$

where the function h is defined like in the proof of Theorem 2.1 and we use also the indicator function $\delta_K : \mathbb{R}^n \to \overline{\mathbb{R}}, \delta_K(x) = \begin{cases} 0, & \text{if } x\in K, \\ +\infty, & \text{if } x \notin K. \end{cases}$
By the definition of the conjugate function, the right-hand side of the relation above is equal to $-\Big(\langle p,\cdot\rangle + h + \delta_K\Big)^*(0)$, which, applying Theorem 20.1 in Ref. 17, can be written as

$$\begin{aligned} & - \inf_{t\in\mathbb{R}^n}\Big[(\langle p,\cdot\rangle + h)^*(t) + \delta_K^*(-t)\Big] \\ = & -\inf_{t\in K^*}\sup_{x\in X}\big\{\langle t,x\rangle - \langle p,x\rangle - \langle q, g(x)\rangle\big\} \\ = & \sup_{t\in K^*}\inf_{x\in X}\big\{\langle p-t,x\rangle + \langle q,g(x)\rangle\big\} \end{aligned}$$

and the existence of a $\bar{t} \in K^*$ where this supremum is attained is granted. Here $K^* = \left\{ y \in \mathbb{R}^n : \langle y, x \rangle \geq 0, \forall x \in K \right\}$ denotes the dual cone of K.

It is obvious that the dual problem (D_K) is equivalent now to (D'_K) and strong duality between (P_K) and (D'_K) is certain. $\square$

Remark 3.1. One can obtain the dual problem (D'_K) also by perturbations, in a similar way we obtained (D) in the previous section by including the cone constraint in the constraint function (cf. Refs. 18-21). We refer further to (D'_K) as the dual problem of (P_K).

The necessary and sufficient optimality conditions are derived from the ones obtained in the general case.

Theorem 3.2.

(a) If the constraint qualification (CQ_K) is fulfilled and the primal problem (P_K) has an optimal solution $\bar{x}$, then the dual problem (D'_K) has an optimal solution $(\bar{p}, \bar{q}, \bar{t})$ and the following optimality conditions are satisfied

  (i) $f(\bar{x}) + f^*(\bar{p}) = \langle \bar{p}, \bar{x} \rangle$,

  (ii) $\inf\limits_{x \in X} \Big[\langle \bar{p} - \bar{t}, x \rangle + \langle \bar{q}, g(x) \rangle \Big] = \langle \bar{p}, \bar{x} \rangle$,

  (iii) $\langle \bar{q}, g(\bar{x}) \rangle = 0$,

  (iv) $\langle \bar{t}, \bar{x} \rangle = 0$.

(b) If $\bar{x}$ is a feasible point to the primal problem (P_K) and $(\bar{p}, \bar{q}, \bar{t})$ is feasible to the dual problem (D'_K) fulfilling the optimality conditions (i)-(iv), then there is strong duality between (P_K) and (D'_K) and the mentioned feasible points turn out to be optimal solutions.

With a suitable choice of the functions and the sets involved in the problem (P_K) one obtains a convex optimization problem equivalent to the primal geometric problem used by Scott and Jefferson in many papers (Refs. 1-2, 4-14),

$$(P_g) \qquad \inf_{\substack{x=(x^0,x^1,...,x^k) \in C_0 \times C_1 \times ... \times C_k, \\ g^i(x^i) \leq 0, i=1,...,k, \\ x \in K}} g^0(x^0),$$

where $C_i \subseteq \mathbb{R}^{l_i}, i = 0, ..., k, \sum\limits_{i=0}^{k} l_i = n$, are convex sets, $g^i : C_i \to \mathbb{R}$, $i = 0, ..., k$, convex functions and $K \subseteq \mathbb{R}^n$ is a closed convex cone.

The proper selection of the mentioned elements follows

$$\begin{cases} X = \mathbb{R}^{l_0} \times C_1 \times ... \times C_k, \\ f : \mathbb{R}^n \to \overline{\mathbb{R}}, \quad g_i : X \to \mathbb{R}, i = 1, ..., k, \\ f(x) = \begin{cases} g^0(x^0), & x \in C_0 \times \mathbb{R}^{n-l_0}, \\ +\infty, & \text{otherwise}, \end{cases} \\ g_i(x) = g^i(x^i), i = 1, ..., k, x = (x^0, x^1, ..., x^k) \in X. \end{cases}$$

Now we can write the Fenchel-Lagrange dual problem to (P_g) (cf. (D'_K))

$$(D_g) \sup_{\substack{p\in\mathbb{R}^n, q\in\mathbb{R}^k_+, \\ t\in K^*}} \left\{ -f^*(p) + \inf_{x\in\mathbb{R}^{l_0}\times C_1\times...\times C_k} \left[\langle p - t, x\rangle + \sum_{i=1}^k q_i g^i(x^i) \right] \right\}.$$

The conjugate of f at $p = (p^0, p^1, ..., p^k) \in \mathbb{R}^{l_0} \times \mathbb{R}^{l_1} \times ... \times \mathbb{R}^{l_k}$ is

$$\begin{aligned} f^*(p) &= \sup_{x\in\mathbb{R}^n} \left\{ \langle p, x\rangle - f(x) \right\} \\ &= \sup_{x=(x^0,x^1,...,x^k)\in C_0\times\mathbb{R}^{l^1}\times...\times\mathbb{R}^{l_k}} \left\{ \sum_{i=0}^k \langle p^i, x^i\rangle - g^0(x^0) \right\} \\ &= \begin{cases} g^{0*}_{C_0}(p^0), & \text{if } p^i = 0, i = 1, ..., k, \\ +\infty, & \text{otherwise}. \end{cases} \end{aligned}$$

As the infimum that appears is separable into a sum of infima, the dual becomes

$$(D_g) \sup_{\substack{p^0\in\mathbb{R}^{l_0}, \\ q\in\mathbb{R}^k_+, \\ t\in K^*}} \left\{ -g^{0*}_{C_0}(p^0) + \inf_{x^0\in\mathbb{R}^{l_0}} \langle p^0 - t^0, x^0\rangle + \sum_{i=1}^k \inf_{x^i\in C_i} [\langle -t^i, x^i\rangle + q_i g^i(x^i)] \right\},$$

if we consider $t = (t^0, t^1, ..., t^k) \in \mathbb{R}^{l_0} \times \mathbb{R}^{l_1} \times ... \times \mathbb{R}^{l_k}$. As

$$\inf_{x^0\in\mathbb{R}^{l_0}} \langle p^0 - t^0, x^0\rangle = \begin{cases} 0, & \text{if } p^0 = t^0, \\ -\infty, & \text{otherwise}, \end{cases}$$

the dual problem to (P_g) turns into

$$(D_g) \qquad \sup_{\substack{q\in\mathbb{R}^k_+, \\ t\in K^*}} \left\{ -g^{0*}_{C_0}(t^0) - \sum_{i=1}^k \sup_{x^i\in C_i} \left[\langle t^i, x^i\rangle - q_i g^i(x^i) \right] \right\}.$$

This is exactly the geometric dual problem encountered in all cited papers by Scott and Jefferson, written without resorting to the homogenous extension of the conjugate functions that can replace the suprema in (D_g).

