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An alternative formulation for a new closed cone constraint qualification

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Abstract. We give an alternative formulation for the so-called closed cone constraint qualification $(CCCQ)$ related to a convex optimization problem in Banach spaces recently introduced in the literature. This new formulation allows to prove in a simple way that $(CCCQ)$ is weaker than some generalized interior-point constraint qualifications given in the past. By means of some insights from the theory of conjugate duality we also show that strong duality still holds under some weaker hypotheses than the ones considered so far in the literature.

Key Words. convex optimization, constraint qualifications, conjugate duality, weak and strong duality

AMS subject classification. 46N10, 42A50, 90C25

1 Introduction

Having a convex optimization problem in Banach spaces with geometric and cone inequality constraints, Jeyakumar, Dinh and Lee (cf. [11]) had recently introduced a new so-called closed cone constraint qualification $(CCCQ)$ in order to obtain strong duality between the primal problem and its Lagrange dual problem. The formulation of $(CCCQ)$ had been inspired by the formula which expresses the epigraph of the support function of the feasible set by the epigraphs of the conjugate functions involving the constraints (see also [10], [12], [15]). Jeyakumar, Dinh and Lee had proved (cf. Proposition 2.1 in [11]) that $(CCCQ)$ turns out to be weaker than some generalized interior-point constraint qualifications given so far in the literature. The proof of the mentioned result is quite sophisticated and is based on an open mapping theorem introduced by Borwein in [1].

In this paper we give an alternative formulation for $(CCCQ)$ which has as a direct consequence the result from Proposition 2.1 in [11]. On the other hand,

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we present some connections between the closed cone constraint qualification and the theory of conjugate duality for convex optimization problems. Therefore we examine the well-known Lagrange and Fenchel duals and the so-called Fenchel-Lagrange dual problem. The last one is a "combination" of the classical Fenchel and Lagrange dual problems and it has been recently introduced and extensively studied by the authors of this paper (cf. [2], [3], [4], [17]). The relations between the optimal objective values of these three dual problems are also given. We show here that the closed cone constraint qualification due to Jeyakumar, Dinh and Lee is strongly connected to the problem of closing the gap between the optimal objective values of the Fenchel and Fenchel-Lagrange dual problems. We also prove that $(CCCQ)$ guarantees strong duality between the primal problem and the Lagrange, Fenchel and Fenchel-Lagrange dual problems, namely that the optimal objective values are equal and the duals have optimal solutions.

Concerning the objective function of the primal problem, we consider then some weaker hypotheses than the ones regarded so far in the literature (cf. [5], [6], [11]) and prove that the results described above remain true.

The paper is organized as follows. In the next section we present some definitions and preliminary results that will be used later in the paper and we introduce the closed cone constraint qualification $(CCCQ)$. In Section 3 we give an alternative formulation for $(CCCQ)$ and prove in a simple way that $(CCCQ)$ is weaker than some generalized interior-point constraint qualifications. Section 4 is devoted to the presentation of the connections between $(CCCQ)$ and the theory of conjugate duality for convex optimization problems. A short concluding section and the list of references close the paper.

2 Notation and preliminary results

In this section we describe the notations used throughout this paper and present some preliminary results. Let X be a Banach space and X^* the continuous dual space of X. X^* will be endowed with the weak* topology and $\langle x^*, x\rangle$ will denote the value at $x \in X$ of the continuous linear functional $x^* \in X^*$. For a set $D \subseteq X$ we shall denote the *closure*, the *interior* and the *affine hull* of D by $cl(D)$, $int(D)$ and $aff(D)$, respectively. Similarly we shall denote the *cone* generated by the set D by $cone(D) = \bigcup\limits_{t\geq 0} tD$.

Furthermore, for the nonempty set $D \subseteq X$, the *indicator function* $\delta_D : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise,} \end{cases}$$

while the *support function* $\sigma_D : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by $\sigma_D(x^*) =$

$\sup_{x\in D}\langle x^*, x\rangle$. Considering now a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by

$$dom(f) = \{x \in X : f(x) < +\infty\}$$

its *effective domain* and by

$$epi(f) = \{(x,r) \in dom(f) \times \mathbb{R} : f(x) \leq r\}$$

its *epigraph*. Moreover, by $cl(f)$ we denote the *lower semicontinuous envelope* of f, namely the function whose epigraph is the closure of $epi(f)$ in $X \times \mathbb{R}$. We say that $f : X \to \mathbb{R} \cup \{+\infty\}$ is *proper* if $dom(f) \neq \emptyset$.

When D is a nonempty subset of X we define for $f : X \to \mathbb{R} \cup \{+\infty\}$ the so-called *conjugate function relative to the set D*

$$f_D^* : X^* \to \mathbb{R} \cup \{+\infty\},\ f_D^*(p) = \sup_{x\in D}\{\langle p, x\rangle - f(x)\}.$$

By taking D equal to the whole space X, the conjugate relative to the set X becomes the classical *conjugate function of f* (the Fenchel-Moreau conjugate)

$$f^* : X^* \to \mathbb{R} \cup \{+\infty\},\ f^*(p) = \sup_{x\in X}\{\langle p, x\rangle - f(x)\}.$$

Two important results, used later within this paper, follow preceded by a necessary definition.

