On the Construction of Gap Functions for Variational Inequalities via Conjugate Duality

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Abstract: In this paper we deal with the construction of gap functions for variational inequalities by using an approach which bases on the conjugate duality. Under certain assumptions we also investigate a further class of gap functions for the variational inequality problem, the so-called dual gap functions.

Key words: variational inequalities, gap functions, conjugate duality, dual gap functions

1 Introduction

Let $K \subseteq \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued mapping. The variational inequality problem is to find a point $x \in K$ such that

$$(VI) \qquad F(x)^T(y - x) \geq 0, \ \forall y \in K,$$

where ^{T} as usual denotes the transposition of a vector in $\mathbb{R}^n$. For a comprehensive survey of the problem (VI) we refer to [10] and [12]. In the literature even though mostly is supposed that K is a closed, convex set and F is a continuous mapping, in this paper we will make such assumptions only if they are required. One of the approaches for solving the problem (VI) is to reformulate it into an equivalent optimization problem. It is well known that the problem (VI) can be transformed into an optimization problem

$$\inf_{x \in K} \phi(x),$$

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in case F is the gradient of a differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$, F is also differentiable and its Jacobian $\nabla F(y)$ is symmetric for all $y \in \mathbb{R}^n$. For asymmetric variational inequalities, several approaches which are based on so-called gap or merit functions have been investigated. For details we refer to [1], [2], [4], [8], [16], [20] and to the survey papers [9], [13]. A function $\gamma : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be a gap function for the problem (VI) if it satisfies the following properties

(i) $\gamma(y) \geq 0, \ \forall y \in K$;

(ii) $\gamma(x) = 0$ if and only if x solves the problem (VI).

Recently, the study of gap functions for the problem (VI) and for some of its special formulations, where the ground set K is defined by means of convex inequality constraints, has been associated to the Lagrange duality (cf. [5], [6], [21]). Moreover, the connections between properties of gap functions and duality have been interpreted in the context of convex optimization and variational inequalities (see [4], [11]).
This paper aims to relate gap functions for variational inequalities to the conjugate duality for an optimization problem. By using conjugate dual problems which have been investigated in [19], we propose some new gap functions for variational inequalities. Under certain assumptions, we discuss a further class of gap functions for the problem (VI), the so-called dual gap functions.

2 Conjugate duality

Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ be given functions. We consider the optimization problem

$$(P) \qquad \inf_{x \in X \cap G} f(x), \quad G = \{x \in \mathbb{R}^n |\ g(x) \leqq 0\}.$$

For $x, y \in \mathbb{R}^m$, by " $\leqq$„ we denote the following ordering relation

$$x \leqq y \ \Leftrightarrow \ y - x \in \mathbb{R}^m_+ = \{z = (z_1, ..., z_m)^T \in \mathbb{R}^m |\ z_i \geq 0, \ i = \overline{1, m}\}.$$

By using a general perturbation approach and the theory of the conjugate duality the following dual problems for (P) have been introduced (see [3], [19])

$$(D_L) \qquad \sup_{q \geqq 0} \inf_{x \in X} [f(x) + q^T g(x)],$$

$$(D_F) \qquad \sup_{p \in \mathbb{R}^n} \Big\{ - f^*(p) + \inf_{x \in X \cap G} p^T x \Big\}$$

and

$$(D_{FL}) \qquad \sup_{\substack{p\in\mathbb{R}^n \\ q\geqq 0}} \Big\{ -f^*(p) + \inf_{x\in X}[p^T x + q^T g(x)]\Big\},$$

or, equivalently,

$$(D_{FL}) \qquad \sup_{\substack{p\in\mathbb{R}^n \\ q\geqq 0}} \Big\{ -f^*(p) - (q^T g)^*_X(-p)\Big\}.$$

Here $h^*_X : \mathbb{R}^n \to \overline{\mathbb{R}}$ we denote the conjugate function of the function h relative to the set X defined by $h^*(\xi) = \sup_{x\in X}[\xi^T x - h(x)]$. If $X = \mathbb{R}^n$, then h^*_X becomes the classical (Fenchel-Moreau) conjugate which will be denoted by h^*. Let us also denote by dom $h = \Big\{x \in \mathbb{R}^n |\ h(x) < +\infty\Big\}$ the effective domain of h and for a subset $C \subseteq \mathbb{R}^n$, by ri(C) its relative interior. The problems (D_L) and (D_F) are the classical Lagrange and Fenchel dual problems, respectively. The dual problem (D_{FL}) is called the Fenchel-Lagrange dual and it is a "combination" of the Fenchel and Lagrange dual problems. By construction weak duality always holds, i.e., the optimal objective values of the mentioned dual problems are less than or equal to the optimal objective value of (P). Denoting by $v(P)$ the optimal objective value of (P) and by $v(D_L)$, $v(D_F)$, $v(D_{FL})$ the optimal objective values of (D_L), (D_F) and (D_{FL}), respectively, the following result has been proved.