The constraint qualification sufficient to guarantee the validity of strong duality for this pair of problems is derived from (CQ_K),

$$(CQ_g) \qquad \exists x' \in \mathrm{ri}(K) : \begin{cases} g^i(x') \leq 0, & i \in L, \\ g^i(x') < 0, & i \in N, \\ x'^i \in \mathrm{ri}(C_i), & i = 0, ..., k. \end{cases}$$

Theorem 3.3. The satisfaction of the constraint qualification (CQ_g) is sufficient to guarantee strong duality regarding (P_g) and (D_g).

Remark 3.2. The cited papers of the mentioned authors do not assert trenchantly any strong duality allegation, containing just the optimality conditions, while for the background of their achievement the reader is referred to Ref. 3. There all the functions and the sets involved are postulated as being closed, alongside their convexity assumptions that proved to be sufficient in our proofs when the constraint qualification is fulfilled. Moreover, the possibility to impose a milder constraint qualification regarding the affine functions whose restrictions to the considered set are among the constraint functions is not taken into consideration at all.

The optimality conditions concerning (P_g) and (D_g) spring directly from Theorem 3.2.

Theorem 3.4.

(a) If the constraint qualification (CQ_g) is fulfilled and the primal problem (P_g) has an optimal solution $\bar{x} = \left(\bar{x}^0, \bar{x}^1, ..., \bar{x}^k\right)$, then the dual problem (D_g) has an optimal solution $(\bar{q}, \bar{t})$ and the following optimality conditions are satisfied

  (i) $g^0(\bar{x}^0) + g^{0*}_{C_0}(\bar{t}^0) = \langle \bar{t}^0, \bar{x}^0 \rangle$,

  (ii) $\left(\bar{q}_i g_i\right)^*_{C_i}(\bar{t}_i) = \langle \bar{t}^i, \bar{x}^i \rangle$, $i = 1, ..., k$,

  (iii) $\bar{q}_i g^i(\bar{x}^i) = 0$, $i = 0, ..., k$,

  (iv) $\langle \bar{t}, \bar{x} \rangle = 0$.

(b) If $\bar{x}$ is a feasible point to the primal problem (P_g) and $(\bar{q}, \bar{t})$ is feasible to the dual problem (D_g) fulfilling the optimality conditions (i)-(iv), then there is strong duality between (P_g) and (D_g) and the mentioned feasible points turn out to be optimal solutions.

Remark 3.3. The optimality conditions we derived are equivalent to the ones displayed by Scott and Jefferson in the cited papers, but have a different form.

4. An overview of some special cases

This last section reviews some of the problems treated during the last quarter of century by Scott and Jefferson, sometimes together with Jorjani, by means of simplified generalized geometric programming. All the problems were artificially transformed into the framework required by geometric programming by introducing new variables in order to separate implicit and explicit constraints and building some cone where the new vector-variable is forced to lie. Then their dual problems arose from the general theory and the optimality conditions came out from the same place. We determine the Fenchel-Lagrange dual problem for each problem, then we specialize the adequate constraint qualification and state the strong duality assertion followed by the optimality conditions all without proofs as they are direct consequences of Theorems 2.1 and 2.2. One may notice that even if the functions and the sets are considered closed in the original papers we remove this redundant property, as we demonstrate that strong duality and optimality conditions stand out even without its presence when the sufficient constraint qualification is valid. We have chosen six problems that we have considered more interesting, but also the problems in Refs. 1, 4-8 or 13 may benefit from the same treatment. The last subsection is dedicated to the well-known posynomial geometric programming which is undertaken into our duality theory. Other papers of Scott and Jefferson treat some problems by means of posynomial geometric programming, so we might have included some of these problems here. During this section, unless otherwise specified, the variables cover the whole space $\mathbb{R}^n$.

4.1. Minmax programs (Ref. 11). The first problem we deal with is the minmax program

$$(P_1) \qquad \inf_{\substack{x\in C, Ax\geqq b,\\ g(x)\leqq 0}} \max_{i=1,...,I} \{f_i(x)\},$$

with $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\mathrm{dom}(f_i) = C$, $i = 1, ..., I$, and $g = (g_1, ..., g_J)^T : C \to \mathbb{R}^J$ convex functions, $C \subseteq \mathbb{R}^n$ a convex set, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. In the original paper the functions $f_i, i = 1, ..., I$, $g_j, j = 1, ..., J$, and the set C are required to be also closed, but strong duality is valid in more general circumstances, i. e. without the closedness assumptions. To treat the problem (P_1) with the method presented in the second section, it is rewritten as

$$(P_1) \qquad \inf_{\substack{x\in C, s\in\mathbb{R},\\ b-Ax\leqq 0, g(x)\leqq 0,\\ f_i(x)-s\leqq 0, i=1,...,I}} s.$$

The Fenchel-Lagrange dual problem to (P_1) is, considering the objective function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, u(x,s) = s$ and $q^f = (q_1^f, ..., q_I^f)^T$,

$$(D_1) \quad \sup_{\substack{p^x \in \mathbb{R}^n, p^s \in \mathbb{R}, \\ q^l \in \mathbb{R}_+^m, q^f \in \mathbb{R}_+^I, q^g \in \mathbb{R}_+^J}} \Bigg\{ - u^*(p^x, p^s) + \inf_{\substack{x \in C, \\ s \in \mathbb{R}}} \Big[\langle p^x, x \rangle + \langle p^s, s \rangle + \sum_{i \in I} q_i^f (f_i(x) - s) + \langle q^g, g(x) \rangle + \langle q^l, b - Ax \rangle \Big] \Bigg\}.$$

Computing the conjugate of the objective function we get

$$u^*(p^x, p^s) = \sup_{\substack{x \in \mathbb{R}^n, \\ s \in \mathbb{R}}} \big\{ \langle (p^x, p^s), (x, s) \rangle - s \big\} = \begin{cases} 0, & \text{if } p^s = 1, p^x = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Noticing that the infimum in (D_1) is separable into a sum of two infima, one concerning $s \in \mathbb{R}$, the other $x \in C$, the dual problem turns into

$$(D_1) \quad \sup_{\substack{q^l \in \mathbb{R}_+^m, q^f \in \mathbb{R}_+^I, \\ q^g \in \mathbb{R}_+^J}} \Bigg\{ \inf_{x \in C} \Big[\sum_{i \in I} q_i^f f_i(x) + \langle q^g, g(x) \rangle - \langle A^T q^l, x \rangle \Big] + \inf_{s \in \mathbb{R}} \Big[s - s \sum_{i \in I} q_i^f \Big] + \langle q^l, b \rangle \Bigg\}.$$