Definition 2.1 *Let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be given. The function $f_1 \Box f_2 : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$f_1 \Box f_2(x) = \inf\{f_1(y) + f_2(x-y) : y \in X\}$$

is called the infimal convolution function of f_1 and f_2. We say that $f_1 \Box f_2$ is exact at $x \in X$ if there exists some $y \in X$ such that $f_1 \Box f_2(x) = f_1(y) + f_2(x-y)$.

Proposition 2.1 *Let $h_1, h_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper functions. Then the following statements are equivalent:*

(i) $epi((h_1+h_2)^*) = epi(h_1^*) + epi(h_2^*)$;

(ii) $(h_1+h_2)^* = h_1^* \Box h_2^*$ and $h_1^* \Box h_2^*$ is exact at every $p \in X^*$.

Proof. "(i) $\Rightarrow$ (ii)" Let be $p \in X^*$. By the definition of the conjugate function, we have

$$(h_1+h_2)^*(p) \leq \sup_{x\in X}\{\langle u, x\rangle - h_1(x)\} + \sup_{x\in X}\{\langle p-u, x\rangle - h_2(x)\} =$$

$$h_1^*(u) + h_2^*(p-u), \forall u \in X^*.$$

If $(h_1+h_2)^*(p) = +\infty$, then (ii) is fulfilled. In case $(h_1+h_2)^*(p) < +\infty$, we have that $(p, (h_1+h_2)^*(p)) \in epi((h_1+h_2)^*) = epi(h_1^*) + epi(h_2^*)$. By (i), there exist $(q, s) \in epi(h_1^*)$ and $(r, t) \in epi(h_2^*)$ such that $p = q + r$ and $(h_1+h_2)^*(p) = s + t$. Therefore $h_1^*(q) \leq s$, $h_2^*(p-q) \leq t$ and $h_1^*(q) + h_2^*(p-q) \leq (h_1+h_2)^*(p)$. This proves (ii).

"$(ii) \Rightarrow (i)$" Let be $(q, s) \in epi(h_1^*)$ and $(r, t) \in epi(h_2^*)$. Then

$$(h_1+h_2)^*(q+r) \leq \sup_{x \in X}\{\langle q, x\rangle - h_1(x)\} + \sup_{x \in X}\{\langle r, x\rangle - h_2(x)\} =$$

$$h_1^*(q) + h_2^*(r) \leq s + t,$$

which implies that $(q+r, s+t) \in epi((h_1+h_2)^*)$. Therefore $epi(h_1^*) + epi(h_2^*) \subseteq epi((h_1+h_2)^*)$.

Taking now $(p, w) \in epi((h_1+h_2)^*)$, we have $(h_1+h_2)^*(p) \leq w$. By (ii), there exists $u \in X^*$ such that $h_1^*(u) + h_2^*(p-u) \leq w$. Then the element (p, w) can be then written as

$$(p, w) = (u, h_1^*(u)) + (p-u, w - h_1^*(u)),$$

which belongs to $epi(h_1^*) + epi(h_2^*)$. Thus $epi((h_1+h_2)^*) = epi(h_1^*) + epi(h_2^*)$. □

Proposition 2.2 (cf. [11], [13]) *Let $h_1 : X \to \mathbb{R}$ be a continuous convex function and let $h_2 : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Then*

(i) $epi(h_1^*) + epi(h_2^*)$ is weak-closed;*

(ii) $epi((h_1+h_2)^*) = epi(h_1^*) + epi(h_2^*)$.

In the last part of this section we introduce a convex optimization problem and present the so-called cone closed constraint qualification $(CCCQ)$ formulated by Jeyakumar, Dinh and Lee in [11].

The primal optimization problem treated throughout this paper will be

$$(P) \quad \inf_{x \in A} f(x),$$

$$A = \{x \in C : g(x) \in -S\} = C \cap g^{-1}(-S),$$

where $f : X \to \mathbb{R}$ is a continuous convex function, $g : X \to Z$ is a continuous and S-convex mapping, Z is another Banach space with Z^* being its continuous dual space, C is a closed convex set in X and S is a convex closed cone in Z which does not necessarily have a nonempty interior. We also assume that the feasible set A is nonempty.

Further, we denote by $S^* = \{\lambda \in Z^* :< \lambda, z >\geq 0, \forall z \in S\}$ the *dual cone* of S and by λg the function defined by $\lambda g(x) = \langle \lambda, g(x)\rangle$, for $\lambda \in Z^*$ and $x \in X$.

Working within these hypotheses, Jeyakumar, Dinh and Lee had given (cf. Lemma 2.1 in [11]) the following formula for the epigraph of the support function of the set A

$$epi(\sigma_A) = cl\Big(\bigcup_{\lambda \in S^*} epi((\lambda g)^*) + epi(\delta_C^*)\Big). \tag{1}$$

One can easily prove that the set $\bigcup_{\lambda \in S^*} epi((\lambda g)^*) + epi(\delta_C^*)$ is a convex cone.

The so-called closed cone constraint qualification introduced in [11] follows

$$(CCCQ) : \bigcup_{\lambda \in S^*} epi((\lambda g)^*) + epi(\delta_C^*) \text{ is a weak* closed set.}$$

The next theorem guarantees the existence of strong duality between the primal problem (P) and its Lagrange dual problem

$$(D_L) \quad \sup_{\lambda \in S^*} \inf_{x \in C} \{f(x) + \lambda g(x)\}.$$

We say that strong duality holds if the optimal objective values of the primal and dual problem coincide and the dual has an optimal solution.

