# Inverse Closedness of Approximation Algebras

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**Abstract.** We prove the inverse closedness of certain approximation algebras based on a quasi-Banach algebra $X$ using two general theorems on the inverse closedness of subspaces of quasi-Banach algebras. In the first theorem commutative algebras are considered while the second theorem can be applied to arbitrary $X$ and to subspaces of $X$ which can be obtained by a general $K$-method of interpolation between $X$ and an inversely closed subspace $Y$ of $X$ having certain properties. As application we present some inversely closed subalgebras of $C(\mathbb{T})$ and $C[-1,1]$. In particular, we generalize Wiener's theorem, i.e., we show that for many subalgebras $S$ of $l^1(\mathbb{Z})$, the property $\{c_k(f)\} \in S$ ($c_k(f)$ being the Fourier coefficients of $f$) implies the same property for $1/f$ if $f \in C(\mathbb{T})$ vanishes nowhere on $\mathbb{T}$.

## 1 Introduction

Let $X$ be a quasi-normed space and let $\{A_n\}_{n=0}^{\infty}$ be a sequence of subsets of $X$. We say that the pair $(X, \{A_n\}_{n=0}^{\infty})$ defines an approximation scheme if the following conditions are satisfied:

$\{0\} = A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots,$

$\lambda A_n \subset A_n$ for all scalars $\lambda$ and all $n \in \mathbb{N}$,

$A_n + A_n \subset A_{K(n)}$, where $K(0) = 0$ and $n \le K(n) \le K(n+1)$ for all $n \in \mathbb{N}$,

$\bigcup_{n=0}^{\infty} A_n$ is a dense subset of $X$.

An approximation space based on $(X, \{A_n\}_{n=0}^{\infty})$ is a set of elements $f$ of $X$ for which the sequences $\{E(f, A_n)\}_{n=0}^{\infty}$ of best approximation errors,

$$E(f, A_n) = E_n(f) := \inf_{f_n \in A_n} \|f - f_n\|,$$

*AMS classification*: 41A65, 46J30, 41A05
*Key words and phrases*: Approximation spaces, Quasi-normed algebras, Wiener-type theorems

belong to a given sequence space $S$. (We use the notation $f$ for the elements of $X$, since we are mainly interested in approximation theory in function spaces.) Particularly, for the case $K(n) = 2n$, the classical approximation spaces (see [Pie]) are defined for each $0 < s < \infty$ and $0 < q \le \infty$ by

$$\mathbf{A}_q^s(X) = \left\{ f \in X : \|f\|_{A_q^s} = \left\| \left\{ (n+1)^{s-(1/q)} E_n(f) \right\}_{n=0}^{\infty} \right\|_q < \infty \right\} ,$$

where $\| \,.\, \|_q$ denotes the $l^q$-norm. (Sometimes we also write $\mathbf{A}_q^s(X, \{A_n\})$, but usually the sequence $\{A_n\}$ is viewed as fixed so that it is not necessary to include it in the notation.)

**Remark 1.1** *In the literature (e.g., in [AL1] and [Pie]) it is often not supposed that $\bigcup A_n$ is dense in $X$. But we do not lose a lot of generality assuming this property, since usually the considered sequence spaces $S$ contain only sequences which converge to zero and this means that, in the case $\overline{\bigcup A_n}^X \neq X$, the corresponding approximation spaces do not change if $X$ is replaced by $\overline{\bigcup A_n}^X$.*

Classical approximation spaces are useful for the study of a classical convergence order like $O(n^{-\gamma})$ for approximation methods in which elements of $A_n$ are used as approximation elements. Another reason for the importance of classical approximation spaces is the fact that, up to a certain upper bound for $s$, they are classical interpolation spaces (obtained by the classical $(\theta, q)$-method) between $X$ and another quasi-normed space $Y \subset X$ if the so-called Jackson and Bernstein inequalities of order $r > 0$,

$$E_n(f) \le c\, n^{-r} \|f\|_Y \quad \text{and} \quad \|f_n\|_Y \le c\, n^r \|f_n\|_X \quad (f \in Y,\ f_n \in A_n,\ n \in \mathbb{N}), \tag{1.1}$$

hold true ([DL, Theorem 7.9.1]). But meanwhile there appeared several applications in which more general approximation spaces are needed (see [AL1, AL2, AL3, JL, LR, Lu2, Lu3, Lu4, Lu5]) which are non-classical interpolation spaces between $X$ and $Y$ if Jackson and Bernstein inequalities hold true (see [Lu]). These spaces are defined as follows.

**Definition 1.2** *Let $S$ be a real linear space of sequences $\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{R}$ (with element-wise defined operations), equipped with a quasi-norm $\| \cdot \|_S$. We say that $S$ is admissible (with respect to the approximation scheme $(X, \{A_n\})$) if*

- *All finite sequences $\{\alpha_n\}_{n=0}^{N}$ belong to $S$ (Here, $\{\alpha_n\}_{n=0}^{N}$ denotes a sequence $\{\alpha_n^*\}_{n=0}^{\infty}$ such that $\alpha_n^* = \alpha_n$ for all $n \le N$ and $\alpha_n^* = 0$ for all $n > N$).*
- *$S$ is a solid. This means that if $0 \le \alpha_n \le \beta_n$ for all $n$ and $\{\beta_n\} \in S$ then also $\{\alpha_n\} \in S$ and $\|\{\alpha_n\}\|_S \le \|\{\beta_n\}\|_S$.*

- *The following control condition holds: If $\alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ and $\{\alpha_{K(n)}\}_{n=0}^{\infty} \in S$ then also $\{\alpha_n\}_{n=0}^{\infty} \in S$ and $\|\{\alpha_n\}_{n=0}^{\infty}\| \leq C_S \|\{\alpha_{K(n)}\}_{n=0}^{\infty}\|_S$, where the constant $C_S > 0$ only depends on $S$ and $\{K(n)\}_{n=0}^{\infty}$.*

*Given an approximation scheme $(X, \{A_n\})$ and an admissible sequence space $S$ (with respect to $(X, \{A_n\})$), we define the approximation space $\mathbf{A}(X, S)$ by*

$$\mathbf{A}(X, S) = \{f \in X : \{E(f, A_n)\}_{n=0}^{\infty} \in S\}$$

*and its quasi-norm by $\|f\|_{\mathbf{A}(X,S)} = \|\{E(f, A_n)\}\|_S$.*

Let us give a list of some properties of these spaces (see [AL1, Prop.3.8, Theo.3.12, Theo.3.17, Rem.3.18, Cor.4.14]). Of course, the admissibility of $S$ is assumed in all of the following theorems.

**Theorem 1.3** *$\mathbf{A}(X, S)$ is a quasi-normed space which is continuously embedded in $X$. If $X$ and $S$ are normed spaces then $\mathbf{A}(X, S)$ is even a normed space.*

**Theorem 1.4** *Suppose that $S$ has the property*

$$\|\{\alpha_n\}_{n=0}^{\infty}\|_S \leq C \lim_{k\to\infty} \|\{\alpha_n\}_{n=0}^{k}\|_S \text{ for all } \{\alpha_n\} \in S \text{ with } \alpha_0 \geq \alpha_1 \geq \cdots \geq 0,$$

*where $C > 0$ is a constant depending only on $S$. If $X$ is complete and one of the following conditions (a), (b) or (c) is satisfied, then $\mathbf{A}(X, S)$ is also complete.*

*(a) $X$ is a Banach space, all $A_n$ are linear subspaces, $S$ is complete, and $\|\alpha_n\|_S = \|\{|\alpha_n|\}\|_S$ for all $\{\alpha_n\} \in S$.*

*(b) A decreasing sequence $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \infty)$ belongs to $S$ if and only if the limit $\lim_{k\to\infty} \|\{\alpha_n\}_{n=0}^{k}\|_S$ is finite.*

*(c) $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$. Moreover, a decreasing sequence $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \infty)$ belongs to $S$ if and only if $\lim_{l\to\infty} \lim_{k\to\infty} \|\{\alpha_n\}_{n=l}^{k}\|_S = 0$.*

**Theorem 1.5** *Let $\mathcal{D}(S)$ be the set of all decreasing sequences $\{E_n\} \subset [0, \infty)$ with $\{E_n\} \in S$. If*

$$\{1\}_{n=0}^{\infty} \notin S \quad \text{and} \quad \lim_{l\to\infty} ||\{E_n\}_{n=l}^{\infty}||_S = 0 \text{ for all } \{E_n\} \in \mathcal{D}(S)\,, \tag{1.2}$$

*then $\bigcup A_n$ is dense in $\mathbf{A}(X, S)$. If, in addition to (1.2), $S$ and $X$ are complete, $\|\alpha_n\|_S = \|\{|\alpha_n|\}\|_S$ for all $\{\alpha_n\} \in S$, and $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$, then $\mathbf{A}(X, S)$ is complete.*

**Theorem 1.6** *Let $0 < q \leq \infty$ and $\mathcal{B} = \{b_n\}_{n=0}^{\infty} \subset (0, \infty)$ such that $\|\mathcal{B}\|_q = \infty$ and $b_{n+1} \leq \text{const}\, \|\{b_m\}_{m=0}^{n}\|_q$. Then, the quasi-Banach space*

$$S = l^q(\mathcal{B}) := \left\{\{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{R} : \|\{\alpha_n\}\|_{l^q(\mathcal{B})} = \|\{\alpha_n\, b_n\}_{n=0}^{\infty}\|_q < \infty\right\}$$

*is admissible if and only if $\left\|\{b_m\}_{m=0}^{K(n)}\right\|_q \leq \text{const}\, \|\{b_m\}_{m=0}^{n}\|_q$.*