The second infimum is equal to 0 when $\sum\limits_{i \in I} q_i^f = 1$, otherwise having the value $-\infty$, while the first, transformed into a supremum, can be viewed as a conjugate function relative to the set C. Applying Theorem 20.1 in Ref. 17 and denoting $q^g = (q_1^g, ..., q_J^g)^T$, the dual problem becomes

$$(D_1) \quad \sup_{\substack{q^l \in \mathbb{R}_+^m, q^g \in \mathbb{R}_+^J, \\ q^f \in \mathbb{R}_+^I, \sum_{i=1}^I q_i^f = 1, \\ \sum_{i=1}^I u_i + \sum_{j=1}^J v_j = A^T q^l}} \Bigg\{ \langle q^l, b \rangle - \sum_{i=1}^I (q_i^f f_i)^*(u_i) - \sum_{j=1}^J (q_j^g g_j)_C^*(v_j) \Bigg\},$$

identical to the dual problem found in Ref. 11. A sufficient circumstance to be able to formulate the strong duality assertion is the following constraint qualification, where the sets L and N are considered as before,

$$(CQ_1) \qquad \exists x' \in \mathrm{ri}(C) : \begin{cases} Ax' \geqq b, \\ g_j(x') \leq 0, & j \in L, \\ g_j(x') < 0, & j \in N, \end{cases}$$

Theorem 4.1. If the constraint qualification (CQ_1) is satisfied, then the strong duality between (P_1) and (D_1) is assured.

Since the optimality conditions are not delivered in Ref. 11, here they are, determined via our method.

Theorem 4.2.

(a) If the constraint qualification (CQ_1) is fulfilled and $\bar{x}$ is an optimal solution to (P_1), then strong duality between the problems (P_1) and (D_1) is attained and the dual problem has an optimal solution $(\bar{q}^l, \bar{q}^f, \bar{q}^g, \bar{u}, \bar{v})$, where $\bar{u} = (\bar{u}_1, ..., \bar{u}_I)^T$ and $\bar{v} = (\bar{v}_1, ..., \bar{v}_J)^T$, satisfying the following optimality conditions

   (i) $f_i(\bar{x}) - \max\limits_{i=1,...,I}\{f_i(\bar{x})\} = 0$ if $\bar{q}_i^f > 0$, $i = 1, ..., I$,

   (ii) $\langle \bar{q}^l, b - A\bar{x}\rangle = 0$,

   (iii) $\langle \bar{q}^g, g(\bar{x})\rangle = 0$,

   (iv) $\left(\bar{q}_i^f f_i\right)^*(\bar{u}_i) + \bar{q}_i^f f_i(\bar{x}) = \langle \bar{u}_i, \bar{x}\rangle, i = 1, ..., I$,

   (v) $\left(\bar{q}_j^g g_j\right)_C^*(\bar{v}_j) + \bar{q}_j^g g_j(\bar{x}) = \langle \bar{v}_j, \bar{x}\rangle, j = 1, ..., J$.

(b) Having a feasible solution $\bar{x}$ to the primal problem and one $(\bar{q}^l, \bar{q}^f, \bar{q}^g, \bar{u}, \bar{v})$ to the dual satisfying the optimality conditions (i)-(v), then the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality stands.

4.2. Entropy constrained programs (Ref. 14). A minute exposition of the way how the Fenchel-Lagrange duality is applicable to the problem treated in Ref. 14 is available in Ref. 15. Let us summarize it here.

The problem

$$(P_2) \qquad \inf_{\substack{Ax \geqq b, \\ -\sum\limits_{i=1}^n x_i \ln x_i \geq H, \\ \sum\limits_{i=1}^n x_i = 1, x \geqq 0}} \langle c, x\rangle,$$

where $x = (x_1, ..., x_n)^T \in \mathbb{R}^n, c = (c_1, ..., c_n)^T \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, prompts the following Fenchel-Lagrange dual problem

$$(D_2) \qquad \sup_{\substack{p\in\mathbb{R}^n, q^x\in\mathbb{R}, \\ q^l\in\mathbb{R}_+^m, q^H\in\mathbb{R}_+}} \Bigg\{ -\langle c, \cdot\rangle^*(p) + \inf_{x\geqq 0}\Big[\langle p, x\rangle + \langle q^l, b - Ax\rangle$$

$$+ \quad q^H\Big(H + \sum_{i=1}^n x_i \ln x_i\Big) + q^x\Big(\sum_{i=1}^n x_i - 1\Big)\Big]\Bigg\}.$$

It is known that $\langle c, \cdot \rangle^*(p) = 0$ if $p = c$, otherwise being equal to $+\infty$. In Ref. 15 we prove that in the constraints of the problem (D_2) one can consider $q^H > 0$ instead of $q^H \in \mathbb{R}_+$. Also, the infimum over $x \geqq 0$ is separable into a sum of infima concerning $x_i \geq 0$, $i = 1, ..., n$. Denoting also by $a_{ji}, j = 1, ..., m, i = 1, ..., n$, the entries of the matrix A and $q^l = (q^l_1, ..., q^l_m)^T$, the dual problem turns into

$$(D_2) \sup_{\substack{q^x \in \mathbb{R}, q^l \in \mathbb{R}^m_+, \\ q^H > 0}} \Bigg\{ q^H H + \langle q^l, b \rangle - q^x + \sum_{i=1}^{n} \inf_{x_i \geq 0} \Big[c_i x_i + q^H x_i \ln x_i + \big(q^x - \sum_{j=1}^{m} q^l_j a_{ji}\big) x_i \Big] \Bigg\}.$$

The infima can easily be computed (cf. Ref. 15) and the dual becomes

$$(D_2) \sup_{\substack{q^x \in \mathbb{R}, q^l \in \mathbb{R}^m_+, \\ q^H > 0}} \Bigg\{ q^H H + \langle q^l, b \rangle - q^x - q^H \sum_{i=1}^{n} \exp\left(\left(\sum_{j=1}^{m} q^l_j a_{ji} - c_i + q^x - q^H \right) / q^H \right) \Bigg\}.$$

The supremum over $q^x \in \mathbb{R}$ is also computable using elementary knowledge regarding the extreme points of functions, so the dual problem turns into its final version

$$(D_2) \qquad \sup_{\substack{q^l \in \mathbb{R}^m_+, \\ q^H > 0}} \left\{ \langle b, q^l \rangle - q^H \ln \sum_{i=1}^{n} \exp\left((A^T q^l - c)_i / q^H \right) + q^H H \right\},$$

almost identical to the dual problem found in Ref. 14. The difference is that the interval variable q^H lies in $\mathbb{R}_+ \backslash \{0\}$ instead of $\mathbb{R}_+$. This does not affect the optimal objective value of the dual problem. We have denoted the i-th entry of the vector $A^T q^l - c$ by $(A^T q^l - c)_i$. With the help of the constraint qualification

$$(CQ_2) \qquad \exists x' \in \operatorname{int}(\mathbb{R}^n_+) : \begin{cases} H + \sum\limits_{i=1}^{n} x'_i \ln x'_i < 0, \\ b - A x' \leqq 0, \\ \sum\limits_{i=1}^{n} x'_i = 1, \end{cases}$$

the strong duality affirmation is ready to be formulated, followed by the optimality conditions, equivalent to the ones in the original paper.

