The potential for ill-posedness of multiplication operators occurring in inverse problems

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Abstract

In this paper, we show the restricted influence of non-compact multiplication operators mapping in $L^2(0,1)$ occurring in linear ill-posed operator equations and in the linearization of nonlinear ill-posed operator equations with compact forward operators. We give examples of nonlinear inverse problems in natural science and stochastic finance that can be written as nonlinear operator equations for which the forward operator is a composition of a simple linear integration operator and a nonlinear Nemytskii operator. Hence, the Fréchet derivative of such a forward operator is a composition of integration and multiplication operators. It is shown for power type functions and conjectured for a wider class of multiplier (weight) functions with essential zeros that the unbounded inverse of the injective multiplication operator does not influence the (local) degree of ill-posedness of inverse problems under consideration.

In a more general Hilbert space setting, we investigate the role of approximate source conditions in the method of Tikhonov regularization for linear and nonlinear ill-posed operator equations. We introduce a distance function measuring the violation of canonical source conditions and derive convergence rates for regularized solutions in the linear case based on that functions. In this context, we formulate cross-connections to convergence rates in Tikhonov regularization with general source conditions as frequently used in the recent literature. By considering the structure of source conditions in Tikhonov regularization it could be expected for the multiplication operators in $L^2(0,1)$ that different decay rates of multiplier functions near a zero, for example the decay as a power or as an exponential function, would lead to completely different ill-posedness situations. Also based on the studies concerning approximate source conditions we indicate that only integrals of multiplier functions and not the specific character of the decay of multiplier functions in a neighborhood of a zero determine the convergence behavior of regularized solutions.

MSC2000 subject classification: 47A52, 47J06, 47B06, 65J20, 47H30, 47B33, 65R30, 45C05

Keywords: Inverse problems, linear and nonlinear ill-posed problems, singular values, degree of ill-posedness, multiplication operator, Nemytskii operator, Tikhonov regularization, Fréchet derivative, convergence rates, source conditions

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1 Introduction

In this paper, we deal with errors and convergence rates of classical Tikhonov regularization applied to ill-posed linear and nonlinear inverse problems written as operator equations in Hilbert spaces, where approximate source conditions are under consideration. In particular, we focus on the potential for ill-posedness of multiplication operators in $L^2(0,1)$ when the associated multiplier (weight) function occurring in the forward operator of the linear inverse problem or in the Fréchet derivative of the forward operator in the nonlinear problem has essential zeros of power type or exponential type.

Let X and Y be infinite dimensional Hilbert spaces over the field of real numbers and let $\|\cdot\|$ denote the generic norm in both spaces. On the one hand we consider inverse problems that can be written as *unconstrained linear operator equations*

$$A\,x \;=\; y \qquad (x \in X,\;\; y \in Y)\,, \tag{1}$$

where the *injective bounded linear forward operators* $A \in \mathcal{L}(X,Y)$ are assumed to have a *non-closed range*, i.e. $R(A) \neq \overline{R(A)}$, implying an *unbounded inverse* $A^{-1} : R(A) \to X$ and leading to *ill-posed* equations (1).

In particular, for $X = Y = L^2(0,1)$, we consider the class of *compact composite linear integral operators* $B = M \circ J$, defined as

$$[B\,x]\,(s) \;=\; m(s) \int\limits_0^s x(t)\,dt \qquad (0 \le s \le 1). \tag{2}$$

Because of the compactness of B we always have $R(B) \neq \overline{R(B)}$ and every operator equation (1) with $A = B$ is *ill-posed*. By definition the operator B is a composition of the *injective simple linear integration operator* J defined as

$$[J\,x]\,(s) \;=\; \int\limits_0^s x(t)\,dt \qquad (0 \le s \le 1) \tag{3}$$

and the *multiplication operator* M defined as

$$[M\,x]\,(t) \;=\; m(t)\,x(t) \qquad (0 \le t \le 1). \tag{4}$$

We only focus on *multiplier functions* m satisfying

$$m \in L^1(0,1)\,, \qquad |m(t)| > 0 \;\text{ a.e. on } [0,1]\,, \tag{5}$$

such that $B = M \circ J$ is a *compact* operator in $L^2(0,1)$. Note that (5) implies die injectivity of the operators M and B.

It is well-known that J is a compact linear operator in $L^2(0,1)$. Moreover, M is a bounded linear operator and hence B a compact linear operator in $L^2(0,1)$ if $m \in L^\infty(0,1)$. For the multiplier functions m we preferably focus on two families both satisfying the condition (5). First we deal with the family of *power type functions*

$$m(t) \;=\; t^r \qquad (0 < t \le 1, \quad r > -1) \tag{6}$$

which have a zero at $t=0$ for $r>0$ and belong to $L^\infty(0,1)$ for $r\ge 0$. Consequently, for $r\ge 0$ the composite operator B is compact. On the other hand, for $-1<r<0$ we have $m\in L^1(0,1)$ and a weak pole at $t=0$, but nevertheless B is compact in $L^2(0,1)$ ([60]). As a second family we consider the *exponential type functions*

$$m(t) \;=\; \exp\left(-\frac{1}{t^\rho}\right) \qquad (0<t\le 1) \tag{7}$$

with exponent $\rho>0$ which can be extended continuously to $[0,1]$ by setting $m(0)=0$. All such functions m belong to $L^\infty(0,1)$ and therefore B is also a compact operator in $L^2(0,1)$.