**Example 1.7** *Theorem 1.6 can be used to construct quite general examples of approximation spaces based on an arbitrary approximation scheme $(X, \{A_n\})$. To do this, let $\mathcal{A} = \{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers such that $0 = a_0 < 1 = a_1 < a_2 < a_3 < \cdots$, $\lim_{n\to\infty} a_n = \infty$ and $a_{K(n)+1} \leq K a_n$ , $n \in \mathbb{N}$, for some constant $K > 1$. Finally, let us set*

$$\mathcal{A}(q) = \{a_n(q)\}_{n=0}^{\infty}\,, \quad \textit{where} \quad a_n(q) = \begin{cases} \left(a_{n+1}^q - a_n^q\right)^{1/q} & \textit{if } \; 0 < q < \infty, \\ a_{n+1} & \textit{if } \; q = \infty. \end{cases}$$

*Then we can apply Theorem 1.6 just taking $\mathcal{B} = \mathcal{A}(q)$. The corresponding approximation space is denoted by*

$$X_q^{\mathcal{A}} = \mathbf{A}(X, l^q(\mathcal{A}(q)))$$

*For example, if we set $a_n = (n+1)^s$ ($s > 0$ fixed), then $a_n(q) \sim (n+1)^{s-(1/q)}$ and we obtain that $X_q^{\mathcal{A}} = \mathbf{A}_q^s(X)$ (in the sense of equivalent quasi-norms).*

**Remark 1.8** *One can easily show that for any admissible sequence space of the form $S = l^q(\mathcal{B})$ ($\mathcal{B}$ as in Theorem 1.6) there exists an $\mathcal{A}$ as in Example 1.7 such that $A(X, S) = X_q^{\mathcal{A}}$ in the sense of equivalent quasi-norms. The notation $X_q^{\mathcal{A}}$ is introduced because of the following embedding theorem which shows that $\mathcal{A}$ (and not $\mathcal{B}$) or, more precisely, the class of all sequences equivalent to $\mathcal{A}$, is the parameter which characterizes the space $X_q^{\mathcal{A}}$ if we identify spaces $X_q^{\mathcal{A}}$ and $X_q^{\widetilde{\mathcal{A}}}$ ($\mathcal{A}$ and $\widetilde{\mathcal{A}}$ as in Example 1.7) with equivalent quasi-norms:*

$$\textit{If } a_n \leq \text{const}\, \widetilde{a}_n\,,\ \textit{then } X_q^{\widetilde{\mathcal{A}}} \textit{ is continuously embedded into } X_q^{\mathcal{A}}$$

*(see [AL1, Theorem 4.12]).*

If $X$ is an unitary algebra and $GX$ denotes the group of its invertible elements, then the following natural question arises: Is $\mathbf{A}(X, S)$ an algebra and does $f \in GX \cap \mathbf{A}(X, S)$ imply $f^{-1} \in \mathbf{A}(X, S)$? (Shortly: Is $\mathbf{A}(X, S)$ an inversely closed subalgebra of $X$?) The question whether $\mathbf{A}(X, S)$ is an algebra can be answered easily (see [AL3, Theorem 1]).

**Theorem 1.9** *Let $\mathbf{A}(X,S)$ be an approximation space based on a quasi-normed algebra $X$ and suppose that $A_nA_n \subset A_{K(n)}$ for all $n \in \mathbb{N}$. Then $\mathbf{A}(X,S)$ is a quasi-normed algebra (a normed algebra, if $X$ and $S$ are normed spaces and the sets $A_n$ are linear subspaces of $X$ for all $n$) with the product induced by the product in $X$.*

**Remark 1.10** *By a quasi-normed algebra $X$ we mean an algebra endowed with a quasi-norm having the property $\|fg\| \le C\|f\|\,\|g\|$, $f,g \in X$, where $C > 0$ is some constant. At some places we will use the fact that $\|\,.\,\|_* = C\|\,.\,\|$ defines an equivalent quasi-norm satisfying $\|fg\|_* \le \|f\|_*\|g\|_*$. But we will not assume $C = 1$ in advance, since, in general, this property gets lost if we consider approximation algebras $\mathbf{A}(X,S)$ based on $X$.*

We reproduce the proof of the above theorem for the sake of completeness. Thereby, we use the notation $c$ for positive constants. Here and in all that follows $c$ may have different values at different places.

**Proof of Theorem 1.9.** Let $f,g \in \mathbf{A}(X,S)$. For all $f_n, g_n \in A_n$ we have

$$\begin{aligned} E_{K(n)}(fg) &\le \|fg - f_ng_n\|_X \le c\big(\|(f-f_n)g\|_X + \|f_n(g-g_n)\|_X\big) \\ &\le c\,[\,\|f-f_n\|_X\|g\|_X + (\|f_n - f\|_X + \|f\|_X)\|g-g_n\|_X]\,. \end{aligned}$$

Consequently, $E_{K(n)}(fg)$ can be estimated by

$$c\left[E_n(f)\|g\|_X + \|f\|_X E_n(g)\right] \le c\left[E_n(f)\|g\|_{\mathbf{A}(X,S)} + \|f\|_{\mathbf{A}(X,S)}E_n(g)\right].$$

The sequence on the right hand side belongs to $S$ and its quasi-norm is bounded by $c\,\|f\|_{\mathbf{A}(X,S)}\|g\|_{\mathbf{A}(X,S)}$. Hence, $\{E_{K(n)}(fg)\} \in S$ with the same upper bound for the quasi-norm. Using the admissibility of $S$ we obtain $\{E_n(fg)\} \in S$ and $\|\{E_n(fg)\}\|_S \le C_S\|\{E_{K(n)}(fg)\}\|_S \le c\,\|f\|_{\mathbf{A}(X,S)}\|g\|_{\mathbf{A}(X,S)}$. □

**Remark 1.11** *If there appear different numbers $K_1(n)$ and $K_2(n)$ in the relations $A_n + A_n \subset A_{K_1(n)}$ and $A_nA_n \subset A_{K_2(n)}$ then we should take $K(n) = \max\{K_1(n), K_2(n)\}$ in the assumption of the theorem above, i.e., $S$ must be admissible with respect to this function $K(n)$. We call this property algebra admissibility. Of course, this property is more restrictive than the usual admissibility condition.*

The aim of this paper is to tackle the problem of inverse closedness of $\mathbf{A}(X,S)$. In the case of a commutative quasi-Banach algebra $X$ there already exists a very general and nice result (see [AL3, Theorem 2]), which we will present, in a more general version, in the next section. Unfortunately, the proof given in [AL3] contains an error. For this reason, we also give in this paper a corrected version of this proof. This proof is based on a general result about the inverse closedness of subalgebras of commutative quasi-Banach algebras which is also of own interest.

In the third section of this paper we introduce a general version of the $K$-method of interpolation of quasi-Banach spaces inspired by the work [BK] of Brudnyi and Krugljak in which the idea of parameter spaces $\Phi$ is used, and we prove that if $X$ is a unitary quasi-Banach algebra and $Y \subset X$ is an inversely closed and dense subspace of $X$ which satisfies certain estimation for $\|f^{-1}\|_Y$, $f \in Y \cap GX$, then the associated interpolation space $(X, Y)_\Phi$ is an inversely closed subspace of $X$.

In the fourth section we will consider the inverse closedness of approximation spaces of the type $X_q^{\mathcal{A}}$, where $X$ is a quasi-Banach algebra which does not have to be commutative. To obtain a corresponding theorem, we will use a result by Luther [Lu] (see Lemma 4.1 of this paper) which shows that, under certain conditions, $X_q^{\mathcal{A}}$ is an interpolation space to which the results of Section 3 can be applied.

As application we will present scales of inversely closed subalgebras of $C(\mathbb{T})$ ($\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$) and $C[-1, 1]$. One of these scales contains the well-known Wiener algebra of all $f \in C(\mathbb{T})$ the Fourier coefficients of which belong to $l^1(\mathbb{Z})$.

Other applications, in particular the study of the inverse closedness of approximation subalgebras of $C(K)$ based on nonlinear subsets $A_n$ (like in the case of $n$-term approximation) will be given in a forthcoming paper.

The inverse closedness problem, in the context of approximation and interpolation theories, has an interesting interpretation. It is well known that both approximation spaces and interpolation spaces, when defined over function spaces, produce scales of function spaces of a certain prescribed degree of smoothness. Thus, when we ask if $\mathbf{A}(X, S)$ or $(X, Y)_\Phi$ are inversely closed subspaces of $X$ what we are asking is: Assuming that $f \in GX$, do $f$ and $f^{-1}$ have the same degree of smoothness? And our answer is: yes, at least for a wide class of definitions of smoothness.

## 2 Inverse closedness of commutative approximation algebras

In this section we give a very general result about inverse closedness for commutative quasi-Banach algebras and we use it to study the corresponding problem for approximation algebras.

**Theorem 2.1** *Let $\mathbb{B}$ be a commutative quasi-Banach algebra with unit $e \neq 0$ and let $\mathbb{A}$ be a dense subalgebra of $\mathbb{B}$ with $e \in \mathbb{A}$. Moreover, let $\mathbb{A}$ be equipped with a quasi-norm which turns $\mathbb{A}$ into a quasi-Banach algebra which is continuously embedded into $\mathbb{B}$. If there exists a dense linear subspace $\mathbb{A}_0 \supset \{e\}$ of $\mathbb{A}$ such that $f \in \mathbb{A}_0$ is invertible in $\mathbb{A}$ if and only if it is invertible in $\mathbb{B}$, then $\mathbb{A}$ is an inversely closed subalgebra of $\mathbb{B}$.*

Before proving this result we present two properties of commutative quasi-Banach algebras $(\mathbb{A}, \| \,.\, \|)$ which are well known if $\| \,.\, \|$ is a norm, but less known if $\| \,.\, \|$ is only a quasi-norm.