Theorem 4.3. If the constraint qualification (CQ_2) is satisfied, then the strong duality between (P_2) and (D_2) is assured.

Theorem 4.4.

(a) If the constraint qualification (CQ_2) is fulfilled and $\bar{x}$ is an optimal solution to (P_2), then strong duality between the problems (P_2) and (D_2) is attained and the dual problem has an optimal solution $(\bar{q}^l, \bar{q}^H)$ satisfying the following optimality conditions

(i) $\langle \bar{q}^l, A\bar{x} - b \rangle = 0$,

(ii) $\bar{q}^H \left(H + \sum\limits_{i=1}^{n} \bar{x}_i \ln \bar{x}_i \right) = 0$,

(iii) $\bar{q}^H \left(\sum\limits_{i=1}^{n} \bar{x}_i \ln \bar{x}_i + \ln \sum\limits_{i=1}^{n} \exp \left((A^T \bar{q}^l - c)_i / \bar{q}^H \right) \right) = \langle \bar{x}, A^T \bar{q}^l - c \rangle$.

(b) Having a feasible solution $\bar{x}$ to the primal problem and one to the dual $(\bar{q}^l, \bar{q}^H)$ satisfying the optimality conditions (i)-(iii), then the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality stands.

4.3. Facility location problem (Ref. 12). In the original paper the authors calculate the geometric duals for some problems involving norms. We have chosen one of them to be presented here, namely

$$(P_3) \qquad \inf_{\substack{\| x - a_j \| \leq d_j, \\ j=1,\dots,m}} \left\{ \sum_{j=1}^{m} w_j \parallel x - a_j \parallel \right\},$$

where $a_j \in \mathbb{R}^n$, $w_j > 0$, $d_j > 0$, for $j = 1, ..., m$. The raw version of its Fenchel-Lagrange dual problem is

$$(D_3) \sup_{\substack{p \in \mathbb{R}^n, \\ q \in \mathbb{R}^m_+}} \left\{ - \left(\sum_{j=1}^{m} w_j \parallel \cdot - a_j \parallel \right)^* (p) + \inf_{x \in \mathbb{R}^n} \left[\langle p, x \rangle + \sum_{j=1}^{m} q_j \big(\parallel x - a_j \parallel - d_j \big) \right] \right\}.$$

By Theorem 20.1 in Ref. 17 it turns into

$$(D_3) \sup_{\substack{p^j \in \mathbb{R}^n, \\ \sum\limits_{j=1}^{m} p^j = p, \\ q \in \mathbb{R}^m_+}} \left\{ - \sum_{j=1}^{m} \Big(w_j \parallel \cdot - a_j \parallel \Big)^* (p^j) + \inf_{x \in \mathbb{R}^n} \left[\langle p, x \rangle + \sum_{j=1}^{m} q_j \big(\parallel x - a_j \parallel - d_j \big) \right] \right\}.$$

Knowing that, for $j = 1, ..., m$,

$$\left(w_j \parallel \cdot - a_j \parallel\right)^*(p^j) = \begin{cases} \langle a_j, p^j \rangle, & \text{if } \parallel p^j \parallel \leq w_j, \\ +\infty, & \text{otherwise}, \end{cases}$$

and turning the infimum into supremum, we get applying again Theorem 20.1 in Ref. 17 the following equivalent formulation of the dual problem, encountered also in Ref. 12,

$$(D_3) \qquad \sup_{\substack{p^j \in \mathbb{R}^n, r^j \in \mathbb{R}^n, \\ \|p^j\| \leq w_j, \|r^j\| \leq q_j, \\ j=1,...,m, \\ \sum\limits_{j=1}^m \left(p^j + r^j\right) = 0, q \in \mathbb{R}^m_+}} \left\{ -\langle q, d \rangle - \sum_{j=1}^m \langle a_j, p^j \rangle - \sum_{j=1}^m \langle a_j, r^j \rangle \right\},$$

rewritable as

$$(D_3) \qquad \sup_{\substack{p^j \in \mathbb{R}^n, r^j \in \mathbb{R}^n, \\ \|p^j\| \leq w_j, \|r^j\| \leq q_j, \\ j=1,...,m, \\ \sum\limits_{j=1}^m \left(p^j + r^j\right) = 0, q \in \mathbb{R}^m_+}} \left\{ -\langle q, d \rangle - \sum_{j=1}^m \langle a_j, p^j + r^j \rangle \right\}.$$

Of course, we have set here $d = (d_1, ..., d_m)^T \in \mathbb{R}^m$ and $q = (q_1, ..., q_m)^T \in \mathbb{R}^m$.

A sufficient background for the existence of strong duality is in this case

$$(CQ_3) \qquad \exists x' \in \mathbb{R}^n : \ \parallel x' - a_j \parallel < d_j, j = 1, ..., m.$$

Theorem 4.5. If the constraint qualification (CQ_3) is satisfied, then the strong duality between (P_3) and (D_3) is assured.

Although there is no mention of the optimality conditions in Ref. 12 for this pair of dual problems we have derived the following result.

Theorem 4.6.

(a) If the constraint qualification (CQ_3) is fulfilled and $\bar{x}$ is an optimal solution to (P_3), then strong duality between the problems (P_3) and (D_3) is attained and the dual problem has an optimal solution $(\bar{p}^1, ..., \bar{p}^m, \bar{r}^1, ..., \bar{r}^m, \bar{q}_1, ..., \bar{q}_m)$ satisfying the following optimality conditions

  (i) $w_j \parallel \bar{x} - a_j \parallel = \langle \bar{p}^j, \bar{x} - a_j \rangle$ and $\parallel \bar{p}^j \parallel = w_j$ when $\bar{x} \neq a_j$, $j = 1, ..., m$.

  (ii) $\bar{q}_j \parallel \bar{x} - a_j \parallel = \langle \bar{r}^j, \bar{x} - a_j \rangle$, $\bar{q}_j \geq 0$, $j = 1, ..., m$, and if $\bar{q}_j > 0$ so is $\parallel \bar{r}^j \parallel = \bar{q}_j$. For $\bar{q}_j = 0$ there is also $\bar{r}^j = 0$. If in particular $\bar{x} = a_j$ for any $j \in \{1, ..., m\}$, then $\bar{q}_j = 0$ and $\bar{r}^j = 0$.