Theorem 2.3 (cf. Theorem 3.1 in [11]) *Let $\alpha \in \mathbb{R}$ and suppose that $(CCCQ)$ holds. Then the following statements are equivalent:*

(i) $\inf\{f(x) : x \in C, g(x) \in -S\} \geq \alpha$,

(ii) $(0, -\alpha) \in epi(f^*) + \bigcup_{\lambda \in S^*} epi((\lambda g)^*) + epi(\delta_C^*)$,

(iii) $(\exists \lambda \in S^*)(\forall x \in C) f(x) + \lambda g(x) \geq \alpha$.

3 An alternative formulation for $(CCCQ)$

In this section we will give an alternative formulation to the closed cone constraint qualification $(CCCQ)$ via duality. We use this result in order to prove that $(CCCQ)$ is implied by some other generalized interior-point constraint qualifications given in the literature. A similar result has been obtained by Jeyakumar, Dinh and Lee (cf. Proposition 2.1 in [11]) in a much more complicated manner.

Lemma 3.1 *The constraint qualification $(CCCQ)$ can be equivalently written as*

$$epi(\delta_A^*) \subseteq \bigcup_{\lambda \in S^*} epi((\lambda g)_C^*).$$

Proof. Because of (cf. (1))

$$epi(\delta_A^*) = epi(\sigma_A) = cl\Big(\bigcup_{\lambda \in S^*} epi((\lambda g)^*) + epi(\delta_C^*)\Big),$$

the constraint qualification $(CCCQ)$ is rewritable as

$$epi(\delta_A^*) \subseteq \bigcup_{\lambda \in S^*} epi((\lambda g)^*) + epi(\delta_C^*) = \bigcup_{\lambda \in S^*} \Big(epi((\lambda g)^*) + epi(\delta_C^*) \Big).$$

Let us notice that δ_C is proper, convex and lower semicontinuous and, on the other hand, that λg is convex and continuous. Thus, by Proposition 2.2 (ii),

$$epi((\lambda g)^*) + epi(\delta_C^*) = epi((\lambda g + \delta_C)^*) = epi((\lambda g)_C^*), \forall \lambda \in S^*.$$

We conclude that $(CCCQ)$ is nothing else than

$$epi(\delta_A^*) \subseteq \bigcup_{\lambda \in S^*} epi((\lambda g)_C^*).$$

□

The next theorem represents the main result of this section.

Theorem 3.2 *The constraint qualification $(CCCQ)$ is fulfilled if and only if $\forall p \in X^*$ between the optimization problem*

$$(P^p) \quad \inf_{x \in A} \langle p, x \rangle,$$

$$A = \{x \in C : g(x) \in -S\},$$

and its Lagrange dual

$$(D_L^p) \quad \sup_{\lambda \in S^*} \inf_{x \in C} \{\langle p, x \rangle + \lambda g(x)\}$$

strong duality holds.

Proof. We denote by $v(P^p)$ and $v(D_L^p)$ the optimal objective values of the problems (P^p) and (D_L^p), respectively. The problem (D_L^p) being the Lagrange dual of (P^p), it is obvious that $v(D_L^p) \leq v(P^p)$.

"$\Rightarrow$" Let be $p \in X^*$. If $v(P^p) = -\infty$, then the theorem is proved. Let us assume now that $v(P^p) \in \mathbb{R}$. Because of $v(P^p) = \inf_{x \in A} \langle p, x \rangle = -\delta_A^*(-p)$, we have that $(-p, -v(P^p)) \in epi(\delta_A^*)$. The constraint qualification $(CCCQ)$ being fulfilled, by Lemma 3.1 there exists a $\bar{\lambda} \in S^*$ such that $(-p, -v(P^p)) \in epi((\bar{\lambda} g)_C^*)$. The last relation can be written equivalently as

$$(\bar{\lambda} g)_C^*(-p) = (\bar{\lambda} g + \delta_C)^*(-p) \leq -v(P^p) \Leftrightarrow$$
$$v(P^p) \leq \inf_{x \in X} \{\langle p, x \rangle + \bar{\lambda} g(x) + \delta_C(x)\} = \inf_{x \in C} \{\langle p, x \rangle + \bar{\lambda} g(x)\} \leq v(P^p).$$

Thus $v(P^p) = v(D^p_L)$ and $\bar{\lambda}$ is an optimal solution of (D^p_L).