Proposition 2.1 *Let X be a convex set and f, g_i, $i = \overline{1,m}$ be convex functions. Then the following assertions are true:*

(i) *If ri$(X \cap G) \cap$ ri(dom $f) \neq \varnothing$, then $v(D_F) = v(P)$. Moreover, if $v(P)$is finite then the problem (D_F) has an optimal solution.*

(ii) *If ri$(X) \cap$ ri(dom $f) \neq \varnothing$, then $v(D_L) = v(D_{FL})$.*

In order to formulate the strong duality theorem for the primal and the three dual problems, we need the following constraint qualification

$$(CQ) \qquad \exists x' \in \text{ri}(X) \cap \text{ri}(\text{dom } f) : \ \left[\begin{array}{l} g_i(x') \leq 0,\ i \in L, \\ g_i(x') < 0,\ i \in N. \end{array} \right.$$

Here

$$L = \{i \in \{1, ..., m\} \mid g_i \text{ is an affine function}\}$$

and

$$N = \{i \in \{1, ..., m\} \mid g_i \text{ is not an affine function}\}.$$

Proposition 2.2 (**Strong duality**)
Let X be a convex set and $f,\ g_i,\ i=\overline{1,m}$ be convex functions.Assume that the constraint qualification (CQ) is fulfilled. If $v(P)$ is finite then $(D_L),\ (D_F)$, (D_{FL}) have optimal solutions and it holds

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}).$$

3 Gap functions for variational inequalities

By using the conjugate duality theory presented in the previous section, we discuss the construction of gap functions for variational inequalities. Before to do this, we recall some well-known gap functions for the problem (VI).

Definition 3.1 (Auslender's gap function, [2])

$$\gamma_A^{VI}(x) := \max_{y \in K} F(x)^T(x-y).$$

Let us now assume that the ground set K is defined by

$$K = \{x \in \mathbb{R}^n |\ g_i(x) \leq 0,\ i = 1,2,..,m\}, \tag{3.1}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $g(x) = (g_1(x), ..., g_m(x))^T$. Giannessi proposed the following gap function which explicitly incorporates the constraints that define the ground set K.

Definition 3.2 (Giannessi's gap function, [5])

$$\gamma_G^{VI}(x) := \inf_{\lambda \geqq 0} \sup_{y \in \mathbb{R}^n} \Big\{ F(x)^T(x-y) - \lambda^T g(y) \Big\}.$$

Note that the formulation of Giannessi's gap function is inspired by the Lagrange duality for the optimization problem

$$(P^{VI}; x) \qquad \inf_{y \in K} F(x)^T(y-x),$$

where K is given by (3.1) and $x \in \mathbb{R}^n$ is fixed. It is easy to see that

$$\gamma_G^{VI}(x) = -v(D_L^{VI}; x),$$

where $v(D_L^{VI}; x)$ denotes the optimal objective value of the Lagrange dual problem for $(P^{VI}; x)$. Now let us state the Fenchel dual problem for $(P^{VI}; x)$ and define a function in the similar way, i.e.,

$$\gamma_F^{VI}(x) := -v(D_F^{VI}; x).$$

Since the conjugate of the objective function for $(P^{VI}; x)$ is

$$\sup_{y \in \mathbb{R}^n} [p^T y - F(x)^T (y - x)] = \begin{cases} F(x)^T x, & \text{if } p = F(x), \\ +\infty, & \text{otherwise}, \end{cases} \tag{3.2}$$

the Fenchel dual problem for $(P^{VI}; x)$ turns out to be

$$(D_F^{VI}, x) \qquad \sup_{p = F(x)} \Big\{ - F(x)^T x + \inf_{y \in K} p^T y \Big\} = \inf_{y \in K} F(x)^T (y - x).$$

Whence we get

$$\gamma_F^{VI}(x) := -v(D_F^{VI}; x) = - \inf_{y \in K} F(x)^T (y - x) = \sup_{y \in K} F(x)^T (x - y).$$