One the other hand we consider *nonlinear ill-posed problems* (see [8, Chapter 10] and [54]) written as an operator equation

$$F(x) \;=\; y \qquad (x\in D(F)\subseteq X,\;\; y\in Y), \tag{8}$$

where the *nonlinear forward operator* $F: D(F)\subseteq X\to Y$ with *closed, convex domain*[1] $D(F)$ is assumed to be *continuous*. If F is smoothing enough, in particular if F is *compact* and *weakly (sequentially) closed*, then *local ill-posedness* of equations (8) at the solution point $x_0\in D(F)$ in the sense of [25, definition 2] (see also [48, definition 7.1.1]) arises, i.e., the solutions x do not stably depend on the data y in a neighborhood of x_0.

We especially consider, for $X=Y=L^2(0,1)$, the class of nonlinear equations (8) with composite nonlinear operators

$$F = N\circ J : D(F)\subset L^2(0,1)\to L^2(0,1)$$

defined as

$$[F(x)](s) \;=\; k(s,[J\,x](s)) \qquad (0\le s\le 1\,;\;\; x\in D(F)) \tag{9}$$

with J from (3) and half-space domains

$$D(F) \;=\; \left\{x\in L^2(0,1):\, x(t)\ge c_0\ge 0 \;\text{ a.e. on } [0,1]\right\}. \tag{10}$$

Here, N defined as

$$[N(z)](t) \;=\; k(t,z(t)) \qquad (0\le t\le 1\,;\;\; z\in D(N)) \tag{11}$$

is a *nonlinear Nemytskii operator* (see, e.g., [3]) generated by a function $k(t,v)$ $((t,v)\in[0,1]\times[0,\infty))$. If the generator function k is sufficiently smooth[2], then

$$N: D(N)\subset L^2(0,1)\to L^2(0,1)$$

defined by formula (11) with $D(N)=\{z\in L^2(0,1):\, z(t)\ge 0 \text{ a.e. on } [0,1]\}$ maps continuously. Moreover, as a consequence of the compactness of J and the continuity of

[1]If $D(F)$ is a closed and convex subset of the Hilbert space X, then $D(F)$ is also weakly closed.

[2]If $|k(t,v)|\le c_1+c_2\,|v|$ for constants $c_1\ge 0$ and $c_2\ge 0$ and $k(t,v)$ is continuous on $(t,v)\in[0,1]\times[0,\infty)$, then the growth condition and the Carathéodory condition are satisfied and the Nemytskii operator N maps continuously in $L^2(0,1)$. If moreover k is continuously differentiable with respect to the second variable v and we have $|k_v(s,v)|\le c_3$ for a constant $c_3\ge 0$, then N is even Gâteaux differentiable with Gâteaux derivative $[N'(z)\,h](t)=k_v(t,z(t))h(t)$ $(0\le t\le 1;\; z\in D(N))$ (see, e.g., [1, pp.15]).

N the composition $F = N \circ J$ is *compact, continuous, weakly continuous* and hence a *weakly (sequentially) closed*[3] nonlinear operator[4]. In view of the compactness and weak (sequential) closedness of F the equation (8) is locally ill-posed on the whole domain (10) (cf. the arguments in the context of [19, corollary 5.2]). Under stronger assumptions on the smoothness[5] of k the operator F from (9) is even Fréchet differentiable with a *Fréchet derivative*[6]

$$F'(x_0) \,=\, B \,=\, M \circ J \tag{12}$$

of the form (2) at the point x_0 satisfying

$$\|F(x) - F(x_0) - F'(x_0)\,(x - x_0)\| \,\le\, \frac{L}{2}\,\|x - x_0\|^2 \qquad \text{for all} \quad x \in D(F)\,, \tag{13}$$

where the corresponding multiplier function m depends on the point $x_0 \in D(F)$ and attains the form

$$m(t) \,=\, k_v(t, [J\,x_0](t)) \qquad (0 \le t \le 1) \tag{14}$$

exploiting the partial derivative k_v of the generator function $k(t, v)$ with respect to the second variable v. In [19, p.1331] it was shown that (12) with multiplier function (14) is a (formal) Gâteaux derivative of the operator (9) at the point $x_0 \in D(F)$. Then $F'(x_0) \in \mathcal{L}(L^2(0,1))$ is even a Fréchet derivative whenever an inequality (13) holds true for some $L > 0$.

2 Inverse problems with multiplication operators

In this paragraph, we recall three examples of nonlinear inverse problems (8) with nonlinear operators of the form (9) and Fréchet derivatives (12) for $X = Y = L^2(0,1)$ and $\|\cdot\| = \|\cdot\|_{L^2(0,1)}$ arising in natural sciences and stochastic finance. Some analysis of the first two examples was already presented in [22] (see also [23, pp.57 and pp.123]). For more details on the third example we refer to [19].

First we consider an example mentioned in the book [16] that aims at determining the growth rate $x(t)$ $(0 \le t \le 1)$ in an O.D.E. model

$$y'(t) \,= x(t)\,y(t), \quad y(0) = c_I \ge c_0 > 0 \tag{15}$$

[3]Provided that N maps continuously with a closed domain $D(N)$ and $R(J) \subseteq D(N)$, then the weak continuity and hence the weak closedness of the nonlinear operator F defined by formula (9) are consequences of the fact that, for sequences $x_n \rightharpoonup x_0$ from $D(F)$ which are weakly convergent in $L^2(0,1)$, we have strong convergence for compact J of the sequences $J\,x_n \to J\,x_0$ and thus $F(x_n) \to F(x_0)$ in $L^2(0,1)$. The same argument provides the compactness of the nonlinear operator F if N is continuous.