"$\Leftarrow$" Let be $(p, v) \in epi(\delta^*_A) \Leftrightarrow \delta^*_A(p) \leq v$. Therefore

$$-v \leq -\delta^*_A(p) = \inf_{x \in A} \langle -p, x \rangle.$$

Using the fact that between the primal problem

$$(P^{-p}) \quad \inf_{x \in A} \langle -p, x \rangle$$

and its Lagrange dual strong duality holds, there exists $\bar{\lambda} \in S^*$ such that

$$-v \leq \inf_{x \in A} \langle -p, x \rangle = \inf_{x \in C} \{\langle -p, x \rangle + \bar{\lambda} g(x)\} =$$

$$\inf_{x \in X} \{\langle -p, x \rangle + \bar{\lambda} g(x) + \delta_C(x)\} = -((\bar{\lambda} g + \delta_C)^*(p)) = -(\bar{\lambda} g)^*_C(p).$$

This is nothing else than $(p, v) \in epi((\bar{\lambda} g)^*_C) \subseteq \bigcup_{\lambda \in S^*} epi((\lambda g)^*_C)$ and so

$$epi(\delta^*_A) \subseteq \bigcup_{\lambda \in S^*} epi((\lambda g)^*_C).$$

Lemma 3.1 implies that $(CCCQ)$ is fulfilled. □

Remark 1. By the last theorem we can conclude that the closed cone constraint qualification $(CCCQ)$ is fulfilled if and only if $\forall p \in X^*$ either $v(P^p) = -\infty$ or the infimal value function of the problem (P^p) $h : Z \to \mathbb{R} \cup \{\pm\infty\}$, $h(q) = \inf\{\langle p, x \rangle : g(x) \in q - S, x \in C\}$ is finite and subdifferentiable at 0 (cf. Proposition 2.2 in [7]).

Further, we show the usefulness of Theorem 3.2 in order to prove that the constraint qualification $(CCCQ)$ is implied by some generalized interior-point constraint qualifications taken from the literature. To arrive there, we need to introduce the following notions first.

For a subset $D \subseteq X$, the *core* of D is defined by $core(D) = \{d \in D : \forall x \in X\ \exists \varepsilon > 0 : \forall \lambda \in [-\varepsilon, \varepsilon]\ d + \lambda x \in D\}$. The core of D relative to $aff(D)$ is called the *intrinsic core* of D and is written $icr(D)$ (cf. [9]). For a convex subset $D \subseteq X$, the *strong quasi-relative interior* of D is the set of those $x \in D$ for which $cone(D - x)$ is a closed subspace and is written $sqri(D)$ (cf. [14]). Consider now the following generalized interior-point constraint qualifications:

(SCQ) : $\exists x' \in C, g(x') \in int(-S)$;

(RCQ) : $0 \in core(g(C) + S)$ (cf. [16]);

$(JWCQ)$: $0 \in sqri(g(C) + S)$ (cf. [14]);

$(GTCQ)$: $0 \in icr(g(C) + S)$ and $aff(g(C) + S)$ is a closed subspace (cf. [8]).

The following relation holds between them (cf. [8]):

$$(SCQ) \Rightarrow (RCQ) \Rightarrow (JWCQ) \Leftrightarrow (GTCQ).$$

Remark 2. Jeyakumar, Dinh and Lee had proved in Proposition 2.1 in [11] that if $0 \in icr(g(C)+S)$ and $aff(g(C)+S)$ is a closed subspace, then $(CCCQ)$ is fulfilled. The proof of this result is quite complicated and is based on an open mapping theorem introduced by Borwein in [1].

On the other hand assuming that $(GTCQ)$ is fulfilled, this is nothing else than assuming that $(JWCQ)$ is fulfilled. Consider now an arbitrary continuous and convex function $f : X \to \mathbb{R}$. For the optimization problem

$$(P) \quad \inf_{x \in A} f(x),$$

$$A = \{x \in C : g(x) \in -S\},$$

where X and Z are Banach spaces, $g : X \to Z$ is continuous and S-convex, C is a closed convex set in X and S is convex closed cone in Z, the fulfilment of $(JWCQ)$ implies the existence of strong duality between (P) and its Lagrange dual. Evidently, taking for a $p \in X^*$, $f(x) = \langle p, x\rangle, \forall x \in X$, the function f will be continuous and convex. Thus there exists strong duality between

$$(P^p) \quad \inf_{x \in A} \langle p, x\rangle,$$

$$A = \{x \in C : g(x) \in -S\},$$

and its Lagrange dual

$$(D_L^p) \quad \sup_{\lambda \in S^*} \inf_{x \in C} \{\langle p, x\rangle + \lambda g(x)\},$$

for all $p \in X^*$. Theorem 3.2 guarantees that $(CCCQ)$ is fulfilled. Example 2.1 in [11] shows that $(CCCQ)$ is really weaker than the generalized interior-point constraint qualifications presented above.

Remark 3. For the sake of completeness we will show how the fulfilment of $(JWCQ)$ implies the existence of strong duality between (P) and its Lagrange dual problem. Therefore let $\Phi : X \times Z \to \mathbb{R} \cup \{+\infty\}$ be the following perturbation function

$$\Phi(x,z) = \begin{cases} f(x), & \text{if } x \in C, g(x) \in z - S, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $z \in Z$ is the so-called perturbation variable. The function Φ is proper, convex and lower semicontinuous and, by $(JWCQ)$, $0 \in sqri(g(C)+S) = sqri(Pr_Z(dom(\Phi))$. By Theorem 2.7.1 (vii) in [19] (see also [18]), follows that

$$\inf_{x \in X} \Phi(x,0) = \max_{\lambda \in Z^*} \{-\Phi^*(0,\lambda)\}$$

or, equivalently,

$$\inf_{x\in A} f(x) = \sup_{\lambda\in S^*} \inf_{x\in C}\{f(x)+\lambda g(x)\}$$

and the supremum in the right-hand side is attained. This provides the existence of strong duality between (P) and its Lagrange dual problem.