**Lemma 2.2** *The following properties hold true for every commutative quasi-Banach algebra $(\mathbb{A}, \| \,.\, \|)$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with unit $e \neq 0$:*

*i) The unit $e$ belongs to the interior of $G\mathbb{A}$.*

*ii) $M \subset \mathbb{A}$ is a maximal ideal of $\mathbb{A}$ if and only if $M = ker(\varphi)$ for some $\varphi \in \mathcal{M}(\mathbb{A})$, $\varphi \neq 0$, where*

$$\mathcal{M}(\mathbb{A}) := \{\tau : \mathbb{A} \to \mathbb{C} : \tau \text{ is } \mathbb{K}\text{-linear, continuous and } \tau(xy) = \tau(x)\tau(y)\}.$$

**Proof.** Without loss of generality we assume that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{A}$ (see Remark 1.10).

Let us prove claim i). From the theory of quasi-normed spaces it is well known that there exists some $p \in (0, 1]$ and an equivalent quasi-norm $\| \cdot \|_*$ such that

$$\|x + y\|_*^p \leq \|x\|_*^p + \|y\|_*^p$$

for all $x, y \in \mathbb{A}$ (this is the so-called Aoki-Rolewicz theorem; see [KP, Theo.1.3]). Let $x \in \mathbb{A}$ such that $\|x\| < 1$. Then

$$\left\| \sum_{k=N}^{M} x^k \right\|^p \leq \frac{1}{C^p} \left\| \sum_{k=N}^{M} x^k \right\|_*^p \leq \frac{1}{C^p} \sum_{k=N}^{M} \|x^k\|_*^p \leq \frac{D^p}{C^p} \sum_{k=N}^{M} \|x^k\|^p \leq \frac{D^p}{C^p} \sum_{k=N}^{M} \|x\|^{kp},$$

where we have obviously used that $C\| \cdot \| \leq \| \cdot \|_* \leq D\| \cdot \|$ and that $\|xy\| \leq \|x\|\|y\|$. Hence the series $\sum_{k=0}^{\infty} x^k$ converges in the quasi-norm $\| \cdot \|$. Moreover, $(e - x) \sum_{k=0}^{N} x^k = e - x^{N+1}$, so that

$$\left\| e - (e - x) \sum_{k=0}^{N} x^k \right\| = \|x^{N+1}\| \leq \|x\|^{N+1} \to 0 \quad \text{for} \quad N \to \infty.$$

Consequently, $e - x$ is invertible $\big($with $(e - x)^{-1} = \sum_{k=0}^{\infty} x^k\big)$ and i) is proved.

Let us now prove ii). To start, let us assume that $\varphi \in \mathcal{M}(\mathbb{A})$ and $\varphi \neq 0$. Obviously, $\operatorname{im} \varphi = \mathbb{R}$ or $\operatorname{im} \varphi = \mathbb{C}$, so that $\mathbb{A}/\ker \varphi$ is a field and $M = \ker \varphi$ is maximal. Now we prove that all maximal ideals of $\mathbb{A}$ are of this form. If $I \neq \mathbb{A}$ is a closed ideal of $\mathbb{A}$ (i.e., it is an ideal which is also a closed subset of $\mathbb{A}$) then it is well known from the theory of quasi-normed spaces that the algebra $\mathbb{A}/I$ is a quasi-Banach space with $\|x + I\|_{\mathbb{A}/I} := \inf_{y \in I} \|x + y\|$. Obviously, $\|(x + I)(y + I)\|_{\mathbb{A}/I} \leq \|x + I\|_{\mathbb{A}/I} \|y + I\|_{\mathbb{A}/I}$, i.e., $\mathbb{A}/I$ is a quasi-Banach algebra. Let $M$ be a maximal ideal of $\mathbb{A}$. Then $M$ is closed, since $M \subset \overline{M}^{\mathbb{A}} \neq \mathbb{A}$ (note that, by i), $\overline{M}^{\mathbb{A}} \neq \mathbb{A}$ since $M \subset \mathbb{A} \setminus \{e\}$, and that $\overline{M}^{\mathbb{A}}$ is an ideal, since $x = \lim_{n\to\infty} x_n$ with $\{x_n\} \subset M$ and $y \in \mathbb{A}$ implies that $\{x_n y\} \subset M$ and $xy = \lim_{n\to\infty} x_n y \in \overline{M}^{\mathbb{A}}$). Hence $\mathbb{A}/M$ is a quasi-Banach field and $M = \ker \varphi$ where $\varphi : \mathbb{A} \to \mathbb{A}/M$ is the natural projection $\varphi(x) = x + M$. It follows from the Gelfand-Mazur-Zelazko theorem [KP, Theorem 7.2] that $\mathbb{A}/M$ is isomorphic to $\mathbb{C}$ if $\mathbb{A}$ is a complex quasi-Banach algebra and $\mathbb{A}/M$ is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$ if $\mathbb{A}$ is a real quasi-Banach algebra. [The idea to prove this theorem is as follows: Firstly, by using the Aoky-Rolewicz theorem, one proves that in any quasi-normed algebra $\mathbb{A}$ the spectral radius $\rho(x) := \overline{\lim}_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ is a seminorm. Secondly, note that $I = \{x \in \mathbb{A} : \rho(x) = 0\}$ is an ideal of $\mathbb{A}$, since $\rho(xy) \leq \rho(x)\rho(y)$. Hence, if $\mathbb{A}$ is a field, then $I = \{0\}$, $\rho$ defines a norm on $\mathbb{A}$ and one can use the classical Gelfand-Mazur Theorem]. This proves ii). □

**Remark 2.3** *Note that if* $(\mathbb{A}, \|\cdot\|)$ *is a complex commutative quasi-Banach algebra with unit* $e \neq 0$ *then it follows from ii) of the Lemma above that there are non-zero continuous linear functionals defined over* $\mathbb{A}$*, a fact that does not hold for arbitrary quasi-Banach vector spaces. Moreover, we remark that every multiplicative linear functional defined on a unitary quasi-Banach algebra* $\mathbb{A}$ *is continuous. This fact can be proved in the same way as in the well known case of a Banach algebra* $\mathbb{A}$ *(using assertion i) of Lemma 2.2 and the proof of this assertion, which also works in the non-commutative case.)*

**Proof of Theorem 2.1.** Let $M$ be a maximal ideal of $\mathbb{A}$ and suppose, without loss of generality, that $\|xy\|_{\mathbb{B}} \leq \|x\|_{\mathbb{B}} \|y\|_{\mathbb{B}}$ for all $x, y \in \mathbb{B}$ (see Remark 1.10). Then, by ii) of Lemma 2.2, there exists $\varphi \in \mathcal{M}(\mathbb{A})$ such that $M = \ker \varphi$.

**Step 1.** $\overline{M}^{\mathbb{B}}$ is an ideal of $\mathbb{B}$.

Let $x \in \overline{M}^{\mathbb{B}}$ and $y \in \mathbb{B}$. Then there exists $\{x_n\} \subset M$ such that $\|x - x_n\|_{\mathbb{B}} \to 0$ and $\{y_n\} \subset \mathbb{A}$ such that $\|y - y_n\|_{\mathbb{B}} \to 0$, since $\mathbb{A}$ is a dense subset of $\mathbb{B}$. Thus, $xy = \mathbb{B}\text{–}\lim_{n\to\infty} x_n y_n \in \overline{M}^{\mathbb{B}}$, since $x_n y_n \in M$.

**Step 2.** $\overline{M}^{\mathbb{B}} \neq \mathbb{B}$.

Assume that $\overline{M}^{\mathbb{B}} = \mathbb{B}$. Then $e \in \overline{M}^{\mathbb{B}}$ and there exists $\{m_k\} \subset M$ such that $\|m_k - e\|_{\mathbb{B}} \to 0$. It follows from the density of $\mathbb{A}_0$ in $\mathbb{A}$ that there exists $\{a_k\} \subset \mathbb{A}_0$ such that $\|a_k - m_k\|_{\mathbb{A}} \to 0$. Now, $M = \ker \varphi$ and the continuity of $\varphi$

$$\varphi(a_k) = \varphi(a_k - m_k) \to 0$$

Now we show that, without loss of generality, it can be assumed that $\varphi(a_k) \in \mathbb{K}$ for all $k$. Indeed, if $\mathbb{K} = \mathbb{R}$ and $\varphi(\mathbb{A}_0) \not\subset \mathbb{R}$, then there exists an $a \in \mathbb{A}_0$ with $\varphi(a) = \mathrm{i}$. Hence, $\varphi(a_k) = \alpha_k + \beta_k \,\mathrm{i} \to 0$ implies that $\widetilde{a}_k = a_k - \beta_k \, a \in \mathbb{A}_0$ satisfies

$$\varphi\big(\widetilde{a}_k\big) = \alpha_k \to 0 \quad \text{and} \quad \big\|\widetilde{a}_k - m_k\big\|_{\mathbb{A}} \le c \left(\|a_k - m_k\|_{\mathbb{A}} + \beta_k \|a\|_{\mathbb{A}}\right) \to 0\,.$$

Since $\varphi\big(\widetilde{a}_k\big) \in \mathbb{R}$, we may suppose $\varphi(a_k) \in \mathbb{K}$ without loss of generality. Set $f_k = a_k - \varphi(a_k)e$. Obviously, $f_k \in M$ since $\varphi(f_k) = 0$. Moreover,

$$f_k = m_k + (a_k - m_k) - \varphi(a_k)e \to e \;\text{ in }\; \mathbb{B}\,,$$

since $m_k \to e$ in $\mathbb{B}$, $a_k - m_k \to 0$ in $\mathbb{B}$ (here we used that the embedding $\mathbb{A} \subset \mathbb{B}$ is continuous) and $\varphi(a_k) \to 0$. It follows from i) of Lemma 2.2 that $f_k$ is invertible in $\mathbb{B}$ for all sufficiently large $k$. On the other hand, $f_k \in \mathbb{A}_0$ for all $k$. Hence, for all $k \ge k_0$, $f_k^{-1} \in \mathbb{A}$ and, consequently, $e = f_k f_k^{-1} \in M$ (since $f_k^{-1} \in \mathbb{A}$ and $f_k \in M$). This is nonsense, since $M$ is a proper ideal of $\mathbb{A}$.