[4]If F is continuous, compact and weakly (sequentially) closed on a convex closed domain $D(F)$, then the assertions on ill-posednes and on Tikhonov regularization formulated in [10] (see also §8 below) hold true for the nonlinear operator equation (8)

[5]$N'(z)$ is in general only a Gâteaux and not a Fréchet derivative (see [1, proposition 2.8]), but the smoothing properties of J ensure that $[F'(x_0)\,h](t) = k_v(t, [J\,x_0](t))\,[J\,h](t)$ $(0 \le t \le 1;\ x \in D(F))$ defines a Fréchet derivative of F at the point x_0 satisfying an inequality (13) with a constant $L > 0$ whenever the second partial derivative of $k(t, v)$ with respect to v exists and is bounded as $|k_{vv}(t, v)| \;\le\; L$ for all $v \ge c_0\,t$ $\ (0 \le t \le 1)$, where $c_0 \ge 0$ is the constant arising in the domain (10) (for more details see [19, p.1332, proof of theorem 5.4]). In such a case, L in (13) is a global constant for all $x_0 \in D(F)$.

[6]In the sense of remark 10.30 of [8] a Fréchet derivative can also be considered if the convex domain $D(F)$ has an empty interior which is the case for the domain (10) in $L^2(0,1)$.

from the data $y(t)$ $(0 \le t \le 1)$, where we have to solve equation (8) with the nonlinear operator

$$[F(x)]\,(s) \;=\; c_I \,\exp\left(\int\limits_0^s x(t)dt\right) \qquad\qquad (0 \le s \le 1) \tag{16}$$

and a positive constant c_0 in the domain (10). The functions $y(s)$ $(0 \le s \le 1)$ can represent, for example, a concentration of a substance in a chemical reaction or the size of a population in a biological system with initial value c_I.

A second example already mentioned in [2, p.190] aims at identifying a heat conduction parameter function $x(t)$ $(0 \le t \le 1)$ from $D(F)$ with positive constant c_0 in a locally one-dimensional P.D.E. model

$$\frac{\partial u(\kappa,t)}{\partial t} = x(t)\,\frac{\partial^2 u(\kappa,t)}{\partial \kappa^2} \qquad (0 < \kappa < 1,\;\; 0 < t \le 1) \tag{17}$$

with the initial condition

$$u(\kappa,0) \;=\; \sin \pi\kappa \qquad\qquad (0 \le \kappa \le 1)$$

and homogeneous boundary conditions

$$u(0,t) \;=\; u(1,t) \;= 0 \qquad\qquad (0 \le t \le 1)$$

from time dependent temperature observations

$$y(t) \;=\; u\left(\frac{1}{2}\,,\,t\right) \qquad\qquad (0 \le t \le 1)$$

at the midpoint $\kappa = 1/2$ of the interval $[0,1]$. Here, we have to solve (8) with an associated nonlinear operator

$$[F(x)]\,(s) \;=\; \exp\left(-\pi^2 \int\limits_0^s x(t)dt\right) \qquad\qquad (0 \le s \le 1). \tag{18}$$

In both examples the operators F with domain (10) are of the form

$$[F(x)]\,(s) \;= c_A \exp\left(c_B\,[J\,x](s)\right) \qquad\qquad (0 \le s \le 1,\;\; c_A \ne 0\,,\;\, c_B \ne 0) \tag{19}$$

and belong to the class (9) of composite operators $F = N \circ J$ with simple integration operator J from (3) and an *injective* Nemytskii operator N with generator function

$$k(s,v) \;= c_A\,\exp\left(c_B\,v\right) \qquad\qquad (0 \le s \le 1,\;\, v \ge 0). \tag{20}$$

In this context, we consider the linear operator $B = M \circ J$ of the form (12) for $x_0 \in D(F)$ with a *continuous* multiplier function

$$m(t) \;=\; k_v(s,[J\,x_0](t)) \;=\; c_A\,c_B\,\exp\left(c_B\,[J\,x_0](t)\right) \;=\; c_B\,[F(x_0)](t) \qquad (0 \le t \le 1) \tag{21}$$

determining the associated multiplication operator M. We easily derive lower and upper bounds c and C such that $0 < c \le C < \infty$ and

$$c = |c_A|\,|c_B| \exp\left(-|c_B|\,\|x_0\|\right) \le |m(t)| = |c_B|\,|[F(x_0)](t)| \le |c_A|\,|c_B| \exp\left(|c_B|\,\|x_0\|\right) = C \tag{22}$$

showing that the continuous multiplier function (21) has *no zeros*. Consequently, $m \in L^\infty(0,1)$ and M is an *injective* and *bounded* linear operator in $L^2(0,1)$. Hence the composite linear operator $B = M \circ J$ is *compact* in $L^2(0,1)$. Then we have:

Theorem 2.1 *Every nonlinear operator $F : D(F) \subset L^2(0,1) \to L^2(0,1)$ of the class (19) with domain (10) is injective, (locally) Lipschitz continuous,[7] compact, weakly continuous and hence weakly (sequentially) closed and possesses for all $x_0 \in D(F)$ a compact Fréchet derivative $F'(x_0) = M \circ J \in \mathcal{L}(L^2(0,1))$ satisfying for all $r > 0$ a local version*

$$\|F(x) - F(x_0) - F'(x_0)\,(x - x_0)\| \;\le\; \frac{L}{2}\,\|x - x_0\|^2 \quad \text{for all} \quad x \in D(F) \quad \text{with} \quad \|x - x_0\| \le r \tag{23}$$

of inequality (13) for some constant $L > 0$ depending on r and x_0, where the multiplication operator M is determined by the multiplier function (21). As a consequence the inverse operator $F^{-1} : R(F) \subset L^2(0,1) \to D(F) \subset L^2(0,1)$ exists, but cannot be continuous and the corresponding operator equation (8) is locally ill-posed everywhere.