γ_F^{VI} is nothing else than Auslender's gap function. Let us notice that, by using the Fenchel duality, we can define a gap function for an arbitrary ground set K. Assuming again that the ground set K is given by (3.1), in view of (3.2), the Fenchel-Lagrange dual problem for $(P^{VI}; x)$ becomes

$$\begin{aligned} (D_{FL}^{VI}; x) \qquad & \sup_{\substack{p = F(x) \\ q \geqq 0}} \Big\{ - F(x)^T x + \inf_{y \in \mathbb{R}^n} [p^T y + q^T g(y)] \Big\} \\ = \qquad & \sup_{q \geqq 0} \inf_{y \in \mathbb{R}^n} [F(x)^T (y - x) + q^T g(y)] \Big\}. \end{aligned}$$

The function $\gamma_{FL}^{VI}(x) := -v(D_{FL}^{VI}; x)$ also reduces to the Giannessi's gap function. The result can be summarized as follows.

Proposition 3.1

(i) *For the problem* (VI), *it holds* $\gamma_F^{VI}(y) = \gamma_A^{VI}(y), \ \forall y \in \mathbb{R}^n$.

(ii) *If the ground set is given by* (3.1), *then it holds*

$$\gamma_{FL}^{VI}(y) = \gamma_L^{VI}(y) = \gamma_G^{VI}(y), \ \forall y \in \mathbb{R}^n.$$

The problem (VI) can be generalized to the following variational inequality problem, find a point $x \in K$ such that

$$(GVI) \qquad F(x)^T (y - x) + f(y) - f(x) \geq 0, \ \forall y \in K,$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper, convex function (see, for instance [8], [23]). As said before, to the problem (GVI) one can associate the following primal problem

$$(P^{GVI}; x) \qquad \inf_{y \in K} \varphi(y),$$

where $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}},\ \varphi(y) := F(x)^T(y-x) + f(y) - f(x)$ and $x \in \mathbb{R}^n$ is fixed. One can derive the conjugate of φ by

$$\begin{aligned}\varphi^*(p) &= \sup_{y\in\mathbb{R}^n}[p^Ty - \varphi(y)] = \sup_{y\in\mathbb{R}^n}[p^Ty - F(x)^T(y-x) - f(y) + f(x))] \\ &= f^*(p - F(x)) + F(x)^Tx + f(x). \end{aligned} \tag{3.3}$$

Therefore the Fenchel dual problem for $(P^{GVI}; x)$ is

$$(D_F^{GVI}; x) \qquad \sup_{p\in\mathbb{R}^n}\Big\{ - f^*(p - F(x)) - F(x)^Tx - f(x) + \inf_{y\in K} p^Ty\Big\}.$$

Likewise the problem (VI), we can introduce the following function

$$\gamma_F^{GVI}(x) := -v(D_F^{GVI}; x) = \inf_{p\in\mathbb{R}^n}\Big\{f^*(p - F(x)) + F(x)^Tx + f(x) + \delta_K^*(-p)\Big\},$$

where δ_K denotes the indicator function of the set K.

Theorem 3.1 *Let $ri(K) \cap ri(dom\ f) \neq \varnothing$ and K be a convex set. Then γ_F^{GVI} is a gap function for the problem (GVI).*

Proof:

(i) Let $x \in K$ be fixed. By weak duality it holds

$$v(D_F^{GVI}; x) \leq v(P^{GVI}; x) \leq 0.$$

Whence $\gamma_F^{GVI}(x) = -v(D_F^{GVI}; x) \geq 0$.

(ii) If $\gamma_F^{GVI}(x) = 0$, then $0 = v(D_F^{GVI}; x) \leq v(P^{GVI}; x) \leq 0$ and so $v(P^{GVI}; x) = 0$. This means that x solves the problem (GVI). On the other hand, if $x \in K$ is a solution to the problem (GVI), then $v(P^{GVI}; x) = 0$. By Proposition 2.1(i) implies that

$$\gamma_F^{GVI}(x) = -v(D_F^{GVI}; x) = -v(P^{GVI}; x) = 0.$$

□

Similarly, by using the formulations of the duals (D_L) and (D_{FL}), we can introduce for $x \in \mathbb{R}^n$ the following functions