**Step 3.** $M = M^* \cap \mathbb{A}$ for a certain maximal ideal $M^*$ of $\mathbb{B}$.

It follows from Steps 1,2 and Zorn's lemma that $\overline{M}^{\mathbb{B}}$ is contained in a certain maximal ideal $M^*$ of $\mathbb{B}$. $M$ is a subset of $M^* \cap \mathbb{A}$ and $M^* \cap \mathbb{A}$ is a proper ideal of $\mathbb{A}$ (since $e \notin M^*$). Thus, $M = M^* \cap \mathbb{A}$ because of the maximality of $M$.

**Step 4.** $\mathbb{A}$ is an inversely closed subalgebra of $\mathbb{B}$.

If $x \in \mathbb{A} \cap G\mathbb{B}$, then $x \notin M^*$ for all maximal ideals $M^*$ of $\mathbb{B}$. By Step 3 this implies $x \notin M = M^* \cap \mathbb{A}$ for all maximal ideals $M$ of $\mathbb{A}$, i.e., $x \in G\mathbb{A}$. $\square$

As a consequence of the result above we can prove the following theorem about approximation algebras.

**Theorem 2.4** *Let $\mathbf{A}(X,S)$ be an approximation space based on a commutative quasi-Banach algebra $X$ over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) with unit $e \neq 0$ and suppose that $A_n A_n \subset A_{K(n)}$ for all $n \in \mathbb{N}$ (compare Theorem 1.9 and Remark 1.11). Moreover, assume that $e \in \mathbf{A}(X,S)$, that $S$ satisfies (1.2), and that*

$$\mathbb{A}_0 := \big\{\lambda e : \lambda \in \mathbb{K}\big\} + \bigcup A_n \quad \textit{satisfies} \quad \mathbb{A}_0 \cap GX \subset G\mathbf{A}(X,S)\,. \tag{2.1}$$

*If $\mathbf{A}(X,S)$ is complete (cf. Theorems 1.4,1.5; for example, this is satisfied if $S$ is complete, $\lim_{k\to\infty}\|\{1\}_{n=0}^k\|_S=\infty$, and $\|\{\alpha_n\}\|_S=\|\{|\alpha_n|\}\|_S$ for all $\{\alpha_n\}\in S$), then $\mathbf{A}(X,S)$ is an inversely closed subalgebra of $X$.*

**Proof.** Just take $\mathbb{B}=X$, $\mathbb{A}=\mathbf{A}(X,S)$, $\mathbb{A}_0=\{\lambda e:\lambda\in\mathbb{K}\}+\bigcup A_n$ and use Theorem 2.1. In view of Theorem 1.9 and the assumed density of $\bigcup A_n$ in $X$, $\mathbb{A}=\mathbf{A}(X,S)$ is a quasi-Banach algebra which is continuously and densely imbedded into $X$. Moreover, by Theorem 1.5 and assumption (2.1), $\mathbb{A}_0$ is dense in $\mathbb{A}$ and $f\in\mathbb{A}_0$ is invertible in $\mathbb{A}$ if and only if it is invertible in $X$. Thus, all assumptions of Theorem 2.1 are satisfied. □

**Corollary 2.5** *Let $X_q^{\mathcal{A}}$ be the approximation space from Example 1.7, based on an approximation scheme $(X,\{A_n\})$ satisfying all assumptions of Theorem 2.4. If $e\in X_q^{\mathcal{A}}$ and $q<\infty$ then $X_q^{\mathcal{A}}$ is an inversely closed subalgebra of $X$.*

**Proof.** $S=l^q(\mathcal{A}(q))$ satisfies (1.2) and $\lim_{k\to\infty}\|\{1\}_{n=0}^k\|_S=\infty$, since $q<\infty$ and $\|\{1\}_{n=0}^k\|_S=a_{k+1}$. Consequently, $X_q^{\mathcal{A}}$ is complete (see Theorem 1.5). Thus, all assumptions of Theorem 2.4 are satisfied. □

The case $q=\infty$ is not considered in Corollary 2.5, since (1.2) is not satisfied for $S=l^\infty(\mathcal{A}(\infty))$. But if we restrict on the subspace

$$l_0^\infty(\mathcal{A}(\infty))=\Big\{\{\alpha_n\}:\lim_{n\to\infty}a_n\alpha_n=0\Big\}$$

(note that $a_n$ is equivalent to $a_{n+1}=a_n(\infty)$) and the corresponding approximation space $X_{\infty,0}^{\mathcal{A}}$, then Theorem 2.4 is applicable and we obtain

$$\lim_{n\to\infty}a_nE_n(f^{-1})=0\quad\text{for all}\ \ f\in X_{\infty,0}^{\mathcal{A}}\cap GX \tag{2.2}$$

if $e\in X_{\infty,0}^{\mathcal{A}}$ and if the approximation scheme $(X,\{A_n\})$ satisfies the assumptions of Theorem 2.4. Now we prove that, since $\mathcal{A}$ can be chosen more or less arbitrary, it even follows that Corollary 2.5 remains true in the case $q=\infty$:

**Corollary 2.6** *If we have $e\in X_{\infty,0}^{\mathcal{A}}$ in the case $q=\infty$, then the restriction $q<\infty$ can be omitted in Corollary 2.5.*

**Proof.** Let $f\in X_\infty^{\mathcal{A}}\cap GX$. For any decreasing sequence $\{\varepsilon_n\}$ with $\lim_{n\to\infty}\varepsilon_n=0$ and

$$1=\varepsilon_1a_1<\varepsilon_2a_2<\varepsilon_3a_3<\dots\,,\qquad\lim_{n\to\infty}\varepsilon_na_n=\infty\,, \tag{2.3}$$

(2.2) can be applied to $f$ and to $\mathcal{B}=\{\varepsilon_na_n\}$ instead of $\mathcal{A}$. Hence,

$$\lim_{n\to\infty}\varepsilon_na_nE_n(f^{-1})=0\,. \tag{2.4}$$

Let us assume that $a_n E_n(f^{-1}) \neq O(1)$. Then,

$$\widetilde{\varepsilon}_n = \left[ \max_{1\le k\le n} a_k E_k(f^{-1}) \right]^{-1}$$

is a decreasing sequence with limit zero. Moreover, it is clear that

$$\widetilde{\varepsilon}_n = \left[a_n E_n(f^{-1})\right]^{-1} \quad \text{for all } \ n \in \mathbb{N}' , \tag{2.5}$$

where $\mathbb{N}' = \{n_j\}_{j=1}^{\infty}$ is some subsequence of $\mathbb{N}$ with $n_1 = 1$. Thus,

$$\{\widetilde{\varepsilon}_n a_n\}_{n\in\mathbb{N}'} = \left\{ \left[E_n(f^{-1})\right]^{-1} \right\}_{n\in\mathbb{N}'}$$

is increasing and converges to infinity, since $E_n(f^{-1}) \to 0$ (because of (2.4) with $\varepsilon_n = (a_n)^{-1/2}$). Without loss of generality we may assume that $\{\widetilde{\varepsilon}_n a_n\}_{n\in\mathbb{N}'}$ is strictly increasing. Now we define $\{\varepsilon_n\}_{n=1}^{\infty}$ by

$$\varepsilon_n = E_1(f^{-1})\, \widetilde{\varepsilon}_{n_j} \quad \text{for} \quad n_j \le n < n_{j+1} \ \ (j = 1, 2, \dots) \, .$$

This sequence is decreasing, converges to zero, and satisfies (2.3). In view of (2.5), $\varepsilon_n a_n E_n(f^{-1}) = E_1(f^{-1})$ for all $n \in \mathbb{N}'$, which is in contradiction with (2.4). □

# 3 Inverse closedness of certain interpolation spaces

Let $X$ and $Y$ be two compatible quasi-normed spaces (i.e., they are continuously embedded in a certain Hausdorff topological vector space) and let $f \in X + Y$. The so-called $K$-functional $K(f, \cdot) : (0,\infty) \to [0,\infty)$ is given by

$$K(f,t) = K_{X,Y}(f,t) := \inf_{f=x+y} (\|x\|_X + t\|y\|_Y),$$

where the infimum is taken over all decompositions $f = x + y$ with $x \in X$ and $y \in Y$.