4 Connections to the theory of conjugate duality

The aim of this section is to show how is the closed cone constraint qualification connected with some results from the theory of conjugate duality obtained in the past by the authors. These insights will allow us to weaken the hypotheses concerning the function f considered in [11].

We start by constructing via the perturbation approach described in the book of Ekeland and Temam ([7]) three dual problems to (P). This approach assumes, in each case, the use of a so-called perturbation function related to the primal problem (P). By calculating the conjugate of this function one gets a dual problem to (P). Throughout this section we denote by $v(P)$ the optimal objective value of (P).

The Lagrange dual

For the beginning, let the perturbation function $\Phi_L : X\times Z \to \mathbb{R}\cup\{+\infty\}$ be defined by

$$\Phi_L(x,z) = \begin{cases} f(x), & \text{if } x\in C, g(x)\in z-S, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variable $z\in Z$. The dual problem to (P) is (cf. [7])

$$(D_L)\quad \sup_{\lambda\in Z^*}\{-\Phi_L^*(0,\lambda)\}$$

or, equivalently,

$$(D_L)\quad \sup_{\lambda\in S^*} \inf_{x\in C}\{f(x)+\lambda g(x)\}.$$

The problem (D_L) is the well-known Lagrange dual problem of (P) and we denote by $v(D_L)$ its optimal objective value.

The Fenchel dual

Considering now the perturbation function $\Phi_F : X\times X \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\Phi_F(x,y) = \begin{cases} f(x+y), & \text{if } x\in C, g(x)\in -S, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variable $y \in X$, one obtains as dual problem to (P)

$$(D_L) \quad \sup_{p \in X^*} \{-\Phi_F^*(0, \lambda)\},$$

which is actually

$$(D_F) \quad \sup_{p \in X^*} \left\{ -f^*(p) + \inf_{x \in A} \langle p, x \rangle \right\} \Leftrightarrow$$

$$(D_F) \quad \sup_{p \in X^*} \{-f^*(p) - \delta_A^*(-p)\}.$$

(D_F) is another well-known dual problem of (P), namely the Fenchel dual. Its optimal objective value is denoted by $v(D_F)$.

The Fenchel-Lagrange dual

The last perturbation function we consider here is $\Phi_{FL} : X \times X \times Z \to \mathbb{R} \cup \{+\infty\}$,

$$\Phi_{FL}(x, y, z) = \begin{cases} f(x+y), & \text{if } x \in C, g(x) \in z - S, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables $y \in X$ and $z \in Z$. Φ_{FL} leads us to a new dual problem

$$(D_{FL}) \quad \sup_{p \in X^*, \lambda \in Z^*} \{-\Phi_F^*(0, p, \lambda)\},$$

which has the following formulation

$$(D_{FL}) \quad \sup_{p \in X^*, \lambda \in S^*} \left\{ -f^*(p) + \inf_{x \in C} [\langle p, x \rangle + \lambda g(x)] \right\} \Leftrightarrow$$

$$(D_{FL}) \quad \sup_{p \in X^*, \lambda \in S^*} \left\{ -f^*(p) - (\lambda g)_C^*(-p) \right\}.$$

The dual problem (D_{FL}) is called the Fenchel-Lagrange dual because it turns out to be a "combination" of the Lagrange and Fenchel duals. The Fenchel-Lagrange dual has been introduced in [17] for the case of finite dimensional optimization problems. The optimal objective value of the Fenchel-Lagrange dual will be denoted by $v(D_{FL})$.

The following relations between the optimal objective values of the primal problem (P) and of the three duals take place

$$v(D_{FL}) \leq \begin{matrix} v(D_F) \\ v(D_L) \end{matrix} \leq v(P). \tag{2}$$

Let us notice that an ordering between $v(D_L)$ and $v(D_F)$ cannot be established in the general case (for a counterexample, see [17]). In case X and Z are finite dimensional spaces, Wanka and Boţ ([17]) and, under much weaker assumptions, Boţ, Kassay and Wanka ([3]), have given sufficient conditions such that all four optimal objective values become equal.

The next statement characterizes the optimal solutions of the Fenchel - Lagrange dual in case strong duality holds between (P) and (D_{FL}).

Theorem 4.1 *Assume that $v(P) = v(D_{FL})$ and that $(\bar{p}, \bar{\lambda})$ is an optimal solution of (D_{FL}). Then $\bar{\lambda}$ is an optimal solution of the Lagrange dual and $\bar{p}$ is an optimal solution of the Fenchel dual.*

Proof. Because of $v(P) = v(D_{FL})$, (2) becomes

$$v(D_{FL}) = v(D_F) = v(D_L) = v(P).$$

By the Young-Fenchel inequality we have $\forall x \in C$,

$$f(x) + \bar{\lambda}g(x) \geq -f^*(\bar{p}) - (\bar{\lambda}g)^*_C(-\bar{p}),$$

which implies

$$v(D_L) = v(D_{FL}) = -f^*(\bar{p}) - (\bar{\lambda}g)^*_C(-\bar{p}) \leq \inf_{x \in C}\{f(x) + \bar{\lambda}g(x)\} \leq v(D_L)$$

and so $\bar{\lambda}$ is optimal to (D_L).