$$\begin{aligned}\gamma_L^{GVI}(x) :&= -v(D_L^{GVI}; x) \\ &= -\sup_{q\geqq 0}\inf_{y\in\mathbb{R}^n}\Big\{F(x)^T(y-x) + f(y) - f(x) + q^Tg(y)\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \inf_{q \geqq 0} \sup_{y \in \mathbb{R}^n} \Big\{ F(x)^T(x-y) - f(y) + f(x) - q^T g(y) \Big\} \\
&= \inf_{q \geqq 0} \Big\{ F(x)^T x + f(x) + \sup_{y \in \mathbb{R}^n} [-F(x)^T y - f(y) - q^T g(y)] \Big\} \\
&= \inf_{q \geqq 0} \Big\{ F(x)^T x + f(x) + (f + q^T g)^*(-F(x)) \Big\}
\end{aligned}
$$

and, in view of (3.3),

$$
\begin{aligned}
\gamma_{FL}^{GVI}(x) : &= -v(D_{FL}^{GVI}; x) \\
&= -\sup_{\substack{p \in \mathbb{R}^n \\ q \geqq 0}} \Big\{ -f^*(p - F(x)) - F(x)^T x - f(x) - (q^T g)^*(-p) \Big\} \\
&= \inf_{\substack{p \in \mathbb{R}^n \\ q \geqq 0}} \Big\{ f^*(p - F(x)) + F(x)^T x + f(x) + (q^T g)^*(-p) \Big\},
\end{aligned}
$$

respectively, if K is given by (3.1).

Theorem 3.2 *Assume that the constraint qualification* (CQ) *is fulfilled. Then* γ_L^{GVI} *and* γ_{FL}^{GVI} *are gap functions for the problem* (GVI).

Proof:

(i) It is easily verified by weak duality (see the proof of the Theorem 3.1(i)).

(ii) By $\gamma_L^{GVI}(x) = \gamma_{FL}^{GVI}(x) = 0$, x is a solution of the problem (GVI). Conversely, let the problem (GVI) be solved by x and the constraint qualification (CQ) be fulfilled. Then by Proposition 2.2, it holds strong duality. This implies that

$$
\gamma_L^{GVI}(x) = \gamma_{FL}^{GVI}(x) = -v(D_L^{GVI}; x) = -v(D_{FL}^{GVI}; x) = -v(P^{GVI}; x) = 0
$$

□

Remark 3.1 Because of $v(P^{GVI}; x) = 0$, where x is fixed, by the strong duality result in Section 2, the dual problems for $(P^{GVI}; x)$ have optimal solutions. Consequently, under the assumptions of Theorem 3.1 and Theorem 3.2, one can use "min" instead of "inf" for the proposed gap functions.

Remark 3.2 If one takes $K = \mathbb{R}^n$ in the formulation of the problem (GVI), then it reduces to the extended variational inequality problem. By using γ_F^{GVI}, we obtain the same gap function for the extended variational inequality as in [4]. This shows that our approach generalizes some previous results.

Indeed, because of

$$\delta^*_{\mathbb{R}^n}(-p) = \sup_{x\in\mathbb{R}^n} [-p^T x] = \begin{cases} 0, & \text{if } p = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \gamma_F^{EVI}(x) &= \inf_{p\in\mathbb{R}^n} \left\{ f^*(p - F(x)) + F(x)^T x + f(x) + \delta^*_{\mathbb{R}^n}(-p) \right\} \\ &= f^*(-F(x)) + F(x)^T x + f(x). \end{aligned}$$

Next let us study the relations between the gap functions for (GVI) introduced above.

Proposition 3.2 *Let the ground set K be given by* (3.1). *Then it holds*

$$\left.\begin{array}{l} \gamma_L^{GVI}(x) \\ \\ \gamma_F^{GVI}(x) \end{array}\right. \le \gamma_{FL}^{GVI}(x), \ \forall x \in \mathbb{R}^n.$$

Proof: Let $x \in \mathbb{R}^n$ be fixed. According to Propositions 2.1 and 2.2 in [3] (see also [19]) implies that

$$v(D_{FL}^{GVI}; x) \le \begin{array}{l} v(D_L^{GVI}; x) \\ \\ v(D_F^{GVI}; x) \end{array},$$

or, equivalently,

$$\begin{array}{l} -v(D_L^{GVI}; x) \\ \\ -v(D_F^{GVI}; x) \end{array} \le -v(D_{FL}^{GVI}; x).$$

which leads to the desired conclusion. □

One of the desirable properties for gap functions is the convexity. Under certain assumptions this property will be fulfilled. Further we need the following definition.