**Definition 3.1** *Let $\mu$ be a non-trivial positive measure on the Borel subsets of $(0,\infty)$ and let $\mathcal{F}$ be the quotient space $\mathcal{F} = \mathcal{V}/\sim$, where $\mathcal{V}$ denotes the real vector space of all functions $K : (0,\infty) \to \mathbb{R}$ and $\sim$ is the equivalence relation given by*

$$K_1 \sim K_2 \text{ if and only if } \exists A \subset (0,\infty)\text{: } \mu(A) = 0, \ \{t : K_1(t) \neq K_2(t)\} \subset A$$

*In all that follows we identify a function $K \in \mathcal{V}$ with its equivalence class $[K] \in \mathcal{F}$. We say that the quasi-normed vector subspace $\Phi \subset \mathcal{F}$ is a parameter space (with respect to the measure $\mu$) if it satisfies the following two conditions:*

- *If $K_1, K_2 : (0,\infty) \to \mathbb{R}$ are increasing functions with $K_2 \in \Phi$ and $0 \le K_1(t) \le K_2(t)$ for all $t \in (0,\infty)$ then $K_1 \in \Phi$ and $\|K_1\|_\Phi \le C_\Phi \|K_2\|_\Phi$ for a certain constant $C_\Phi$ depending only on $\Phi$.*
- *The function* $\min\{1,t\}$ *belongs to $\Phi$.*

Given a parameter space $\Phi$ and a compatible couple of quasi-normed spaces $(X,Y)$, the space

$$(X,Y)_\Phi = \{f \in X+Y : K(f,\cdot) \in \Phi\}\,, \quad \|f\|_{(X,Y)_\Phi} = \|K(f,\cdot)\|_\Phi$$

is called interpolation space associated to the parameter space $\Phi$. (This name is justified since it is possible to prove (see [Lu, Theorem 3.3], [BK, Proposition 3.3.1]) that the map $(X,Y) \to (X,Y)_\Phi$ defines an interpolation method.) For interpolation theory, the parameter spaces $\Phi$ play a role quite similar to the role of the admissible sequence spaces $S$ in the context of approximation spaces.

**Lemma 3.2** *Let $X$ be a quasi-Banach algebra with unit $e$ and let $Y$ be a quasi-normed space which is densely and continuously embedded into $X$ and which contains $e$. Moreover, suppose that $Y$ is inversely closed in $X$ and that there is a constant $C = C(X,Y)$ such that*

$$\|f^{-1}\|_Y \le C\,\|f\|_Y \quad \textit{for all } f \in Y \cap GX \textit{ with } \|f^{-1}\|_X \le 1\,. \tag{3.1}$$

*Then there is a constant $c = c(X,Y)$ such that the $K$-functional with respect to $X$ and $Y$ satisfies*

$$K(f^{-1},t) \le c\,\|f^{-1}\|_X^2 K(f,t) \quad \textit{for all } f \in GX \textit{ and all } t \in (0, t_0(f)]\,,$$

*where $t_0(f)$ is some positive constant depending on $f$.*

**Proof.** Since $Y$ is inversely closed in $X$, we have

$$K(f^{-1},t) \le \inf_{g \in Y\cap GX} \left(\|f^{-1}-g^{-1}\|_X + t\,\|g^{-1}\|_Y\right), \quad t>0\,. \tag{3.2}$$

From assumption (3.1) (applied to $\|g^{-1}\|_X\, g$ instead of $f$) it follows that

$$\|g^{-1}\|_Y \le C\,\|g^{-1}\|_X^2\,\|g\|_Y \quad \text{for all } g \in Y \cap GX\,. \tag{3.3}$$

Moreover,

$$\|f^{-1}-g^{-1}\|_X = \|f^{-1}(g-f)g^{-1}\|_X \le c\,\|f^{-1}\|_X \|g-f\|_X \|g^{-1}\|_X\,. \tag{3.4}$$

If we fix some arbitrary constant $D > 0$, then (3.2) obviously implies

$$K(f^{-1},t) \le \inf_{g\in Y\cap GX,\, \|g^{-1}\|_X \le D\|f^{-1}\|_X} \left(\|f^{-1}-g^{-1}\|_X + t\,\|g^{-1}\|_Y\right)$$

and together with the estimates (3.3) and (3.4) we obtain

$$K(f^{-1},t) \le c\,\|f^{-1}\|_X^2 \inf_{g\in Y\cap GX,\, \|g^{-1}\|_X \le D\|f^{-1}\|_X} \left(\|f-g\|_X + t\,\|g\|_Y\right)\,, \quad t>0\,.$$

It remains to prove that, at least for a sufficiently large constant $D$ independ of $f$, the last infimum equals $K(f,t)$ for $t \le t_0(f)$. For this aim we remark that

$$\lim_{t\to 0} K(f,t) = 0\,. \tag{3.5}$$

Indeed, choose $g_m \in Y$ with $g_m \to f$ in $X$ and $t_1 > t_2 > \ldots > 0$ with $t_m \to 0$ and $\|g_m\|_Y \le (t_m)^{-1/2}$. Then, $K(f,t_m) \le \|f-g_m\|_X + t_m\,\|g_m\|_Y$ converges to zero, which implies (3.5), since $K(f,t)$ is increasing in $t$. Now it is clear that, for sufficiently small $t \le t_0(f)$ and for all $g \in Y$ which play a role in the infimum

$$\inf_{g\in Y} \left(\|f-g\|_X + t\,\|g\|_Y\right) = K(f,t)\,,$$

the distance $\|f-g\|_X$ is small enough such that $g \in GX$ and $\|g^{-1}\|_X \le D\|f^{-1}\|_X$ if $D$ is chosen large enough (because of $f \in GX$, $g = f(e - f^{-1}(f-g))$ and the proof of assertion i) of Lemma 2.2 which remains true in the non-commutative case). □

**Theorem 3.3** *Let $\Phi$ be a parameter space and let $X, Y$ satisfy the assumptions of Lemma 3.2. Then $(X,Y)_\Phi$ is inversely closed in $X$.*

**Proof.** Let $f \in (X,Y)_\Phi \cap GX$. Then, in view of Lemma 3.2,

$$K(f^{-1},t) \le c\,\|f^{-1}\|_X^2 K(f,t) \quad \text{for } 0 < t \le t_0(f)$$

and, since $\min\{1,t\}\,\|f^{-1}\|_X \le c\,(\|f^{-1}-g\|_X + t\|g\|_X) \le c\,(\|f^{-1}-g\|_X + t\|g\|_Y)$ for all $g \in Y$,

$$\begin{aligned} K(f^{-1},t) &\le \|f^{-1}\|_X \le c\,\min\{1,t_0(f)\}^{-1} K(f^{-1},t_0(f)) \\ &\le c\,\min\{1,t_0(f)\}^{-1}\|f^{-1}\|_X^2 K(f,t) \quad \text{for } t > t_0(f)\,. \end{aligned}$$

Thus, $K(f^{-1},\,.\,) \in \Phi$, since $K(f,\,.\,) \in \Phi$. □

# 4 Noncommutative approximation algebras

Until now, to study the inverse closedness of approximation spaces $\mathbf{A}(X,S)$ we assumed that $X$ is a commutative quasi-Banach algebra. In this section we treat the analogous problem for approximation algebras based on arbitrary quasi-Banach algebras $X$ which do not have to be commutative. Gaining such a generality, we will loose in the possible choice of the sequence space $S$. Indeed, what we do is to use the fact that many approximation spaces can be identified with an interpolation space of the type just described in the preceding section.

In [Lu], the author proved the following nice result (even under more general assumptions; here we consider the special case $b_n = a_n(q)$, $v_m = 1$ of [Lu, Theorem 4.3]):

**Lemma 4.1** *Let $X_q^{\mathcal{A}}$ be an approximation space as in Example 1.7 and let $Y$ be a quasi-normed space which is continuously embedded into $X$ and which satisfies $\bigcup_{n=0}^{\infty} A_n \subset Y$. Further, suppose that the so-called Jackson and Bernstein inequalities*

*(J) $E_n(f) \le c\, h_n \|f\|_Y$, $f \in Y$, $n \in \mathbb{N}$,*

*(B) $\|f_n\|_Y \le c\, h_n^{-1} \|f_n\|_X$ , $f_n \in A_n$, $n \in \mathbb{N}$*

*are satisfied with certain constants $c > 0$ and $h_1 > h_2 > h_3 > \ldots > 0$. If there is an $\alpha \in (0,1)$ such that $h_n^{\alpha} a_n$ is equivalent to some decreasing sequence, then*

$$\Phi = \left\{ K : \{h_n\}_{n=1}^{\infty} \to \mathbb{R} : \|K\|_\Phi = \left\| \{a_n(q) K(h_n)\}_{n=1}^{\infty} \right\|_q < \infty \right\}$$

*is a parameter space (with respect to the measure $\mu(A) = \#(A \cap \{h_n\}_{n=1}^{\infty})$) and*

$$X_q^{\mathcal{A}} = (X,Y)_\Phi$$

*in the sense of equivalent quasi-norms.*

**Remark 4.2** *We would like to recall that two sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ of nonnegative numbers are said to be equivalent if there are constants $C, D > 0$ such that $C\alpha_n \le \beta_n \le D\alpha_n$ for all $n$. Moreover, a sequence $\{\alpha_n\}_{n=1}^{\infty} \subset [0,\infty)$ is named almost decreasing if it is equivalent to some decreasing sequence. Obviously, this is the case if and only if $\alpha_m \le \mathrm{const}\, \alpha_n$ for all $n \le m$.*

Theorem 3.3 and Lemma 4.1 yield the following main result of this section. We mention that we do not have to suppose the density of $Y \supset \bigcup A_n$ in $X$, since we have already assumed that $\bigcup A_n$ is dense in $X$.