On the other hand, from the Young-Fenchel inequality, we have $\forall x \in C$, $\bar{\lambda}g(x) \geq -\langle \bar{p}, x\rangle - (\bar{\lambda}g)^*_C(-\bar{p})$. Because of $\bar{\lambda}g(x) \leq 0, \forall x \in A$, we obtain $\langle \bar{p}, x\rangle \geq -(\bar{\lambda}g)^*_C(-\bar{p}), \forall x \in A$, and from here

$$v(D_F) = v(D_{FL}) = -f^*(\bar{p}) - (\bar{\lambda}g)^*_C(-\bar{p}) \leq -f^*(\bar{p}) + \inf_{x \in A}\langle \bar{p}, x\rangle \leq v(D_F).$$

Thus $\bar{p}$ is optimal to (D_F) and the theorem is proved. □

The next theorem proves that under the hypotheses given for the problem (P), which are the same with the ones considered by Jeyakumar, Dinh and Lee in [11], between (P) and (D_F) strong duality holds.

Theorem 4.2. *The Fenchel dual problem (D_F) has an optimal solution and $v(P) = v(D_F)$.*

Proof. The function g being continuous and the set S being closed it follows that $g^{-1}(-S)$ is also a closed set. From the closedness of C we have that $A = C \cap g^{-1}(-S)$ is closed. Moreover, A is convex. Thus the indicator function of the set A

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise,} \end{cases}$$

is a proper, convex and lower semicontinuous function.

The optimal objective value of (P) can be equivalently written as

$$v(P) = \inf_{x\in X}(f(x) + \delta_A(x)) = -(f + \delta_A)^*(0).$$

By Proposition 2.2, we have that $epi((f + \delta_A)^*) = epi(f^*) + epi(\delta_A^*)$. Further, Proposition 2.1 states that this relation is equivalent to the fact that $\forall r \in X^*$, $(f+\delta_A)^*(r) = \inf\{f^*(p)+\delta_A^*(r-p) : p \in X^*\}$ and this infimum is attained. For $r = 0$, we obtain that there exists a $\bar{p} \in X^*$ such that

$$v(P) = -(f + \delta_A)^*(0) = \sup_{p\in X^*}\{-f^*(p) - \delta_A^*(-p)\} = -f^*(\bar{p}) - \delta_A^*(-\bar{p}) = v(D_F).$$

□

Remark 4. Similarly one can prove that the optimal objective values of (D_L) and (D_{FL}) are equal

$$v(D_L) = \sup_{\lambda\in S^*}\inf_{x\in C}\{f(x) + \lambda g(x)\} = \sup_{\lambda\in S^*}\inf_{x\in X}\{f(x) + \lambda g(x) + \delta_C(x)\}$$

$$= \sup_{\lambda\in S^*} -\Big((f + \lambda g + \delta_C)^*(0)\Big) = \sup_{\lambda\in S^*} -\Big(f^*\Box(\lambda g + \delta_C)^*(0)\Big)$$

$$= \sup_{\lambda\in S^*}\Big(-\inf_{p\in X^*}\{f^*(p) + (\lambda g + \delta_C)^*(-p)\}\Big)$$

$$= \sup_{\lambda\in S^*}\sup_{p\in X^*}\{-f^*(p) - (\lambda g)_C^*(-p)\} = v(D_{FL}).$$

By the last remark and Theorem 4.2, relation (2) becomes

$$v(D_{FL}) = v(D_L) \leq v(D_F) = v(P) \tag{3}$$

and it is obvious that in order to close the gap between $v(D_L)$ and $v(P)$ it is sufficient to close the gap between $v(D_{FL})$ and $v(D_F)$. Theorem 4.3 shows that the closed cone constraint qualification $(CCCQ)$ is a sufficient assumption in order to obtain equality in (3).

Theorem 4.3. *If $(CCCQ)$ is fulfilled, then between (P) and (D_{FL}) strong duality holds. Therefore also between (P) and (D_L) strong duality holds.*

Proof. By Theorem 4.2, there exists a $\bar{p} \in X^*$ such that

$$v(P) = v(D_F) = -f^*(\bar{p}) - \delta_A^*(-\bar{p}) = -f^*(\bar{p}) + \inf_{x\in A}\langle \bar{p}, x\rangle.$$

The constraint qualification $(CCCQ)$ being fulfilled, by Theorem 3.2, there exists $\bar{\lambda} \in S^*$ such that $\inf_{x \in A} \langle \bar{p}, x \rangle = \inf_{x \in C} \{\langle \bar{p}, x \rangle + \bar{\lambda} g(x)\}$ and so

$$v(P) = -f^*(\bar{p}) + \inf_{x \in C} \{\langle \bar{p}, x \rangle + \bar{\lambda} g(x)\} = -f^*(\bar{p}) - (\bar{\lambda} g)_C^*(-\bar{p}) = v(D_{FL}).$$

Thereby the dual (D_{FL}) has an optimal solution and the optimal objective values of (P) and (D_{FL}) are equal. From (3) and Theorem 4.1 follows that also between (P) and (D_L) strong duality holds. □

At a careful reading of the proof of Theorem 4.3, one can observe that the applicability of the closed cone constraint qualification $(CCCQ)$ is strongly connected with the existence of strong duality between (P) and (D_F). As we had shown in the proof of Theorem 4.2, this is guaranteed by the continuity of the convex function f.