Proof: To prove this theorem we exploit the auxiliary function

$$\Psi(s) = c_B\,[J\,(x - x_0)](s) \;\; (0 \le s \le 1) \quad \text{with} \quad |\Psi(s)| \le |c_B|\,\|x - x_0\| \;\; (0 \le s \le 1).$$

Then we can write

$$[F(x) - F(x_0)](s) \;=\; [F(x_0)](s)\,(\exp(\Psi(s)) - 1) \qquad (0 \le s \le 1)$$

with $|\exp(\Psi(s)) - 1| \le |\Psi(s)|\,\exp(r|c_B|)$. This yields

$$\|F(x) - F(x_0)\| \le C\,\exp(r|c_B|)\,\|x - x_0\| \quad \text{for all} \quad x \in D(F) \quad \text{with} \quad \|x - x_0\| \le r$$

proving the (local) Lipschitz continuity of F. On the other hand, we can write

$$[F(x) - F(x_0) - F'(x_0)(x - x_0)](s) \;=\; [F(x_0)](s)\,(\exp(\Psi(s)) - 1 - \Psi(s)) \quad (0 \le s \le 1)$$

for $F'(x_0) = M \circ J$ with multiplier function (21). From

$$|\exp(\Psi(s)) - 1 - \Psi(s)| \le |\Psi(s)|\,|\exp(\Psi(s)) - 1| \qquad \text{for all} \quad s \in [0,1]$$

we obtain, for $x \in D(F)$ with $\|x - x_0\| \le r$, the estimates

$$|[F(x) - F(x_0) - F'(x_0)(x - x_0)](s)| \;\le\; |c_B|\,|[F(x) - F(x_0)](s)|\,\|x - x_0\| \quad (0 \le s \le 1)$$

[7] Here the generator function (20) fails to satisfy a growth condition $|k(s,v)| \le c_1 + c_2|v|$ which would provide the continuity of N. Moreover, neither the absolute value of the first partial derivative $k_v(s,v) = c_A\,c_B \exp(c_B\,v)$ nor the absolute value of the second partial derivative $k_{vv}(s,v) = c_A\,c_B^2 \exp(c_B\,v)$ are bounded from above. So a global inequality (13) cannot be shown, but nevertheless because of the smoothing properties of J the composite operator $F = N \circ J$ is continuous and at least a local version (23) of (13) holds true.

and

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\| \le |c_B| \, \|F(x) - F(x_0)\| \, \|x - x_0\|. \tag{24}$$

Note that (24) is an η-*inequality* (see [8, p.279, formula (11.6)], [17] and [25]) that provides (23) with

$$L = 2\,C\,|c_B| \, \exp(r\,|c_B|).$$

Using the compactness of the imbedding operator from $H^1(0,1)$ to $L^2(0,1)$ the nonlinear operator F is compact, since $F(x) \in H^1(0,1)$ for all $x \in D(F)$ and by formula (22) and an analogous formula for the derivative $[F(x_0)]'(t)$ it can be shown that $\|F(x)\|_{H^1(0,1)}$ is uniformly bounded for $\|x\| \le K$ and any constant $K > 0$. Now it remains to show the weak continuity of F implying the weak (sequential) closedness of the operator. We assume, for a bounded sequence $x_n \in D(F)$, weak convergence $x \rightharpoonup x_0$ in $L^2(0,1)$ exploiting the fact that this is equivalent to $\int_0^s (x_n - x_0)(t)dt \to 0$ and hence to $\Psi_n(s) = c_B \int_0^s (x_n - x_0)(s) \to 0$ for all $s \in [0,1]$ and show weak convergence $F(x_n) \rightharpoonup F(x_0)$ in $L^2(0,1)$ by exploiting the same fact for the sequence $F(x_n)$ which is bounded since F is compact. Namely, we can estimate above for all $\overline{s} \in [0,1]$

$$\left| \int_0^{\overline{s}} [F(x_n) - F(x_0)](s)ds \right| \le \int_0^{\overline{s}} |[F(x_0)](s)| \, |\Psi_n(s)| ds \le C\,\|x - x_0\|.$$

This estimate provides with Lebesgue's theorem on dominated convergence the required weak convergence $F(x_n) \rightharpoonup F(x_0)$ in $L^2(0,1)$ taking into account that a weak convergent sequence is always bounded. Note that the weak limit x_0 assumed here belongs to $D(F)$, since the domain (10) is weakly closed ■