Definition 3.3 A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(i) monotone if $[F(x) - F(y)]^T (x - y) \ge 0, \ \forall x, y \in \mathbb{R}^n$;

(ii) pseudo-monotone if $F(y)^T (x - y) \ge 0$ implies

$$F(x)^T (x - y) \ge 0, \ \forall x, y \in \mathbb{R}^n.$$

Proposition 3.3 (Convexity of γ_F^{GVI})
Assume that K is a convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ is an affine and monotone mapping. Then γ_F^{GVI} is convex.

Proof: Let us verify first that the the function

$$(x, p) \mapsto f^*(p - F(x)) + F(x)^T x + f(x) + \delta_K^*(-p) \tag{3.4}$$

is convex with respect to (x, p). If F is affine and monotone then $F(x)^T x$ is convex. Moreover the conjugate functions f^* and δ_K^* are also convex. By the assumptions f is also convex. If F is affine then $p - F(x)$ is affine. The composition of a convex function with an affine function $f^*(p - F(x))$ is also convex. So, the function given by (3.4) is convex. Therefore, by Theorem 1 in [18], γ_F^{GVI} is convex. $\Box$

Proposition 3.4 (Convexity of γ_L^{GVI} and γ_{FL}^{GVI})
Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is an affine and monotone mapping. Then γ_L^{GVI} and γ_{FL}^{GVI} are convex.

Proof: Because of the functions

$$(f + q^T g)^*(-F(x)) = \sup_{y \in \mathbb{R}^n} [-F(x)^T y - f(y) - q^T g(y)]$$

and $(q^T g)_X^*(-p) = \sup_{y \in X} [-p^T y - q^T g(y)]$ are convex as the pointwise supremum of affine functions with respect to (x, q) and (p, q), respectively, the convexity of γ_L^{GVI} and γ_{FL}^{GVI} follows from Theorem 1 in [18]. $\Box$

4 Dual gap functions for the problem (VI)

In this section we introduce another class of gap functions for the problem (VI), the so-called dual gap functions. Under assumptions that K is a closed, convex set and F is a pseudo-monotone (monotone) and continuous mapping (see, for instance [10], [12]), the problem (VI) is equivalent to the problem of finding $x \in K$ such that

$$(VI') \qquad F(y)^T (y - x) \geq 0, \ \forall y \in K.$$

The function $\gamma_A^{VI'} : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\gamma_A^{VI'}(x) := \sup_{y \in K} F(y)^T (x - y)$$

is called the dual gap function for the problem (VI). Remark that $\gamma_A^{VI'}$ is the gap function in the sense of Auslender for the problem (VI') and has been studied, for instance, in [14] and [22]. On the other hand, by using the duals $(D_F^{VI'}; x)$, $(D_L^{VI'}; x)$ and $(D_{FL}^{VI'}; x)$ of the optimization problem

$$(P^{VI'}; x) \qquad \inf_{y \in K} F(y)^T(y - x),$$

where $x \in \mathbb{R}^n$ is fixed, we formulate the corresponding functions as follows

$$\begin{aligned}
\gamma_F^{VI'}(x) :&= -v(D_F^{VI'}; x) = \inf_{p \in \mathbb{R}^n} \Big\{ \sup_{y \in \mathbb{R}^n} [p^T y + F(y)^T(x - y)] + \delta_K^*(-p) \Big\}, \\
\gamma_L^{VI'}(x) :&= -v(D_L^{VI'}; x) = \inf_{q \geqq 0} \sup_{y \in K} \Big\{ F(y)^T(x - y) - q^T g(y) \Big\}. \\
\gamma_{FL}^{VI'}(x) :&= -v(D_{FL}^{VI'}; x) = \inf_{\substack{p \in \mathbb{R}^n \\ q \geqq 0}} \Big\{ \sup_{y \in \mathbb{R}^n} [p^T y + F(y)^T(x - y)] + (q^T g)^*(-p) \Big\}.
\end{aligned}$$

In case of the functions $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$, K is given by (3.1). Before we show that the proposed functions are gap functions for the problem (VI), let us prove some relations between them.