The following theorem states the existence of strong duality between (P) and (D_F) under some weaker assumptions concerning the function f.

Theorem 4.4

(a) Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function with $dom(f) \cap A \neq \emptyset$ and the function $f^* \Box \delta_A^*$ is lower semicontinuous on X^* and exact at 0. Then the Fenchel dual problem (D_F) has an optimal solution and $v(P) = v(D_F)$.

(b) If $f : X \to \mathbb{R}$ is a continuous and convex function, then the assumptions in (a) are fulfilled.

Proof.

(a) Applying the Moreau-Rockafellar Theorem we get

$$(f + \delta_A)^*(p) = cl(f^* \Box \delta_A^*)(p) = f^* \Box \delta_A^*(p), \forall p \in X^*. \tag{4}$$

Because the infimal convolution of f^* and δ_A^* is exact at zero, there exists a $\bar{p} \in X^*$ such that

$$v(P) = -(f + \delta_A)^*(0) = -\Big(f^* \Box \delta_A^*(0)\Big) = -f^*(\bar{p}) - \delta_A^*(-\bar{p}) = v(D_F).$$

(b) By Proposition 2.2 we have that $epi((f + \delta_A)^*) = epi(f^*) + epi(\delta_A^*)$ and, further, by Proposition 2.1 that $(f + \delta_A)^* = f^* \Box \delta_A^*$ and $f^* \Box \delta_A^*$ is exact at every $p \in X^*$. Regarding (4) it follows that the assumptions in (a) are fulfilled.

□

Remark 5. Let us notice that between (P) and (D_F) there is strong duality if and only if $cl(f^*\square\delta_A^*)(0) = f^*\square\delta_A^*(0)$ and $f^*\square\delta_A^*$ is exact at 0.

In two very recent papers Burachik and Jeyakumar (cf. [5] and [6]) have given a dual condition for the convex subdifferential sum formula. They proved that if $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semicontinuous functions such that $dom(f) \cap dom(g) \neq \emptyset$ and if $epi(f^*) + epi(g^*)$ is weak* closed, then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \forall x \in dom(f) \cap dom(g).$$

In case of the primal problem (P), if $epi(f^*) + epi(\delta_A^*)$ is weak* closed, then the strong duality between (P) and (D_F) is evidently fulfilled and one has a weaker assumption than the continuity of f. On the other hand, this condition is stronger than the assumption imposed in Theorem 4.4 (a). In order to see this, one should notice than the weak* closedness of $epi(f^*) + epi(\delta_A^*)$ implies that $epi((f+\delta_A)^*) = epi(f^*) + epi(\delta_A^*)$ (by the Moreau-Rockafellar Theorem). As we have seen in the proof of Theorem 4.4 (b), this implies that $f^*\square\delta_A^*$ is lower semicontinuous on X^* and exact at 0. An example, which underlines the fact that the assumption made in Theorem 4.4 (a) is weaker even than the recent condition of Burachik and Jeyakumar, follows.

Example. Let be $X = \mathbb{R}^2$, $Z = \mathbb{R}$, $C = \{(x_1, x_2)^T \in \mathbb{R}^2 : 2x_1 + x_2^2 \leq 0\}$, $S = \mathbb{R}_+$, $g : \mathbb{R}^2 \to \mathbb{R}, g(x) = 0, \forall x \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1, & \text{if } x_1 \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously, $A = C$ and $dom(f) \cap A = \{(0,0)^T\}$. For the conjugate functions f^* and δ_A^* we have

$$f^*(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 \leq 1, u_2 = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\delta_A^*(v_1, v_2) = \begin{cases} \frac{v_2^2}{v_1}, & \text{if } v_1 > 0, \\ 0, & \text{if } v_1 = v_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $p = (p_1, p_2)^T \in \mathbb{R}^2$, $(f + \delta_A)^*(p_1, p_2) = \sup\limits_{x_1 = x_2 = 0} \{p_1 x_1 + p_2 x_2 - x_1\} = 0$

and, on the other hand,

$$\begin{aligned} f^*\Box\delta_A^*(p_1,p_2) &= \inf_{\substack{u_1+v_1=p_1\\ u_2+v_2=p_2}} \{f^*(u_1,u_2)+\delta_A^*(v_1,v_2)\} \\ &= \inf_{\substack{u_1+v_1=p_1\\ u_2+v_2=p_2}} \begin{cases} \frac{v_2^2}{v_1}, & \text{if } u_1\leq 1, u_2=0, v_1>0, \\ 0, & \text{if } u_1\leq 1, u_2=0, v_1=v_2=0. \end{cases} \\ &= \inf_{\substack{p_1-1\leq v_1\\ v_2=p_2}} \begin{cases} \frac{v_2^2}{v_1}, & \text{if } v_1>0, \\ 0, & \text{if } v_1=v_2=0. \end{cases} \\ &= 0. \end{aligned}$$

Thus $(f+\delta_A)^* = f^*\Box\delta_A^*$, which, combined with the Moreau-Rockafellar Theorem, provides the lower semicontinuity of $f^*\Box\delta_A^*$ on $\mathbb{R}^2$. Moreover, $f^*\Box\delta_A^*$ is exact at $(0,0)$ (the infimum is attained for $(v_1,v_2)^T=(0,0)$) and so the assumption of Theorem 4.4 (a) is fulfilled.