As the third example, we present an inverse problem of option pricing. In particular, at time $t = 0$ we have a risk-free interest rate $r \ge 0$ and we consider a family of European standard call options for varying maturities $t \in [0,1]$ and a fixed strike price $S > 0$ written on an asset with asset price $X > 0$, where $y(t)$ $(0 \le t \le 1)$ is the function of option prices observed at an arbitrage-free financial market. From that function we are going to determine the unknown volatility term-structure. We denote the squares of the volatility at time t by $x(t)$ $(0 \le t \le 1)$ and neglect a possible dependence of the volatilities from asset price. Using a generalized *Black-Scholes formula* (see, e.g. [34, pp.71]) we obtain

$$[F(x)](t) = U_{BS}(X, S, r, t, [J\,x](t)) \qquad (0 \le t \le 1) \tag{25}$$

as the fair price function for the family of options, where the nonlinear operator F with domain (10) and $c_0 > 0$ maps in $L^2(0,1)$ and U_{BS} is the Black-Scholes function defined as

$$U_{BS}(X, S, r, \tau, s) := \begin{cases} X\Phi(d_1) - Se^{-r\tau}\Phi(d_2) & (s > 0) \\ \max\left(X - Se^{-r\tau}, 0\right) & (s = 0) \end{cases}$$

with

$$d_1 = \frac{\ln\frac{X}{S} + r\tau + \frac{s}{2}}{\sqrt{s}}, \quad d_2 = d_1 - \sqrt{s}$$

and the cumulative density function of the standard normal distribution

$$\Phi(\zeta) \; = \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} e^{-\frac{\xi^2}{2}} d\xi.$$

Obviously, we have $F = N \circ J$ as in (9) with Nemytskii operator

$$[N(z)](t) \;=\; k(t, z(t)) \;=\; U_{BS}(X, S, r, t, z(t)) \qquad (0 \le t \le 1).$$

If we exclude *at-the money options*, i.e. for

$$S \neq X\,, \tag{26}$$

we have a *compact* Fréchet derivative $F'(x_0) \;=\; M \circ J$ with *continuous, nonnegative* multiplier function

$$m(0) = 0, \quad m(t) = \frac{\partial U_{BS}(X, S, r, t, [J\,x_0](t))}{\partial s} \quad (0 < t \le T)\,, \tag{27}$$

for which we can show the formula

$$m(t) = \frac{X}{2\sqrt{2\pi [J\,x_0](t)}} \exp\left(-\frac{(v + rt)^2}{2[J\,x_0](t)} - \frac{(v + rt)}{2} - \frac{[J\,x_0](t)}{8}\right) > 0 \qquad (0 < t \le 1),$$

where $v = \ln \frac{X}{S} \neq 0$. Note that in view of $c_0 > 0$ we have $\underline{c}\,t \le [J\,x_0](t) \le \overline{c}\,t$ $(0 \le t \le 1)$ with $\underline{c} = c_0 > 0$ and $\overline{c} = \|x_0\|$. Then we may estimate

$$\underline{C}\,\frac{\exp\left(-\frac{v^2}{2\,\underline{c}\,t}\right)}{\sqrt[4]{t}} \;\le\; m(t) \;\le\; \overline{C}\,\frac{\exp\left(-\frac{v^2}{2\,\overline{c}\,\sqrt{t}}\right)}{\sqrt{t}}, \qquad (0 < t \le 1) \tag{28}$$

for some positive constants $\underline{C}$ and $\overline{C}$. The function $m \in L^\infty(0,1)$ of this example has a uniquely determined essential zero at $t = 0$. In the neighborhood of this zero the multiplier function decreases to zero exponentially, i.e., faster than any power of t.

In [19] we find the proof of the following theorem:

Theorem 2.2 *Under the assumption (26) the nonlinear operator $F : D(F) \subset L^2(0,1) \to L^2(0,1)$ from (25) with domain (10) and $c_0 > 0$ is injective, continuous, compact, weakly continuous and hence weakly (sequentially) closed and possesses for all $x_0 \in D(F)$ a compact Fréchet derivative $F'(x_0) = M \circ J \in \mathcal{L}(L^2(0,1))$ satisfying (13) for some $L > 0$ independent of x_0, where the multiplication operator M is determined by the multiplier function (27). As a consequence the inverse operator $F^{-1} : R(F) \subset L^2(0,1) \to D(F) \subset L^2(0,1)$ exists, but cannot be continuous and the corresponding operator equation (8) is locally ill-posed everywhere.*

In all three examples the local ill-posedness of the occurring nonlinear operator equations and the ill-posedness of the linearized equations require the use of a regularization method for the stable approximate solution. In this paper we consider the method of Tikhonov regularization for linear and nonlinear ill-posed operator equations more detailed.

3 On measures of ill-posedness

In the past twenty years in the literature of linear inverse and ill-posed problems (1) many authors considered measures for the ill-posedness and its consequences for condition numbers of discretized problems and appropriate regularization approaches (see, e.g., [5], [6], [8], [29], [32], [38], [40], [41], [44], [46], [51], [52], [57], [58] and [59]). Numerous papers used in this context Hilbert scale techniques. If only the smoothing properties of an injective compact linear forward operator A mapping between the infinite dimensional Hilbert spaces X and Y are considered, then the decay rate of the positive, non-increasing sequence $\{\sigma_n(A)\}_{n=1}^{\infty}$ of singular values of A tending to zero as $n \to \infty$ is a frequently used measure of ill-posedness (see, e.g., [8, p.40], [20, p.31] and [33, p.235]). It defines a finite degree $\mu \in (0, \infty)$ of ill-posedness if

$$\sigma_n(A) \asymp n^{-\mu} \tag{29}$$

is valid. [8] For small μ, e.g. $\mu \leq 1$, the equation (1) is called *mildly ill-posed* and for finite μ *moderately ill-posed*. If, however, the singular values σ_n fall exponentially, i.e., faster than any power of n, then (1) is called *severely ill-posed*.