Proposition 4.1 *It holds*

$$\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x), \ \forall x \in \mathbb{R}^n.$$

Proof: Let $x \in \mathbb{R}^n$ be fixed. For any $p \in \mathbb{R}^n$ it holds

$$\sup_{z \in \mathbb{R}^n} [p^T z - F(z)^T(z - x)] \geq p^T y - F(y)^T(y - x), \ \forall y \in \mathbb{R}^n,$$

or, equivalently,

$$\sup_{z \in \mathbb{R}^n} [p^T z - F(z)^T(z - x)] - p^T y \geq F(y)^T(x - y), \ \forall y \in \mathbb{R}^n.$$

Taking the supremum in both sides over all $y \in K$

$$\sup_{z \in \mathbb{R}^n} [p^T z - F(z)^T(z - x)] + \delta_K^*(-p) \geq \sup_{y \in K} F(y)^T(x - y).$$

After taking the infimum in both sides over all $p \in \mathbb{R}^n$ we conclude that

$$\gamma_F^{VI'}(x) \geq \gamma_A^{VI'}(x), \ \forall x \in \mathbb{R}^n. \qquad \square$$

Proposition 4.2 *Let the ground set be given by* (3.1). *Then it holds*

$$\gamma_A^{VI'}(x) \leq \begin{matrix} \gamma_L^{VI'}(x) \\ \\ \gamma_F^{VI'}(x) \end{matrix} \leq \gamma_{FL}^{VI'}(x), \ \forall x \in \mathbb{R}^n.$$

Proof: Like in Proposition 3.2, by Propositions 2.1 and 2.2 in [3] (see also [19]), one can conclude that

$$\left.\begin{array}{l}\gamma_L^{VI'}(x)\\ \\ \gamma_F^{VI'}(x)\end{array}\right. \leq \gamma_{FL}^{VI'}(x),\ \forall x \in \mathbb{R}^n.$$

On the other hand by Proposition 4.1, one has $\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x),\ \forall x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $q \geqq 0$ be fixed. Because of $-q^T g(y) \geq 0, \forall y \in K$, adding $F(y)^T(x-y)$ in both sides, we have

$$F(y)^T(x-y) - q^T g(y) \geq F(y)^T(x-y).$$

Taking the supremum over all $y \in K$ and that the infimum over all $q \geqq 0$ implies that $\gamma_L^{VI'}(x) \geq \gamma_A^{VI'}(x),\ \forall x \in \mathbb{R}^n$. Thus the proof is completed. □

At next we show that under monotonicity assumptions the functions introduced above can be related also to the Auslender's and Giannessi's gap functions.

Proposition 4.3 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone mapping. Then it holds*

$$\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x) \leq \gamma_A^{VI}(x),\ \forall x \in \mathbb{R}^n.$$

Proof: By Proposition 4.1 there is $\gamma_A^{VI'}(x) \leq \gamma_F^{VI'}(x),\ \forall x \in \mathbb{R}^n$. Taking into account the monotonicity of F, it holds

$$[F(y) - F(x)]^T(y-x) \geq 0,\ \forall x, y \in \mathbb{R}^n,$$

or

$$F(y)^T(y-x) \geq F(x)^T(y-x),\ \forall x, y \in \mathbb{R}^n.$$

Let $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ be fixed. Adding $-p^T y$ and taking the infimum in both sides over all $y \in \mathbb{R}^n$, we get

$$\inf_{y \in \mathbb{R}^n}[-p^T y + F(y)^T(y-x)] \geq \inf_{y \in \mathbb{R}^n}[-p^T y + F(x)^T(y-x)],$$

or, equivalently,

$$\sup_{y \in \mathbb{R}^n}[p^T y - F(y)^T(y-x)] \leq \sup_{y \in \mathbb{R}^n}[p^T y - F(x)^T(y-x)], \tag{4.1}$$

Then, after adding $\delta_K^*(-p)$ and taking the infimum in both sides over all $p \in \mathbb{R}^n$

$$\begin{aligned} &\inf_{p\in\mathbb{R}^n}\left\{\sup_{y\in\mathbb{R}^n}[p^Ty - F(y)^T(y-x)] + \delta_K^*(-p)\right\} = \gamma_F^{VI'}(x) \\ \le\ &\inf_{p\in\mathbb{R}^n}\left\{\sup_{y\in\mathbb{R}^n}[p^Ty - F(x)^T(y-x)] + \delta_K^*(-p)\right\} = \gamma_F^{VI}(x). \end{aligned}$$

In view of Proposition 3.1(i) implies that $\gamma_F^{VI'}(x) \le \gamma_A^{VI}(x),\ \forall x \in \mathbb{R}^n$. □