(iii) $\| \bar{x} - a_j \| = d_j$, for $j \in \{1, ..., m\}$ such that $\bar{q}_j > 0$.

(iv) $\sum\limits_{j=1}^{m} \left(\bar{p}^j + \bar{r}^j \right) = 0$.

(b) Having a feasible solution $\bar{x}$ to the primal problem and one to the dual $(\bar{p}^1, ..., \bar{p}^m, \bar{r}^1, ..., \bar{r}^m, \bar{q}_1, ..., \bar{q}_m)$ satisfying the optimality conditions (i)-(iv), then the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality holds.

4.4. Quadratic concave fractional programs (Ref. 9). Another problem artificially pressed into the selective framework of geometric programming is

$$(P_4) \qquad \inf_{Cx \leqq b} \big(Q(x)/f(x) \big),$$

where $Q(x) = (1/2)x^T Ax, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, $C \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ a concave function, strictly positive over the feasible set of the problem. Because no analytic representation of the conjugate of the objective function is available, the problem is rewritten

$$(P_4) \qquad \inf_{\substack{sQ\left((1/s)x\right) - f(x) \leq 0, \\ Cx \leqq b, s \in \mathbb{R}_+ \setminus \{0\}}} s.$$

To compute the Fenchel-Lagrange dual problem, we need first the conjugate of the objective function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, u(x, s) = s$. Using the result presented for the same objective function in section 4.1, the dual problem becomes

$$(D_4) \qquad \sup_{\substack{q^x \in \mathbb{R}^m_+, \\ q^s \in \mathbb{R}_+}} \left\{ \inf_{\substack{x \in \mathbb{R}^n, \\ s > 0}} \left[s + q^s \left(sQ\big((1/s)x\big) - f(x) \right) + \langle q^x, Cx - b \rangle \right] \right\}.$$

The infimum over (x, s), transformed into a supremum, can be viewed as a conjugate function that is determined after some standard calculations. The formula that results for the dual problem is identical to the geometric dual obtained by the cited authors,

$$(D_4) \qquad \sup_{\substack{q^x \in \mathbb{R}^m_+, q^s \in \mathbb{R}_+, \\ (1/2)u^T A^{-1} u \leq q^s, \\ u + v = -C^T q^x}} \Big\{ -b^T q^x - (-q^s f)^*(v) \Big\},$$

moreover simplifiable even to

$$(D_4) \qquad \sup_{\substack{q^x \in \mathbb{R}^m_+, q^s \in \mathbb{R}_+, \\ (1/2)(-v - C^T q^x)^T A^{-1} (-v - C^T q^x) \leq q^s}} \Big\{ -b^T q^x - (-q^s f)^*(v) \Big\}.$$

Of course we have removed the assumption of closedness that has been imposed on the function f before. Because of the linearity of the constraints of (P_4) no constraint qualification is required in this case.

Theorem 4.7. Provided that the primal problem has at least a feasible point, strong duality between problems (P_4) and (D_4) is assured.

The optimality conditions, equivalent to the ones given in Ref. 9, are presented in the following statement.

Theorem 4.8.

(a) If the problem (P_4) has an optimal solution $\bar{x}$ then strong duality between the problems (P_4) and (D_4) is attained and the dual problem has an optimal solution $(\bar{v}, \bar{q}^x, \bar{q}^s)$ satisfying the following optimality conditions

  (i) $(-\bar{q}^s f)^*(\bar{v}) - \bar{q}^s f(\bar{x}) = \langle \bar{v}, \bar{x} \rangle$,

  (ii) $(1/2)(-\bar{v} - \bar{C}^T q^x)^T A^{-1}(-\bar{v} - \bar{C}^T q^x) + (1/2)\bar{x}^T A \bar{x} = \langle -\bar{v} - \bar{C}^T q^x, \bar{x} \rangle$,

  (iii) $\langle \bar{q}^x, b - C\bar{x} \rangle = 0$,

  (iv) $(1/2)\big(-\bar{v} - \bar{C}^T q^x\big)^T A^{-1}\big(-\bar{v} - \bar{C}^T q^x\big) = \bar{q}^s$.

(b) Having a feasible solution $\bar{x}$ to the primal problem and one to the dual $(\bar{v}, \bar{q}^x, \bar{q}^s)$ satisfying the optimality conditions (i)-(iv) the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality stands.

4.5. Sum of convex ratios (Ref. 10). An extension to vector optimization of the problem treated here can be found in Ref. 22. Here we consider as primal problem

$$(P_5) \qquad \min_{Cx \leqq b} \left\{ h(x) + \sum_{i=1}^{N} \big(f_i^2(x)/g_i(x) \big) \right\},$$

where $f_i, h : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex functions, $g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ concave, $f_i(x) \geq 0$, $g_i(x) > 0$, $i = 1, ..., N$, for all x feasible to (P_5), $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{m \times n}$, that is equivalent to

$$(P_5) \qquad \min_{\substack{f_i(x) \leq s_i, s_i \in \mathbb{R}_+, \\ g_i(x) \geq t_i, t_i \in \mathbb{R}_+ \setminus \{0\}, \\ i=1,...,N, Cx \leqq b}} \left\{ h(x) + \sum_{i=1}^{N} \big(s_i^2/t_i \big) \right\}.$$

The Fenchel-Lagrange dual problem arises naturally from its basic formula, where we denote the objective function by $u(x, s, t)$, $s = (s_1, ..., s_N)^T$, $t = (t_1, ..., t_N)^T$

and also the functions $f=(f_1,...,f_N)^T$ and $g=(g_1,...,g_N)^T$,

$$(D_5)\quad \sup_{\substack{p^x\in\mathbb{R}^n, p^s,p^t\in\mathbb{R}^N,\\ q^x\in\mathbb{R}^m_+, q^s,q^t\in\mathbb{R}^N_+}} \Bigg\{ -u^*(p^x,p^s,p^t)+\inf_{\substack{x\in\mathbb{R}^n, s\in\mathbb{R}^N_+,\\ t\in\mathrm{int}(\mathbb{R}^N_+)}}\Big[\langle p^x,x\rangle+\langle p^s,s\rangle$$

$$+\langle p^t,t\rangle+\langle q^s,f(x)-s\rangle+\langle q^t,t-g(x)\rangle+\langle q^x,Cx-b\rangle\Big]\Bigg\}.$$