Since a condition (29) is only valid for a specific class of compact operators A, we defined the more general interval of ill-posedness

$$[\underline{\mu}(A), \overline{\mu}(A)] = \left[\liminf_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n} \, , \, \limsup_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n} \right]$$

in [26], where $\underline{\mu}$ and $\overline{\mu}$ also can also be zero and infinity. As proposed and motivated in [14], [21] and [22] such measures are also helpful for evaluating the local behaviour of ill-posedness for nonlinear operator equations (8) at a point $x_0 \in D(F)$ by considering a linearized version of (8) as an equation (1) with the Fréchet derivative $A = F'(x_0)$ which is compact whenever F is compact ([7, Theorem 4.19]). For non-compact linear operators A, in particular multiplication operators, some ideas concerning measures of ill-posedness for (1) were presented in [24]. Some interdependencies between an ill-posed nonlinear equation (8) and its *linearization* with respect to the *local degree of ill-posedness* characterized by $F'(x_0)$ and the *degree of nonlinearity* of F at x_0 including consequences for regularization were formulated in [25].

Now we turn towards the composite integral operators B from (2). From the explicitly given singular values

$$\sigma_n(J) \;=\; \frac{1}{\pi\,\left(n - \frac{1}{2}\right)} \;\sim\; \frac{1}{\pi\, n} \qquad (n = 1, 2, ...)$$

of the compact integration operator $J \in \mathcal{L}(L^2(0,1))$ and for multiplier functions

$$0 < c \leq |m(t)| \leq C \qquad \text{a.e. on } [0,1] \tag{30}$$

we derive the inequalities $c\,\sigma_n(J) \leq \sigma_n(B) \leq C\,\sigma_n(J)$ and the asymptotics

$$\sigma_n(B) \asymp n^{-1} \tag{31}$$

[8] Here the notation $a_n \asymp b_n$ for sequences of positive numbers a_n and b_n denotes the existence of positive constants c_1 and c_2 such that $c_1 \leq a_n/b_n \leq c_2$ for all sufficiently large n. If moreover $\lim_{n\to\infty} a_n/b_n = 1$ we write $a_n \sim b_n$. If only an estimate $a_n \leq c_3\, b_n$ is under consideration for a positive constant c_3, then as obvious we write $a_n = \mathcal{O}(b_n)$.

based on the Poincaré-Fischer extremum principle

$$\sigma_n(B) \; = \; \max_{X_n \subset X} \; \min_{x \in X_n,\, x \neq 0} \; \frac{\|A\,x\|}{\|x\|}, \tag{32}$$

where X_n represents an arbitrary n-dimensional subspace of the Hilbert space X (cf., e.g., [5, Lemma 4.18] or [20, Lemma 2.44]). That means, the degree of ill-posedness is $\mu = 1$ for all such multiplier functions. The first two examples (see formulae (21) and (22)) presented in the preceding paragraph were concerned with nonlinear operators (9) leading to the situation (30). Consequently, nonlinear equations (8) with F from the class (19) have uniformly a local degree one of ill-posedness at any point $x_0 \in D(F)$.

Note that (30) implies a continuous (non-compact) multiplication operator $M \in \mathcal{L}(L^2(0,1))$ of the form (4) and hence the compactness of the linear operator $B = M \circ J$. Moreover, from (32) it follows that the operator B is also compact whenever we have

$$|m(t)| \le C\,t^r \quad \text{a.e. on } [0,1] \tag{33}$$

for a constant $C > 0$ and some exponent $r > -1$. However, we have a non-closed range of M, i.e. $R(M) \neq \overline{R(M)}$, whenever m has essential zeros and (30) cannot hold. Then a *new factor of ill-posedness* occurs in the operator B in addition to the ill-posedness coming from the compactness of J. It could be of some interest to evaluate the strength of this new factor. In particular, it seems to be an interesting question whether the *non-compact* operator M with *non-closed range* can *destroy* the degree of ill-posedness $\mu = 1$ defined by the integration operator J. By considering source conditions in regularization (see §4 and §7 below) there are apparently strong arguments that the ill-posedness effect of M can be significant. However, in the next paragraph we formulate a stringent result that multiplier functions (6) of power type do not change the degree of ill-posedness and we have a degree of ill-posedness $\mu(B) = \mu(J) = 1$ for all exponents $r > -1$. We also provide some arguments to conjecture that this assertion even remains true for the family (7) of multiplier functions with exponential decay.

4 Specific assertions for composite linear operators

A standard method for the stable approximate solution of ill-posed equations (1) is the *Tikhonov regularization* (see, e.g., [53], [4], [15] and [8, Chapter 5]), where *regularized solutions* x_α depending on a *regularization parameter* $\alpha > 0$ are obtained by solving the extremal problem

$$\|A\,x - y\|^2 + \alpha\,\|x\|^2 \;\longrightarrow\; \min\,, \qquad \text{subject to} \quad x \in X. \tag{34}$$

As discussed more detailed in §5 below *convergence rates* $\|x_\alpha - x_0\| = \mathcal{O}(\sqrt{\alpha})$ as $\alpha \to 0$ of regularized solutions x_α to the exact solution x_0 with $y = A\,x_0$ can be ensured provided that x_0 satisfies a *source condition*

$$x_0 \; = \; A^*\,v \qquad (v \in Y). \tag{35}$$

In general, a growing degree of ill-posedness of (1) corresponds to a growing strength of the condition (35) that has to be imposed on the solution element x_0 (see, e.g., [14] and [21]).