**Theorem 4.3** *Let $X_q^{\mathcal{A}}$ be an approximation space as in Example 1.7, based on a quasi-Banach algebra $X$ with unit $e$. Further, let $Y \supset \{e\} \cup \bigcup A_n$ be a quasi-normed space which is continuously and inversely closed embedded into $X$ and which satisfies*

$$\|f^{-1}\|_Y \leq C\,\|f\|_Y \quad \textit{for all } f \in Y \cap GX \ \textit{ with } \|f^{-1}\|_X \leq 1\,, \tag{4.1}$$

*where $C = C(X, Y)$ is some constant. If (J) and (B) (see Lemma 4.1) hold true with certain numbers $h_1 > h_2 > h_3 > \ldots > 0$ for which, with some $\alpha \in (0, 1)$, $\{h_n^\alpha a_n\}$ is almost decreasing, then $X_q^{\mathcal{A}}$ is inversely closed in $X$.*

**Remark 4.4** *Note that we have not assumed the inclusions $A_n A_n \subset A_{K(n)}$ to hold. Thus, the above theorem is also applicable to approximation spaces which are no algebras.*

We should mention that the condition (4.1) is very restrictive. In many applications in which several spaces $Y$ satisfying (J) and (B) (with different $h_n$) are known, (4.1) means that $Y$ cannot be chosen too small, which implies that only sufficiently large spaces $X_q^{\mathcal{A}} \supset Y$, i.e., sufficiently slow increasing sequences $\mathcal{A}$ can be considered. For example, if $X = C[a, b]$ and $Y = C^{(r)}[a, b]$ ($r \in \mathbb{N}$), then (4.1) is only satisfied for $r = 1$ and, consequently, only spaces $X_q^{\mathcal{A}}$ bigger than $C^{(1)}[a, b]$ can be considered in Theorem 4.3. However, the following trick is often useful to come from slowly increasing sequences $\mathcal{A} = \{a_n\}$ to faster increasing sequences $\{n^r a_n\}$:

Let $X$ be a quasi-normed algebra with unit $e$, $Y \supset \{e\}$ a quasi-normed space continuously embedded into $X$, and suppose that there is a quasi-Banach algebra $X_0$ between $Y$ and $X$ (with continuous embeddings) such that

i) $(X_0, \{A_n\})$ is an approximation scheme with $K(n) = Kn$ ($K \in \mathbb{N}$ some constant),

ii) all assumptions of Theorem 4.3 are satisfied with $X$ replaced by $X_0$,

iii) $X_0$ is inversely closed in $X$,

iv) there exists some $r > 0$ such that (1.1) is satisfied with $Y$ replaced by $X_0$,

v) $a_n \leq \operatorname{const} n^{-r} \big\| \{m^r a_m(q)\}_{m=1}^n \big\|_q$ for all $n \in \mathbb{N}$ and $\big\{n^\varepsilon a_n^{-1}\big\}$ is almost decreasing for some $\varepsilon > 0$. (Later (see (5.3)) we will see that the second of these two conditions implies the first one, at least if $a_n$ is replaced by a certain equivalent sequence which has no influence on the considered approximation spaces because of Remark 1.8.)

Then $\mathbf{A}^r_p(X) \subset X_0 \subset \mathbf{A}^r_\infty(X)$ for some $p \in (0,1]$ (see [AL1, Theorem 4.9]) and, consequently, $(\mathbf{A}^r_p(X))^{\mathcal{A}}_q \subset (X_0)^{\mathcal{A}}_q \subset (\mathbf{A}^r_\infty(X))^{\mathcal{A}}_q$. Moreover, $(X_0)^{\mathcal{A}}_q$ is inversely closed in $X_0$ and, hence, inversely closed in $X$. But $(\mathbf{A}^r_p(X))^{\mathcal{A}}_q = (\mathbf{A}^r_\infty(X))^{\mathcal{A}}_q = X_q^{\{n^r a_n\}}$ ([Lu, Cor.6.4]) and we conclude that $X_q^{\{n^r a_n\}}$ is inversely closed in $X$.

For example, let $X = C[0,2\pi] \cap \{f : f(0) = f(2\pi)\}$, $X_0 = C^{(1)}[0,2\pi] \cap X$, $Y = C^{(2)}[0,2\pi] \cap X$, $a_n = n^s$ with $s \in (1,2)$, and let $A_n$ be the set of all trigonometric polynomials of degree less than $n$. Then (4.1) is satisfied with $X_0$ instead of $Y$, but $X_0 \not\subset X^{\mathcal{A}}_q$. On the other hand $Y \subset X^{\mathcal{A}}_q$, but (4.1) is not satisfied for $Y$. Thus, Theorem 4.3 cannot be applied directly to obtain the inverse closedness of $X^{\mathcal{A}}_q = \mathbf{A}^s_q(X)$. But with the above trick this is possible, since all assumptions i)–v) are satisfied with $\widetilde{\mathcal{A}} = \{n^{s-1}\}$ instead of $\mathcal{A}$, $h_n = n^{-1}$, and $r = 1$.

# 5 Applications

We restrict on results of the following type: "If $f$ is a non-vanishing function with certain properties, then $1/f$ has the same properties". More precisely, we will consider the inverse closedness of certain spaces of the type $X^{\mathcal{A}}_q$ (see Example 1.7), where $X$ is an algebra of continuous functions. Although the multiplication of functions is commutative, we will also need results from Section 4 to interpret $f \in X^{\mathcal{A}}_q$ as a smoothness property of $f$.

## 5.1 Smoothness of $1/f$

In the present subsection we study the inverse closedness of generalized Hölder-Zygmund spaces based on

$$X = C_{2\pi} := \{f \in C(\mathbb{R}) : f = f(\,.\,+2\pi)\}\,, \quad \|f\| = \max\{|f(x)| : x \in \mathbb{R}\}\,.$$

These spaces are defined with the help of the classical modulus of smoothness

$$\omega_r(f,t) = \sup_{0<h\le t} \|\Delta^r_h f\|\,, \quad (\Delta^r_h f)(x) = \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f\left(x + \left(\frac{r}{2} - k\right) h\right)$$

as follows:

$$H^{\mathcal{A},q}_{2\pi} = \left\{ f \in C_{2\pi} : \left\{\omega_r(f,n^{-1})\right\}_{n=1}^\infty \in l^q\big(\mathcal{A}(q)\big) \right\},$$

where the norm in $H^{\mathcal{A},q}_{2\pi}$ is given by

$$\|f\|_{\mathcal{A},q} = \|f\| + \left\| \left\{a_n(q)\, \omega_r(f,n^{-1})\right\}_{n=1}^\infty \right\|_q .$$

Here $\mathcal{A} = \{a_n\}$ is a sequence as in Example 1.7 (where $K(n) = n$) such that

$$n^{-c}a_n \text{ is almost decreasing for some constant } c > 0$$

and $r \in \mathbb{N}$ is chosen sufficiently large such that, for some $\varepsilon > 0$,

$$n^{\varepsilon - r}a_n \text{ is almost decreasing.} \tag{5.1}$$

The following proposition shows that $H_{2\pi}^{\mathcal{A},q}$ is an approximation space which is independent of the choice of $r$ (in the sense of equivalent quasi-norms) and that the classical Hölder-Zygmund space

$$H_{2\pi}^s = \left\{ f \in C_{2\pi} : \omega_{1+[\nu]}\big(f^{(k)}, t\big) = O(t^\nu) \ \ (t \downarrow 0) \right\}, \quad s = k + \nu, \ 0 < \nu \le 1$$

corresponds to the case $a_n = n^s$, $q = \infty$.

**Proposition 5.1** *Take $\mathcal{A}$ and $\mathcal{A}(q)$ as in Example 1.7 (where $K(n) = n$) and let (5.1) be satisfied. Then, in the sense of equivalent quasi-norms,*

$$H_{2\pi}^{\mathcal{A},q} = (C_{2\pi})_q^{\mathcal{A}}(\{T_n\}), \quad \textit{where} \quad T_n = \operatorname{span}\left\{e^{i\,k\,\cdot}\right\}_{k=-n+1}^{n-1}. \tag{5.2}$$

*Moreover, for every fixed $k \in \mathbb{N}_0$ ($k < r$) for which, with some constant $\varepsilon > 0$,*

$$n^{-k-\varepsilon}a_n \textit{ is almost increasing},$$

*the assertions*

$$f \in H_{2\pi}^{\mathcal{A},q} \quad \textit{and} \quad f^{(k)} \in H_{2\pi}^{\{n^{-k}\}\mathcal{A},q}$$

*are equivalent, where $\{b_n\} = \{n^{-k}\}\mathcal{A}$ is any strictly increasing sequence with*

$$b_1 = 1 \quad \textit{and} \quad b_n \sim n^{-k}a_n$$

*("$\sim$" means equivalent; in view of (5.2) and Remark 1.8, $H_{2\pi}^{\{n^{-k}\}\mathcal{A},q}$ does not depend on the choice of $b_n$) and $\|f\|_{\mathcal{A},q} \sim \|f\| + \|f^{(k)}\|_{\{n^{-k}\}\mathcal{A},q}$.*

**Proof.** It is well known (see [DL, Sect. 3.4,7.2]) that, for $X = C_{2\pi}$, $A_n = T_n$, and

$$Y = C_{2\pi}^{(r)} := C_{2\pi} \cap C^{(r)}(\mathbb{R}), \quad \|f\|_Y = \|f\| + \|f^{(r)}\|$$

the assumptions (J) and (B) of Lemma 4.1 are satisfied with $h_n = n^{-r}$. Thus,

$$(C_{2\pi})_q^{\mathcal{A}}(\{T_n\}) = \left\{ f \in C_{2\pi} : \left\| \{a_n(q)K(f, n^{-r})\}_{n=1}^\infty \right\|_q < \infty \right\}$$

in the sense of equivalent quasi-norms, where

$$K(f,t) = \inf_{g\in C_{2\pi}^{(r)}} \Big(\|f-g\| + t\big(\|g\| + \|g^{(r)}\|\big)\Big) .$$

From the estimate $t\|f\| \le t(\|f-g\| + \|g\|) \le \|f-g\| + t\|g\| \le 2\|g-f\| + t\|f\|$, $t \in (0,1]$, and a well known result on the equivalence of K-functionals and moduli of smoothness (see [DL, Theorem 6.2.4]) it follows

$$K(f,t) \sim \inf_{g\in C_{2\pi}^{(r)}} \Big(\|f-g\| + t\big\|g^{(r)}\big\|\Big) + t\|f\| \sim \omega_r(f, t^{1/r}) + t\|f\| , \quad t \in (0,1] .$$

Taking into account that the space $\Phi$ from Lemma 4.1 is a parameter space, i.e., $\|\{n^{-r}a_n(q)\}\|_q < \infty$, we conclude (5.2).