For the conjugate function one has (consult Ref. 22 for computational details), denoting $p^s=(p^s_1,...,p^s_N)^T$ and $p^t=(p^t_1,...p^t_N)^T$,

$$u^*(p^x,p^s,p^t)=\begin{cases} h^*(p^x), & (p^s_i)^2+4p^t_i\leq 0, i=1,...,N,\\ +\infty, & \text{otherwise,}\end{cases}$$

while the infimum over (x,s,t) is separable into a sum of three infima each of them concerning a variable. The dual problem becomes

$$(D_5)\quad \sup_{\substack{p^x\in\mathbb{R}^n, p^s,p^t\in\mathbb{R}^N,\\ (p^s_i)^2+4p^t_i\leq 0, i=1,...,N,\\ q^x\in\mathbb{R}^m_+, q^s,q^t\in\mathbb{R}^N_+}} \Bigg\{ -h^*(p^x)+\inf_{s\in\mathbb{R}^N_+}\Big[\langle p^s,s\rangle-\langle q^s,s\rangle\Big]-\langle q^x,b\rangle+$$

$$\inf_{x\in\mathbb{R}^n}\Big[\langle p^x+C^Tq^x,x\rangle+\langle q^s,f(x)\rangle-\langle q^t,g(x)\rangle\Big]+\inf_{t\in\mathbb{R}^N_+\setminus\{0\}}\Big[\langle p^t,t\rangle+\langle q^t,t\rangle\Big]\Bigg\}.$$

The infimum regarding $s\in\mathbb{R}^N_+$ has a negative infinite value unless $p^s-q^s\geqq 0$, when it nullifies itself, while the one regarding $t\in\mathbb{R}^N_+\setminus\{0\}$ is zero when $p^t+q^t\geqq 0$, otherwise being equal to $-\infty$. The infimum regarding $x\in\mathbb{R}^n$ can be turned into a supremum and computed as a conjugate of a sum of functions at $-(p^x+C^Tq^x)$. Applying Theorem 20.1 in Ref. 17 to this conjugate, the dual develops denoting $q^s=(q^s_1,...,q^s_N)^T$ and $q^t=(q^t_1,...,q^t_N)^T$ into

$$(D_5)\quad \sup_{\substack{q^x\in\mathbb{R}^m_+, q^s,q^t\in\mathbb{R}^N_+,\\ p^s,p^t\in\mathbb{R}^N, p^s\geqq q^s, p^t\geqq -q^t,\\ p^t_i\leq -\left(p^s_i/2\right)^2, i=1,...,N,\\ a_i,d_i,p^x\in\mathbb{R}^n, i=1,...,N,\\ \sum_{i=1}^N (a_i+d_i)=-p^x-C^Tq^x}} \Bigg\{ -h^*(p^x)-\sum_{i=1}^N (q^s_if_i)^*(a_i)$$

$$-\sum_{i=1}^N(-q^t_ig_i)^*(d_i)-\langle q^x,b\rangle\Bigg\},$$

that can be simplified, renouncing the variables p^s and p^t, to

$$(D_5) \quad \sup_{\substack{q^x \in \mathbb{R}^m_+, q^s, q^t \in \mathbb{R}^N_+, \\ q^t_i \geq \left(q^s_i/2\right)^2, i=1,...,N, \\ a_i, d_i, p^x \in \mathbb{R}^n, i=1,...,N, \\ \sum\limits_{i=1}^N (a_i + d_i) = -p^x - C^T q^x}} \left\{ - h^*(p^x) - \sum_{i=1}^N (q^s_i f_i)^*(a_i) - \sum_{i=1}^N (-q^t_i g_i)^*(d_i) - \langle q^x, b \rangle \right\}.$$

Writing the homogenous extensions of the conjugate functions one gets the dual problem obtained in the original paper. Let us stress that we have ignored the hypotheses of closedness associated to the functions f, g and h in Ref. 10, as strong duality is valid even in their absence.

Theorem 4.9. Provided that the primal problem (P_5) has a finite optimal objective value, strong duality between problems (P_5) and (D_5) is assured.

The optimality conditions we determined in this case are richer than the ones presented in Ref. 10.

Theorem 4.10.

(a) If the problem (P_5) has an optimal solution $\bar{x}$ where its objective function is finite, then strong duality between the problems (P_5) and (D_5) is attained and the dual problem has an optimal solution $(\bar{p}^x, \bar{q}^x, \bar{q}^s, \bar{q}^t, \bar{a}, \bar{d})$ with $\bar{a} = (\bar{a}_1, ..., \bar{a}_N)^T$ and $\bar{d} = (\bar{d}_1, ..., \bar{d}_N)^T$ satisfying the following optimality conditions

(i) $\left(\bar{q}^s_i f_i\right)^*(\bar{a}_i) + \bar{q}^s_i f_i(\bar{x}) = \langle \bar{a}_i, \bar{x} \rangle, i = 1, ..., N,$

(ii) $\left(- \bar{q}^t_i g_i\right)^*(\bar{d}_i) - \bar{q}^t_i g_i(\bar{x}) = \langle \bar{d}_i, \bar{x} \rangle, i = 1, ..., N,$

(iii) $h^*(\bar{p}^x) + h(\bar{x}) = \langle \bar{p}^x, \bar{x} \rangle,$

(iv) $\bar{q}^s_i = 2\big(f_i(\bar{x})/g_i(\bar{x})\big), i = 1, ..., N,$

(v) $\bar{q}^t_i = f_i^2(\bar{x})/g_i^2(\bar{x}), i = 1, ..., N,$

(vi) $\langle \bar{q}^x, b - C^T \bar{x} \rangle = 0.$

(b) Having a feasible solution $\bar{x}$ to the primal problem and one to the dual $(\bar{p}^x, \bar{q}^x, \bar{q}^s, \bar{q}^t, \bar{a}, \bar{d})$ satisfying the optimality conditions (i)-(vi), the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality holds.

4.6. Quasiconcave multiplicative programs (Ref. 2). Despite its intricateness, geometric programming seems to be still very popular, as its direct applications still get published. The latest we could find is on a class of quasiconcave multiplicative programs that originally look like

$$(P_6) \qquad \sup_{Ax \leqq b} \left\{ \prod_{i=1}^{k} [f_i(x)]^{a_i} \right\},$$

with $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ concave functions, positive over the feasible set of the problem, $a_i > 0$, $i = 1, ..., k$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. The problem is brought into another layout in order to be properly treated,

$$(\widetilde{P}_6) \qquad \inf_{\substack{f_i(x) \geq s_i, i=1,...,k,\\ Ax \leqq b, s \in \operatorname{int}(\mathbb{R}^k_+)}} \left\{ -\sum_{i=1}^{k} a_i \ln s_i \right\}.$$