Now we compare the strength of condition (35) for the case $A = J$ with the simple integration operator J defined by formula (3) that can be written as

$$x_0(t) \;=\; [J^*\,v](t) \;=\; \int_t^1 v(s)\,ds \qquad (0 \le t \le 1;\; v \in L^2(0,1)\,,\; \|v\| \le R_0) \tag{36}$$

and for the case $A = B = M \circ J$ with the composite integral operator B with weights m defined by formula (2). In this case we can write (35) as

$$x_0(t) \;=\; [J^* M^*\,v](s) \;=\; [J^* M\,v](t) \;=\; \int_t^1 m(s)\,v(s)\,ds \quad (0 \le t \le 1;\; v \in L^2(0,1)). \tag{37}$$

If we assume that the multiplier function m has an essential zero, say only at $t = 0$, then the condition (36) that can be reformulated as

$$x_0' \in L^2(0,1), \quad \text{i.e.} \quad x_0 \in H^1(0,1) \quad \text{with} \quad x_0(1) = 0 \tag{38}$$

is obviously weaker than the condition (37) which is equivalent to

$$\frac{x_0'}{m} \in L^2(0,1) \quad \text{with} \quad x_0(1) = 0, \tag{39}$$

since we have $\frac{1}{m} \notin L^\infty(0,1)$ for the new factor occurring in (39). Consequently in order to satisfy the source condition (39), the (generalized) derivative of the function x_0 has to compensate in some sense the pole of $\frac{1}{m}$ at $t = 0$. The requirement of compensation grows when the decay rate of $m(t) \to 0$ as $t \to 0$ grows. Hence the strength of the requirement (39) imposed on x_0 grows for the families (6) and (7) of multiplier functions m when the exponents r and ρ increase. Moreover, for the exponential type functions (7) the condition (39) is stronger than for the power type functions (6). Note that exponential functions can really arise as multiplier functions in applications as the inverse option pricing problem (see third example in §2) shows. In that example, we have an estimate from below and above (see formula (28)) of the form

$$K_1\,\exp\left(-\frac{c_1}{t}\right) \;\le\; m(t) \;\le K_2\,\exp\left(-\frac{c_2}{\sqrt{t}}\right) \qquad (0 < t \le 1\,;\; c_1, c_2, K_1, K_2 > 0).$$

We will conjecture below that the asymptotics (31) remains true for all multiplier functions m satisfying (5) and (33) with $r > -1$ and that consequently the non-compact operator M does not destroy the degree of ill-posedness determined by the compact operator J.

In the recent paper [27] we could prove that linear ill-posed problems (1) with composite operators (2) have a constant degree of ill-posedness $\mu = 1$ if the multiplier function m is a power function (6). For completeness we will formulate the results and a conjecture which had been presented in [27]. Moreover, we will repeat the proof of the main theorem 4.1 in the appendix of this paper.

Theorem 4.1 *For the non-increasing sequence $\{\sigma_n(B)\}_{n=1}^\infty$ of singular values of the compact linear operator B in $L^2(0,1)$ defined by*

$$[B\,x]\,(s) \;=\; s^{\,r} \int_0^s x(t)\,dt \qquad (0 < s \le 1) \tag{40}$$

with exponent $r > -1$, we have the asymptotics

$$\sigma_n(B) \sim \frac{1}{(r+1)\,\pi\,n} = \left(\int_0^1 m(t)dt\right) \frac{1}{\pi\,n} \quad as \quad n \to \infty. \tag{41}$$

Corollary 4.2 *For the singular values of a compact linear operator $B = M \circ J$ defined by the formulae (2) ,(3) and (4) with a multiplier function m satisfying the inequalities*

$$c\,t^{r_2} \,\le\, |m(t)| \,\le\, C\,t^{r_1} \quad a.e.\ on\ [0,1] \tag{42}$$

for some constants $-1 < r_1 \le r_2$ and $c,\, C > 0$, we have

$$\sigma_n(B) \asymp n^{-1}. \tag{43}$$

Proof: By considering theorem 4.1 and the Poincaré-Fischer extremum principle (32) the asymptotics (43) is an immediate consequence of spectral equivalence in the sense of the inequalities

$$c\sqrt{\int_0^1 s^{2\alpha_2}[(J\,x)(s)]^2 ds} \le \|B\,x\| \le C\sqrt{\int_0^1 s^{2\alpha_1}[(J\,x)(s)]^2 ds} \qquad \text{for all} \quad x \in L^2(0,1)$$

that follow from (42) ∎

Conjecture 4.3 *We conjecture that for all the compact linear operators B from (2) the asymptotic behaviour*

$$\sigma_n(B) \sim \left(\int_0^1 m(t)dt\right) \sigma_n(J) \quad as \quad n \to \infty \tag{44}$$

remains true whenever the multiplier function m satisfies for some $r > -1$ the inequalities

$$0 < m(t) \le C\,t^r \quad a.e.\ on\ [0,1]. \tag{45}$$

This conjecture is based on three comprehensive numerical case studies in the diploma thesis [13] of Melina Freitag using the numerical solution of corresponding Sturm-Liouville problems, moreover a Galerkin approximation of B along the lines of [59] and a Rayleigh-Ritz ansatz for B^*B solving general eigenvalue problems. She compared the two families (6) and (7) and pointed out that the decay of singular values $\sigma_n(B)$ was uniformly proportional to $1/n$ in all three studies, where as in formula (44) the integral $\int_0^1 m(t)dt$ occurred as essential factor in all studies. A stringent proof of formula (44), however, seems to be missing up to now for the family (7).