To prove the second assertion we choose the sequence $b_n$ in such a way that

$$b_n = n^{\varepsilon} c_n \quad \text{for all} \quad n \in \mathbb{N} ,$$

where $c_n$ is an increasing sequence equivalent to $n^{-k-\varepsilon}a_n$. In this case we have

$$n^k b_n(q) \sim (n^k b_n)(q) , \quad n \in \mathbb{N} . \tag{5.3}$$

Indeed, $n^k b_n(q) \le c\,(n^k b_n)(q)$ is obvious. For $q = \infty$, also the reverse inequality is clear. If $q < \infty$ then we use the mean value theorem $\ln B/A = \xi^{-1}(B-A)$ $(0 < A < \xi < B)$ and the monotonicity of $n^{\varepsilon} b_n^{-1} = c_n^{-1}$ to obtain

$$\begin{aligned} (n+1)^{kq} b_{n+1}^q - n^{kq} b_n^q &\le (n+1)^{kq} b_{n+1}^q \ln \frac{(n+1)^{kq} b_{n+1}^q}{n^{kq} b_n^q} \\ &\le (n+1)^{kq} b_{n+1}^q \left(1 + \frac{k}{\varepsilon}\right) \ln \frac{b_{n+1}^q}{b_n^q} \\ &\le c\, n^{kq}\big(b_{n+1}^q - b_n^q\big). \end{aligned}$$

Together with the known equivalence

$$\big\|\{n^k b_n(q) E_n(f)\}_{n=1}^{\infty}\big\|_q \sim \big\|\{b_n(q) E_n(f^{(k)})\}_{n=1}^{\infty}\big\|_q ,$$

which is proved in [Lu2, Corollary 1] for the case $q \ge 1$ and which can be proved in almost the same way as in [Lu2] for the case $0 < q < 1$, (5.3) yields $f^{(k)} \in (C_{2\pi})_q^{\mathcal{B}}$ if and only if $f \in (C_{2\pi})_q^{\{n^k b_n\}} = (C_{2\pi})_q^{\mathcal{A}}$ (see Remark 1.8), where $\|f\|_{(C_{2\pi})_q^{\mathcal{A}}} \sim \|f\| + \|f^{(k)}\|_{(C_{2\pi})_q^{\mathcal{B}}}$. □

From the above proposition and Corollaries 2.5 and 2.6 we conclude the following result. Here we take into account that the condition $a_{2n} \le c\, a_n$ (which is needed because of $T_n T_n \subset T_{K(n)}$ for $K(n) = 2n-1$) is automatically satisfied if (5.1) holds true.

**Theorem 5.2** *Take $\mathcal{A}$ as in Proposition 5.1. Then, $H_{2\pi}^{\mathcal{A},q}$ is an inversely closed subalgebra of $C_{2\pi}$, i.e., $f \in H_{2\pi}^{\mathcal{A},q}$ and $f(x) \ne 0$ for all $x$ imply $1/f \in H_{2\pi}^{\mathcal{A},q}$.*

**Proof.** It remains to show that $1/f \in H_{2\pi}^{\mathcal{A},q}$ for every $f \in \bigcup T_n$ with $f(x) \ne 0$ for all $x$. This is very easy if we recall that $H_{2\pi}^{\mathcal{A},q}$ is an interpolation space between the spaces $Y$ and $X$ from the proof of Proposition 5.1. In particular, $Y \subset H_{2\pi}^{\mathcal{A},q}$. Clearly, $1/f \in Y$. □

**Remark 5.3** *If we consider the case $q = \infty$ and if we choose some increasing function $G : (0,1] \to (0,\infty)$ such that*

$$a_n = \frac{1}{G(1/n)} \quad \textit{for all} \quad n \in \mathbb{N}\,,$$

*then the assertion of Theorem 5.2 can be rewritten as follows:*

$$\begin{aligned} &\textit{If } \omega_r(f,t) = O(G(t)) \ (t \downarrow 0) \textit{ and } f(x) \ne 0 \textit{ for all } x, \textit{ then} \\ &\omega_r(1/f,t) = O(G(t)) \ (t \downarrow 0). \end{aligned} \tag{5.4}$$

*Results of this type are surely known, since they can be proved directly without using approximation spaces. For example, if $r = 1$, then one can use that $\Delta_h^1(1/f)(x) = -[f(x+(h/2))f(x-(h/2))]^{-1}\Delta_h^1 f(x)$ and, consequently,*

$$\omega_1(1/f,t) \le \|1/f\|^2 \omega_1(f,t)\,.$$

*If $r = 2$, then $\Delta_h^2(1/f) = \Delta_h^1(\Delta_h^1(1/f))$ and the above formula for $\Delta_h^1(1/f)$ can be used to show that*

$$\omega_2(1/f,t) \le \|1/f\|^2 \omega_2(f,t) + 2\|1/f\|^3 \omega_1(f,t)^2\,.$$

*Together with the Marchaud inequality $\omega_1(f,t) \le c\,t \int_t^\infty s^{-2}\omega_2(f,s)\,ds$ ([DL, Theorem 2.8.1]) this leads to (5.4) for certain functions $G(t)$. For bigger values of $r$ the direct proof of (5.4) is involved. So it is more elegant to use the approach via approximation spaces.*

## 5.2 Weighted smoothness of $1/f$

Now we study the inverse closedness of weighted Hölder-Zygmund type spaces based on

$$X = C[-1,1]\,, \quad \|f\| = \max\{|f(x)| : x \in [-1,1]\}\,.$$

These spaces are defined with the help of the modulus with step-weight function $\varphi(x) = \sqrt{1-x^2}$ (see [DT])

$$\omega_\varphi^r(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|\,,\ (\Delta_{h\varphi}^r f)(x) = \begin{cases} (\Delta_{h\varphi(x)}^r f)(x)\,, & x \pm \frac{rh}{2}\varphi(x) \in (-1,1) \\ 0\,, & \text{otherwise} \end{cases}$$

as follows:

$$H_\varphi^{\mathcal{A},q} = \Big\{ f \in C[-1,1] : \big\{\omega_\varphi^r(f,n^{-1})\big\}_{n=1}^\infty \in l^q\big(\mathcal{A}(q)\big) \Big\}\,,$$

where the norm in $H_\varphi^{\mathcal{A},q}$ is given by

$$\|f\|_{\mathcal{A},q} = \|f\| + \big\|\big\{a_n(q)\,\omega_\varphi^r(f,n^{-1})\big\}_{n=1}^\infty\big\|_q\,.$$

Here $\mathcal{A} = \{a_n\}$ and $r \in \mathbb{N}$ have to satisfy the same assumptions as in the preceding section.

In the proof of (5.2) we have used Lemma 4.1 together with the Jackson and Bernstein inequalities (J) and (B) for trigonometric polynomials and the equivalence of unweighted $K$-functionals and unweighted moduli of smoothness. To obtain the analogous result for $H_\varphi^{\mathcal{A},q}$, we need (J),(B) for algebraic polynomials and equivalences between weighted $K$-functionals and moduli of smoothness.

**Lemma 5.4 ([DT], Theorems 7.2.1, 6.1.1, 8.4.7)** *Let* $\Pi_n = \operatorname{span}\{x^k\}_{k=0}^{n-1}$ *and let* $\mathcal{K}(f,t)$ *be the* $K$*–functional with respect to* $X = C[-1,1]$ *and the space*

$$Y = \big\{ f \in C[-1,1] : \varphi^r f^{(r)} \in L^\infty(-1,1) \big\} \quad \textit{with seminorm} \quad |f|_Y = \|\varphi^r f^{(r)}\|_{L^\infty}\,,$$

*i.e.,* $\mathcal{K}(f,t) = \inf_{g\in Y}\big(\|f-g\| + t\,|g|_Y\big)$. *Then, the following assertions hold true:*

*(i)* $\inf_{p_n\in\Pi_n} \|g - p_n\| \le \text{const}\, n^{-r} |g|_Y$ *for all* $g \in Y$ *and all* $n \in \mathbb{N}$,

*(ii)* $|p_n|_Y \le \text{const}\, n^r \|p_n\|$ *for all* $p_n \in \Pi_n$ *and all* $n \in \mathbb{N}$,

*(iii)* $\mathcal{K}(f,t^r) \sim \omega_\varphi^r(f,t)$ *for all* $f \in C[-1,1]$ *and all* $t \in (0,1]$.