Denoting $f = (f_1, ..., f_k)^T$, $s = (s_1, ..., s^k)^T$ and $u : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$u(x, s) = \begin{cases} -\sum\limits_{i=1}^{k} a_i \ln s_i, & \text{if } (x, s) \in \mathbb{R}^n \times \operatorname{int}(\mathbb{R}^k_+), \\ +\infty, & \text{otherwise,} \end{cases}$$

the raw formula of the Fenchel-Lagrange dual to $(\widetilde{P}_6)$ is

$$(\widetilde{D}_6) \qquad \sup_{\substack{p^x \in \mathbb{R}^n, p^s \in \mathbb{R}^k,\\ q^l \in \mathbb{R}^m_+, q^f \in \mathbb{R}^k_+}} \left\{ -u^*(p^x, p^s) + \inf_{\substack{x \in \mathbb{R}^n,\\ s \in \operatorname{int}(\mathbb{R}^k_+)}} \Big[\langle p^x, x \rangle + \langle p^s, s \rangle + \langle q^l, Ax - b \rangle + \langle q^f, s - f(x) \rangle \Big] \right\}.$$

Regarding the conjugate of the objective function the following result is available for $p^s = (p^s_1, ..., p^s_k)^T$

$$u^*(p^x, p^s) = \begin{cases} -\sum\limits_{i=1}^{k} a_i \Big(1 - \ln \big(a_i / (-p^s_i) \big) \Big), & \text{if } p^x = 0, p^s < 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The infimum in the dual problem can also be separated into a sum of two infima, one concerning $s \in \operatorname{int}(\mathbb{R}^k_+)$, the other $x \in \mathbb{R}^n$. Let us write again the dual using the last observations and denoting $-p^s_i$ by p^s_i, $i = 1, ..., k$,

$$(\widetilde{D}_6) \quad \sup_{\substack{p^s \in \operatorname{int}(\mathbb{R}^k_+),\\ q^l \in \mathbb{R}^m_+, q^f \in \mathbb{R}^k_+}} \left\{ \sum_{i=1}^{k} a_i \Big(1 - \ln \big(a_i / p^s_i \big) \Big) + \inf_{s \in \operatorname{int}(\mathbb{R}^k_+)} \langle q^f - p^s, s \rangle + \inf_{x \in \mathbb{R}^n} \Big[\langle q^l, Ax \rangle - \langle q^f, f(x) \rangle \Big] - \langle q^l, b \rangle \right\}.$$

The infimum regarding s is equal to 0 when $q^f - p^s \geqq 0$, otherwise being $-\infty$, while the one over $x \in \mathbb{R}^n$ can be rewritten as a supremum and viewed as a conjugate of a sum of functions. The dual problem yields

$$(\widetilde{D}_6) \quad \sup_{\substack{p^s \in \operatorname{int}(\mathbb{R}^k_+), q^f \geqq p^s, \\ q^l \in \mathbb{R}^m_+, \sum\limits_{i=1}^k v_i = -A^T q^l}} \left\{ \sum_{i=1}^k \Big(a_i - a_i \ln \big(a_i / p_i^s \big) \Big) - \sum_{i=1}^k q_i^f (-f_i)^* \big((1/q_i^f) v_i \big) - \langle b, q^l \rangle \right\},$$

where $q^f = (q_1^f, ..., q_k^f)^T$, and as the supremum regarding the variable p^s can be easily computed, being attained for $p^s = -q^f$, we get the following final version of the dual, equivalent to the one found in Ref. 2,

$$(\widetilde{D}_6) \quad \sup_{\substack{q^l \in \mathbb{R}^m_+, q^f \in \operatorname{int}(\mathbb{R}^k_+), \\ \sum\limits_{i=1}^k v_i = -A^T q^l}} \left\{ \sum_{i=1}^k \Big(a_i - a_i \ln \big(a_i / q_i^f \big) \Big) - \sum_{i=1}^k q_i^f (-f_i)^* \Big((1/q_i^f) v_i \Big) - \langle b, q^l \rangle \right\}.$$

For strong duality a constraint qualification would normally be required because within the constraints of $(\widetilde{P}_6)$ there are affine as well as non-affine functions. But when the problem (P_6) has a non-empty feasible set, the existence of a feasible point x' so that $f_i(x') > 0, i = 1, ..., k$, is assured and there is also an $s' > 0$ so that $f_i(x') > s' > 0$, $i = 1, ..., k$. So the constraint qualification that comes from the general case is automatically fulfilled for $(\widetilde{P}_6)$. Without any additional assumption, such as closedness, required for the functions $f_i, i = 1, ..., k$, in the cited paper, one may formulate the strong duality statement.

Theorem 4.11. Provided that the primal problem has a feasible point, strong duality between problems $(\widetilde{P}_6)$ and $(\widetilde{D}_6)$ is assured.

No surprises appear when we derive the optimality conditions concerning the pair of dual problems in discussion.

Theorem 4.12.

(a) If the problem $(\widetilde{P}_6)$ has an optimal solution $\bar{x}$, then strong duality between the problems $(\widetilde{P}_6)$ and $(\widetilde{D}_6)$ is attained and the dual problem has an optimal solution $(\bar{v}_1, ..., \bar{v}_k, \bar{q}^l, \bar{q}^f)$ satisfying the following optimality conditions

(i) $(-f_i)^*\Big(\big(1/\bar{q}_i^f\big)\bar{v}_i\Big) - f_i(\bar{x}) = \Big\langle \big(1/\bar{q}_i^f\big)\bar{v}_i, \bar{x}\Big\rangle, i = 1, ..., k,$

(ii) $\langle A^T\bar{x} - b, \bar{q}^l\rangle = 0,$

(iii) $\sum\limits_{i=1}^{k} \bar{v}_i = -\bar{A}^T q^l,$

(iv) $\ln(f_i(\bar{x})) + \Big(\bar{q}_i^f/a_i\Big) f_i(\bar{x}) = \ln\Big(a_i/\bar{q}_i^f\Big) - 1, i = 1, ..., k.$

(b) Having a feasible solution $\bar{x}$ to the primal problem and one to the dual $(\bar{v}_1, ..., \bar{v}_k, \bar{q}^l, \bar{q}^f)$ satisfying the optimality conditions (i)-(iv), the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality holds.

4.7. Posynomial geometric programming (Ref. 24). We are going to prove now that also the posynomial geometric programming duality can be viewed as a special case of the Fenchel-Lagrange duality. As it has been already proved (cf. Ref. 1) that the generalized geometric programming includes the posynomial instance as special case, our result is not so surprising within the framework of this paper. The primal-dual pair of posynomial geometric problems is composed by

$$(P_7) \qquad \inf_{\substack{t\in\mathbb{R}_+,\\ g_j(t)\leq 1,\\ j=1,...,s}} g_0(t),$$

where

$$g_k(t) = \sum_{i\in J[k]} c_i \prod_{j=1}^{m} t^{a_{ij}}, k = 0, ..., s,$$

$$a_{ij} \in \mathbb{R}, j = 1, ..., m, c_i > 0, i = 1, ..., n,$$

$$J[k] = \{m_k, m_k + 1, ..., n_k\}, k = 0, ..., s,$$

$$m_0 = 1, m_1 = n_0 + 1, ..., m_k = n_{k-1} + 1, ..., n_s = n,$$

and

$$(D_7) \qquad \sup_{\substack{\delta\in\mathbb{R}_+,\\ \sum\limits_{i\in J[0]}\delta_i=1,\\ \sum\limits_{i=1}^{n}\delta_i a_{ij}=0,\\ j=1,...,m}} \left[\prod_{i=1}^{n}(c_i/\delta_i)^{\delta_i}\right] \prod_{k=1}^{s} \lambda_k(\delta)^{\lambda_k(\delta)},$$