Proposition 4.4 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone mapping and the ground set K be given by* (3.1)*. Then it holds*

$$\gamma_L^{VI'}(x) \le \gamma_{FL}^{VI'}(x) \le \gamma_G^{VI}(x),\ \forall x \in \mathbb{R}^n.$$

Proof: By Proposition 4.2 one has

$$\gamma_L^{VI'}(x) \le \gamma_{FL}^{VI'}(x),\ \forall x \in \mathbb{R}^n.$$

Let $x, p \in \mathbb{R}^n$ and $q \geqq 0$ be fixed. Since F is a monotone mapping, in the same way we can obtain the relation (4.1). Whence, adding $(q^Tg)^*(-p)$ and taking the infimum in both sides over all $p \in \mathbb{R}^n$ and $q \geqq 0$ implies that

$$\begin{aligned} &\inf_{\substack{p\in\mathbb{R}^n\\ q\geqq 0}}\left\{\sup_{y\in\mathbb{R}^n}[p^Ty - F(y)^T(y-x)] + (q^Tg)^*(-p)\right\} = \gamma_{FL}^{VI'}(x) \\ \le\ &\inf_{\substack{p\in\mathbb{R}^n\\ q\geqq 0}}\left\{\sup_{y\in\mathbb{R}^n}[p^Ty - F(x)^T(y-x)] + (q^Tg)^*(-p)\right\} = \gamma_{FL}^{VI}(x). \end{aligned}$$

Taking into account Proposition 3.1(ii) we conclude that

$$\gamma_{FL}^{VI'}(x) \le \gamma_G^{VI}(x),\ \forall x \in \mathbb{R}^n.$$ □

Theorem 4.1 *Let K be a nonempty, closed, convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone and continuous mapping. Then $\gamma_F^{VI'}$ is a gap function for the problem (VI).*

Proof:

(i) Let $x \in K$ be fixed. By weak duality it holds

$$\gamma_F^{VI'}(x) = -v(D_F^{VI'}; x) \ge -v(P^{VI'}; x) \ge 0.$$

(ii) If $\gamma_F^{VI'}(x) = 0$, then

$$0 = v(D_F^{VI'}; x) \leq v(P^{VI'}; x) \leq 0.$$

Thus $v(P^{VI'}; x) = 0$ and this means that also $v(P^{VI}; x) = 0$ and so x is a solution of (VI). Conversely, let $x \in K$ be a solution of the problem (VI). Then it holds $\gamma_A^{VI}(x) = 0$. By Proposition 4.3 and according to (i) we conclude that $\gamma_F^{VI'}(x) = 0$. □

Theorem 4.2 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone and continuous mapping. Assume that for the problem (VI) the constraint qualification (CQ) is fulfilled. Then $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$ are gap functions for the problem (VI).*

Proof:

(i) Let $x \in K$ be fixed. By weak duality and in view of Proposition 4.2 it holds

$$\gamma_{FL}^{VI'}(x) \geq \gamma_L^{VI'}(x) = -v(D_L^{VI'}; x) \geq -v(P^{VI'}; x) \geq 0.$$

(ii)

Since $\gamma_{FL}^{VI'}(x) = \gamma_L^{VI'}(x) = 0$ implies that

$$0 = v(D_{FL}^{VI'}; x) = v(D_L^{VI'}; x) \leq v(P^{VI'}; x) \leq 0.$$

Consequently $v(P^{VI'}; x) = v(P^{VI}; x) = 0$. So x is a solution of (VI). Let $x \in K$ be a solution of the problem (VI) and the constraint qualification (CQ) be fulfilled. Then it holds $\gamma_G^{VI}(x) = 0$. By Proposition 4.4 and in view of (i), implies that $\gamma_{FL}^{VI'}(x) = \gamma_L^{VI'}(x) = 0$. □

Remark 4.1 Since the functions

$$\sup_{y \in \mathbb{R}^n} \left\{ p^T y + F(y)^T (x - y) \right\} \text{ and } \sup_{y \in K} \left\{ F(y)^T (x - y) - q^T g(y) \right\}$$

are convex as the pointwise supremum of affine functions with respect to (p, x) and (q, x), respectively, by Theorem 1 in [18] one can easily verify the convexity of the functions $\gamma_F^{VI'}$, $\gamma_L^{VI'}$ and $\gamma_{FL}^{VI'}$.

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