Now it is clear that (5.2) can be proved analogously for the spaces $H_\varphi^{\mathcal{A},q}$:

**Proposition 5.5** *Take $\mathcal{A}$ and $\mathcal{A}(q)$ as in Example 1.7 (where $K(n) = n$) and let (5.1) be satisfied. Then, in the sense of equivalent quasi-norms,*

$$H_\varphi^{\mathcal{A},q} = C[-1,1]_q^{\mathcal{A}}(\{\Pi_n\})\,. \tag{5.5}$$

Now one can use Corollaries 2.5 and 2.6 to prove the inverse closedness of the spaces $H_\varphi^{\mathcal{A},q}$ in $C[-1,1]$. Alternatively, the known identity

$$\inf_{P\in\Pi_n} \|f-P\|_{C[-1,1]} = \inf_{T\in T_n} \|f(\cos(\,.\,)) - T\|_{C[0,2\pi]}\,, \quad f \in C[-1,1]$$

(see [Na]) leads to the inverse closedness of $H_\varphi^{\mathcal{A},q} = C[-1,1]_q^{\mathcal{A}}(\{\Pi_n\})$ because of the inverse closedness of $H_{2\pi}^{\mathcal{A},q} = (C_{2\pi})_q^{\mathcal{A}}(\{T_n\})$:

**Theorem 5.6** *Take $\mathcal{A}$ as in Proposition 5.5. Then, $f \in H_\varphi^{\mathcal{A},q}$ if and only if $f(\cos(\,.\,)) \in H_{2\pi}^{\mathcal{A},q}$. In particular, $H_\varphi^{\mathcal{A},q}$ is an inversely closed subalgebra of $C[-1,1]$ (compare Theorem 5.2), i.e., $f \in H_\varphi^{\mathcal{A},q}$ and $f(x) \neq 0$ for all $x \in [-1,1]$ imply $1/f \in H_\varphi^{\mathcal{A},q}$.*

**Remark 5.7** *An interpretation of the inverse closedness of the approximation space $H_\varphi^{\mathcal{A},q} = \mathbf{A}(C[-1,1], l^q(\mathcal{B}); \{\Pi_n\})$ ($\mathcal{B} = \mathcal{A}(q)$) is as follows: If $f \in \mathbf{C}[-1,1]$ vanishes nowhere on $[-1,1]$ then*

$$\{E(f,\Pi_n)\}_{n=0}^\infty \in l^q(\mathcal{B}) \iff \{E(f,\Gamma_n)\}_{n=0}^\infty \in l^q(\mathcal{B})\,, \tag{5.6}$$

*where $\Gamma_n := \{1/p : p \in \Pi_n$ and $p(t) \neq 0$ for all $t \in [-1,1]\}$, $n = 0,1,\cdots$. Indeed, from the equality*

$$p_n\left(f - \frac{1}{p_n}\right) = f\left(p_n - \frac{1}{f}\right)\,, \quad \frac{1}{p_n} \in \Gamma_n$$

*it follows easily that the sequences $\{E(f,\Gamma_n)\}_{n=n_0}^\infty$ and $\{E(1/f,\Pi_n)\}_{n=n_0}^\infty$ ($n_0 = n_0(f)$ sufficiently large) are equivalent. Thus, (5.6) is only a reformulation of the equivalence $f \in H_\varphi^{\mathcal{A},q} \Leftrightarrow 1/f \in H_\varphi^{\mathcal{A},q}$.*

### 5.3 Wiener type theorems

We start this section by showing that classical Wiener's inversion theorem is an easy corollary of Theorem 2.1.

**Theorem 5.8 (N. Wiener, (see [K]))** *Let us assume that $f \in C(\mathbb{T})$ is a continuous function defined on the unit circle $\mathbb{T}$ which vanishes nowhere on $\mathbb{T}$. If the Fourier coefficients of $f$ satisfy $\{c_k(f)\} \in l^1(\mathbb{Z})$ then $1/f$ also satisfies $\{c_k(1/f)\} \in l^1(\mathbb{Z})$ .*

**Proof.** We apply Theorem 2.1 to

$$\mathbb{B} = C(\mathbb{T}), \quad \|f\|_{\mathbb{B}} = \max_{z\in\mathbb{T}} |f(z)|, \qquad \text{and}$$

$$\mathbb{A} = \{f \in C(\mathbb{T}) : \{c_k(f)\} \in l^1(\mathbb{Z})\}, \quad \|f\|_{\mathbb{A}} = \|\{c_k(f)\}\|_{l^1(\mathbb{Z})}.$$

Clearly, $\mathbb{A}$ is isometrically isomorphic to the Banach algebra $l^1(\mathbb{Z})$, where the product in $l^1(\mathbb{Z})$ is the convolution of sequences. The subalgebra

$$\mathbb{A}_0 = \text{span}\{z^k : k \in \mathbb{Z}\}$$

of $\mathbb{A}$ corresponds to the dense subalgebra $\{\{c_k\} : c_k \neq 0$ only for finitely many $k\}$ of $l^1(\mathbb{Z})$. Hence, $\mathbb{A}_0$ is dense in $\mathbb{A}$. It is well known that $\mathbb{A}_0$ is also dense in $\mathbb{B}$. Thus, $\mathbb{A}$ is a dense subalgebra of $\mathbb{B}$. Clearly, the embedding $\mathbb{A} \subset \mathbb{B}$ is continuous. It remains to show that $1/f \in \mathbb{A}$ for every $f \in \mathbb{A}_0$ without zeros on $\mathbb{T}$. For this end, we recall that $g(t) := 1/f(\mathrm{e}^{\mathrm{i}t})$ belongs to all spaces $H_{2\pi}^{\mathcal{A},q}$ considered in Section 5.1 (see the proof of Theorem 5.2). In particular, $E(g, T_n) = O(n^{-s})$ for every fixed $s > 0$. If we take into account that

$$c_k(1/f) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt}\, g(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt}\, [g(t) - g_k(t)]\, dt$$

for all $g_k \in T_{|k|}$, then we obtain $|c_k(1/f)| = O(|k|^{-s})$ ($|k| \to \infty$) for every fixed $s > 0$, which implies $\{c_k(1/f)\} \in l^1(\mathbb{Z})$. □

The above proof can be written down word by word with $l^1(\mathbb{Z})$ replaced by a general sequence space $S$ satisfying certain assumptions. In this way we obtain the following main result of this section.

**Theorem 5.9** *Let $S$ be a subalgebra of $l^1(\mathbb{Z})$ (where the multiplication in $l^1(\mathbb{Z})$ is the convolution of sequences) and assume that $\{c_k\}_{k\in\mathbb{Z}} \in S$ for all $\{c_k\}$ which satisfy, for every fixed $s > 0$, $|c_k| = O(|k|^{-s})$ ($|k| \to \infty$). Further, suppose that there is a quasi-norm on $S$ such that $S$ is a quasi-Banach algebra which is continuously embedded into $l^1(\mathbb{Z})$ and in which the set of all finite sequences (i.e. sequences having only finitely many non-zero entries) is dense. If $f \in C(\mathbb{T})$ vanishes nowhere on $\mathbb{T}$ and $\{c_k(f)\} \in S$, then $\{c_k(1/f)\} \in S$.*

**Remark 5.10** *Theorem 5.9 says that the subalgebra*

$$\mathbf{W}(S) = \{f \in C(\mathbb{T}) : \{c_k(f)\} \in S\}$$

*of $C(\mathbb{T})$ is inversely closed. We mention that, if $S$ has the property $\{c_k\} \in S \Leftrightarrow \{|c_k|\} \in S$, $\mathbf{W}(S)$ is an approximation space based on the Wiener algebra*

$$\mathbf{W} = \mathbf{W}(l^1(\mathbb{Z}))\,, \quad \|f\|_{\mathbf{W}} = \|\{c_k(f)\}\|_{l^1(\mathbb{Z})}\,.$$

*Indeed, the absolute values of the Fourier coefficients of $f \in \mathbf{W}$ can be obtained from the errors $E_m^{\mathbf{W}}(f)$ of best approximation in the norm of $\mathbf{W}$ by elements of $A_m$, where $A_1 = \mathrm{span}\{1\}\,,\ A_2 = \mathrm{span}\{1, z\}\,,\ A_3 = \mathrm{span}\{z^{-1}, 1, z, \}\,,\ A_4 = \mathrm{span}\{z^{-1}, 1, z, z^2\}\,,\ \ldots\,,$ as follows:*

$$|c_{-n}(f)| = E_{2n}^{\mathbf{W}}(f) - E_{2n+1}^{\mathbf{W}}(f)\,,\ n \in \mathbb{N}_0\,,\ \ |c_n(f)| = E_{2n-1}^{\mathbf{W}}(f) - E_{2n}^{\mathbf{W}}(f)\,,\ n \in \mathbb{N}.$$

*If we use these equations to define a corresponding linear operator $\Delta$ mapping sequences with index set $\mathbb{N}_0$ into sequences with index set $\mathbb{Z}$, then we obtain $\mathbf{W}(S) = \mathbf{A}(\mathbf{W}, \mathcal{S})$, where $\mathcal{S} = \{\{\alpha_n\}_{n=0}^{\infty} : \Delta\{\alpha_n\} \in S\}$. Consequently, for certain spaces $S$ also Theorem 2.4 can be used to prove the inverse closedness of $\mathbf{W}(S)$ in $C(\mathbb{T})$ (using the inverse closedness of $\mathbf{W}$ in $C(\mathbb{T})$). But, clearly, the direct application of Theorem 2.1 is more elegant.*

We finish this section with giving a concrete class of sequence spaces $S$ for which the assumptions of Theorem 5.9 are satisfied.

**Example 5.11** *Let $\{b_k\}_{k\in\mathbb{Z}}$ be a sequence of positive numbers satisfying $b_{k+l} \le \mathrm{const}\, b_k b_l$ for all $k, l \in \mathbb{Z}$ and $0 < \mathrm{const} \le b_k = O(|k|^\alpha)$ ($|k| \to \infty$) for a certain constant $\alpha > 0$. Then it is easy to prove that*

$$S = \left\{ \{c_k\}_{k\in\mathbb{Z}} : \|\{c_k\}\|_S = \sum_{k=-\infty}^{\infty} |c_k|\, b_k < \infty \right\}$$

*satisfies the assumptions of Theorem 5.9.*

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