where

$$\lambda_k(\delta) = \sum_{i\in J[k]} \delta_i, k = 1, ..., s.$$

The primal posynomial problem is equivalent to (cf. Refs. 1 and 24)

$$(\widetilde{P}_7) \qquad \inf_{\substack{\ln\left(\sum\limits_{i\in J[k]} c_i \exp(x_i)\right)\leq 0,\\ k=1,\dots,s, x\in\mathcal{A}}} \left\{ \ln\left(\sum_{i\in J[0]} c_i \exp(x_i)\right)\right\},$$

where $\mathcal{A}$ denotes the linear subspace generated by the columns of the exponent matrix (a_{ij}). Let us name also the primal objective function $u(x)$. We determine the Fenchel-Lagrange dual problem to $(\widetilde{P}_7)$ from the formula of (D'_K), with $\mathcal{A}^\perp$ indicating the orthogonal subspace of $\mathcal{A}$, for $q=(q_1,...,q_s)^T$ and $t=(t_1,...,t_n)^T$

$$(\widetilde{D}_7) \quad \sup_{\substack{p\in\mathbb{R}^n, t\in\mathcal{A}^\perp,\\ q\in\mathbb{R}^s_+}} \left\{ -u^*(p) + \inf_{x\in\mathcal{A}} \left[\langle p-t, x\rangle + \sum_{k=1}^s q_k \ln\left(\sum_{i\in J[k]} c_i \exp(x_i)\right)\right]\right\}.$$

For the conjugate of the objective function we have

$$u^*(p) = \begin{cases} \sum\limits_{i\in J[0]} p_i \ln(p_i/c_i), & \text{if } p_j = 0, j\in J[k], k=1,...,s, \\ & \sum\limits_{i\in J[0]} p_i = 1, p=(p_1,...,p_n)^T \in \mathbb{R}^n_+, \\ +\infty, & \text{otherwise} \end{cases}$$

and similar results can be derived if we write the infimum within the dual as a sum of suprema over $(x_i)_{i\in J[k]}$, $k=1,...,s$, just with the changed constraints $\sum\limits_{i\in J[k]} t_i = q_k$. Also there follows $p_i = t_i, i\in J[0]$. Like in entropy optimization we consider $0\ln(0/c_i)=0, c_i>0, i=1,...,n$. After these, the dual problem becomes

$$(\widetilde{D}_7) \qquad \sup_{\substack{t\in\mathcal{A}^\perp, q\in\mathbb{R}^s_+,\\ t\geqq 0, \sum\limits_{i\in J[0]} t_i = 1,\\ \sum\limits_{i\in J[k]} t_i = q_k, k=1,\dots,s}} \left\{ \sum_{i=1}^n t_i \ln(c_i/t_i) + \sum_{k=1}^s q_k \ln q_k \right\}$$

that is equivalent to the problem (D_7), namely they have the same optimal solution and the objective function of (D_7) is the exponential of the objective function of $(\widetilde{D}_7)$.

Finally, the condition that guarantees strong duality, derived from the constraint qualification (CQ), is actually the so-called superconsistency introduced in Ref. 24, i. e.

$$(CQ_7) \qquad \exists t' > 0 : g_k(t') < 1, k=1,...,s.$$

Theorem 4.13. If the constraint qualification (CQ_7) is satisfied, then the strong duality between (P_7) and (D_7) is assured.

Consequently we present also the optimality conditions concerning this pair of problems.

Theorem 4.14.

(a) If the constraint qualification (CQ_7) is fulfilled and $\bar{x}$ is an optimal solution to (P_7), then strong duality between the problems (P_7) and (D_7) is attained and the dual problem has an optimal solution $(\bar{t}, \bar{q})$ satisfying the following optimality conditions

(i) $\ln\left\{\sum\limits_{i\in J[0]} c_i \exp(\bar{x}_i)\right\} + \sum\limits_{i\in J[0]} \bar{t}_i \ln(\bar{t}_i/c_i) = \langle \bar{t}^{J[0]}, \bar{x}^{J[0]}\rangle$,

(ii) $\bar{q}_k \ln\left\{\sum\limits_{i\in J[k]} c_i \exp(\bar{x}_i)\right\} + \sum\limits_{i\in J[k]} \bar{t}_i \ln(\bar{t}_i/c_i) - \bar{q}_k \ln \bar{q}_k = \langle \bar{t}^{J[k]}, \bar{x}^{J[k]}\rangle$, $k = 1, ..., s$,

(iii) $\sum\limits_{i\in J[k]} c_i \exp(\bar{x}_i) = 1$ when $\bar{q}_k > 0$, $k = 1, ..., s$,

(iv) $\langle \bar{x}, \bar{t}\rangle = 0$,
where $x^{J[k]} = (x_{m_k}, ..., x_{n_k})^T$ and $t^{J[k]} = (t_{m_k}, ..., t_{n_k})^T$, $k = 0, ..., s$.

(b) Having a feasible solution $\bar{x}$ to the primal problem and one to the dual $(\bar{t}, \bar{q})$ satisfying the optimality conditions (i)-(iv), then the mentioned feasible solutions turn out to be optimal solutions to the corresponding problems and strong duality holds.

5. Conclusions

Let us sum up now the main results in this paper. First we have introduced a new duality approach, named Fenchel-Lagrange duality because of the way it is established and of the properties it has. Strong duality and optimality conditions for a general convex optimization primal problem and its Fenchel-Lagrange dual are presented, provided that a constraint qualification is valid. Then we prove that the geometric duality used by Scott and Jefferson in many of their papers turns out to be a special case of our Fenchel-Lagrange duality. In all the invoked papers the mentioned authors present the geometric dual problem to the primal and give the necessary and sufficient optimality conditions that are true under assumptions of convexity and closedness concerning the functions and the sets involved there, together with a constraint qualification. We established the same dual problem to the primal exploiting the Fenchel-Lagrange duality we presented earlier. Strong duality and optimality conditions are revealed to stand in much weaker circumstances, i. e. the closedness can be cancelled from the initial assumptions, while the constraint qualification can be generalized and weakened, respectively. We review some convex optimization problems treated by the cited authors by means of geometric programming duality, showing how their duals can be obtained easier and more directly via Fenchel-Lagrange duality. Of course strong duality is accomplished under weaker sufficient conditions than in the original papers, namely the closedness assumptions for the functions and sets involved can be removed. Necessary and sufficient optimality conditions are derived for each problem.

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