On the other hand, the function $f^*\Box\delta_A^*$ is not exact at every point of $\mathbb{R}^2$. Taking, for example, $(p_1,p_2)^T=(1,1)^T$, the infimum in the infimal convolution formula of $f^*\Box\delta_A^*(p_1,p_2)$ is not attained. By Proposition 2.1, the sets $epi(f^*)+epi(\delta_A^*)$ and $epi((f+\delta_A)^*)$ are not equal, which means that $epi(f^*)+epi(\delta_A^*)$ can not be weak* closed.

Theorem 4.5 provides weaker sufficient conditions for strong duality between (P) and its Lagrange, Fenchel and Fenchel-Lagrange duals, respectively, than given so far and represents in the same time a generalization of Theorem 4.3.

Theorem 4.5 *Assume that $f: X\to\mathbb{R}\cup\{+\infty\}$ is a proper, convex and lower semicontinuous function with $dom(f)\cap A\neq\emptyset$, the function $f^*\Box\delta_A^*$ is lower semicontinuous on X^* and exact at 0 and that the constraint qualification $(CCCQ)$ is fulfilled. Then $v(P)=v(D_L)=v(D_F)=v(D_{FL})$ and the dual problems have optimal solutions.*

5 Conclusions

In this paper we give an alternative formulation for the so-called closed cone constraint qualification $(CCCQ)$ introduced by Jeyakumar, Dinh and Lee in [11] related to a convex optimization problem with cone-inequality constraints. This new formulation allows us to prove in a simpler way that $(CCCQ)$ is weaker than some generalized interior-point constraint qualifications given until now in the literature. Further, we employ these insights by giving some sufficient conditions which ensure the existence of strong duality between the primal problem (P) and its Lagrange, Fenchel and Fenchel-Lagrange duals, respectively, under weaker hypotheses than those given in [11].

References

[1] J. M. Borwein, *A Lagrange multiplier theorem and a sandwich theorem for convex relations*, Mathematica Scandinavica, 48, 189–204, 1981.

[2] R. I. Boţ, S. M. Grad and G. Wanka, *Fenchel-Lagrange versus geometric duality in convex optimization*, 2003 (submitted for publication).

[3] R. I. Boţ, G. Kassay and G. Wanka, *Strong duality for generalized convex optimization problems*, Journal of Optimization Theory and Applications (to appear), 2003.

[4] R. I. Boţ and G. Wanka, *Farkas-type results with conjugate functions*, SIAM Journal on Optimization (to appear), 2003.

[5] R. S. Burachik and V. Jeyakumar, *A dual condition for the convex subdifferential sum formula with applications*, Journal of Convex Analysis (to appear), 2004.

[6] R. S. Burachik and V. Jeyakumar, *A simpler closure condition for the normal cone intersection formula*, Proceedings of the American Mathematical Society (to appear), 2004.

[7] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland Publishing Company, Amsterdam, 1976.

[8] M. S. Gowda and M. Teboulle, *A comparison of constraint qualifications in infinite-dimensional convex programming*, SIAM Journal on Control and Optimization, 28 (4), 925–935, 1990.

[9] R. B. Holmes, *Geometric functional analysis*, Springer-Verlag, Berlin, 1975.

[10] V. Jeyakumar, *Characterizing set containments involving infinite convex constraints and reverse-convex constraints*, SIAM Journal of Optimization, 13 (4), 947–959, 2003.

[11] V. Jeyakumar, N. Dinh and G. M. Lee, *A new closed cone constraint qualification for convex optimization*, Applied Mathematics Report AMR04/8, University of New Soth Wales, Sydney, Australia, 2004.

[12] V. Jeyakumar, G. M. Lee and N. Dinh, *New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs*, SIAM Journal of Optimization, 14 (2), 534–547, 2003.

[13] V. Jeyakumar, A. M. Rubinov, B. M. Glover and Y. Ishizuka, *Inequality systems and global optimization*, Journal of Mathematical Analysis and Applications, 202, 900–919, 1996.

[14] V. Jeyakumar and H. Wolkowicz, *Generalizations of Slater's constraint qualification for infinite convex programs*, Mathematical Programming, 57 (1), 85–102, 1992.

[15] V. Jeyakumar and A. Zaffaroni, *Asymptotic conditions for weak and proper optimality in infinite dimensional convex vector optimization*, Numerical Functional Analysis and Optimization, 17 (3&4), 323–343, 1996.

[16] R. T. Rockafellar, *Conjugate duality and optimization*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics 16, Society for Industrial and Applied Mathematics, Philadelphia, 1974.

[17] G. Wanka, R. I. Boţ, *On the relations between different dual problems in convex mathematical programming*, in: P. Chamoni, R. Leisten, A. Martin, J. Minnemann and H. Stadtler (Eds.), "Operations Research Proceedings 2001", Springer Verlag, Berlin, 255–262, 2002.

[18] C. Zălinescu, *A comparison of constraint qualifications in infinite - dimensional convex programming revisited*, Journal of Asutralian Mathematical Society, Series B, 40 (3), 353–378, 1999.

[19] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, Singapore, 2002.