At the end of this paragraph we should mention that for the class of composite linear compact operators $A_\gamma = M \circ J_\gamma$ mapping in $L^2(0,1)$ and defined by

$$[A_\gamma\,x]\,(s) \,=\, s^{-\beta} \int_0^s \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)}\,x(t)\,dt \qquad (0 < s \le 1\,;\ \gamma > \beta > 0), \tag{46}$$

where J_γ are fractional integral operators of order $\gamma > 0$ with degree of ill-posedness $\mu(J_\gamma) = \gamma$ and the multiplier functions m are power functions with a weak pole, considerations on non-changing degree of ill-posedness $\mu(A_\gamma) = \gamma$ were performed by Vu Kim Tuan and R. Gorenflo in 1994 (see [56]). Using Gegenbauer polynomials they proved the asymptotics

$$\sigma_n(A_\gamma) \asymp n^{-\gamma} \tag{47}$$

for $0 \le \beta < \frac{\gamma}{2}$ and conjectured that (47) also remains true for $\frac{\gamma}{2} \le \beta < \gamma$. To our knowledge results on the degree of ill-posedness of the class (46) seem not to be published in the inverse problems literature for multiplier functions m with zeros ($\beta < 0$). In theorem 4.1, corollary 4.2 and conjecture 4.3 formulated in this paragraph we have focused on the particular case $\gamma = 1$ of operator (46).

In the following four paragraphs we are going to interpret error estimates for the *Tikhonov regularization method* applied to the inverse problems (1) and (8), where $y \in Y$ denotes the exact right-hand side $y = Ax_0$ ($x_0 \in X$) and $y = F(x_0)$ ($x_0 \in D(F) \subset X$), respectively, for a fixed solution element x_0 under consideration. Furthermore, $y^\delta \in Y$ is assumed to be an approximation (noisy data) of y with $\|y^\delta - y\| \le \delta$ and noise level $\delta > 0$.

5 Measuring the violation of source conditions yields convergence rates for Tikhonov regularization

In this and the two subsequent paragraphs we focus on the linear equation (1) and distinguish *Tikhonov regularized solutions* for regularization parameters $\alpha > 0$

$$x_\alpha = (A^*A + \alpha I)^{-1} A^* y \tag{48}$$

in the noiseless case and

$$x_\alpha^\delta = (A^*A + \alpha I)^{-1} A^* y^\delta \tag{49}$$

in the case of noisy data that represent the uniquely determined minimizers of the extremal problem (34) for right-hand sides $y = Ax_0$ and y^δ with $\|y^\delta - y\| \le \delta$, respectively.

In the sequel we call the noiseless error function

$$f(\alpha) := \|x_\alpha - x_0\| = \|\alpha (A^*A + \alpha I)^{-1} x_0\| \qquad (\alpha > 0) \tag{50}$$

profile function for fixed A and x_0. In combination with the noise level δ this function determines the total regularization error of Tikhonov regularization

$$e(\alpha) := \|x_\alpha^\delta - x_0\| \le \|x_\alpha - x_0\| + \|x_\alpha^\delta - x_\alpha\| = f(\alpha) + \|(A^*A + \alpha I)^{-1} A^*(y^\delta - y)\| \tag{51}$$

with the estimate

$$e(\alpha) \le f(\alpha) + \frac{\delta}{2\sqrt{\alpha}}. \tag{52}$$

The inequality (52) is a consequence of the spectral inequality $\| \left(A^*A+\alpha I\right)^{-1} A^*\| \le \frac{1}{2\sqrt{\alpha}}$ following from $\frac{\sqrt{\lambda}}{\lambda+\alpha} \le \frac{1}{2\sqrt{\alpha}}$ for all $\lambda \ge 0$ and $\alpha > 0$ in the sense of [8, p.45, formula (2.48)].

Note that $\lim\limits_{\alpha\to 0} f(\alpha)=0$ for all $x_0 \in X$ as proven by using spectral theory for general linear regularization schemes in [8, p.72, theorem 4.1] (see also [55, p.45, theorem 5.2]), but the decay rate of $f(\alpha)\to 0$ as $\alpha \to 0$ depends on x_0 and can be arbitrarily slow (see [50] and [8, Prop. 3.11]). However, the analysis of profile functions $f(\alpha)$ expressing the *relative smoothness of x_0 with respect to the operator A* yields *convergence rates* of regularized solutions. In this paragraph we first recall the standard approach for analyzing f exploiting *source conditions* imposed on x_0 and secondly we present an alternative theoretical approach using functions d measuring how far the element x_0 is away from the source condition

$$x_0 \;=\; A^*\,v \qquad\qquad (v \in Y,\ \|v\| \le R_0) \tag{53}$$

is the element x_0. Note that (53) would imply

$$f(\alpha)=\|\alpha\left(A^*A+\alpha I\right)^{-1} A^* v\| \le \frac{\sqrt{\alpha}}{2}\,\|v\| \le \frac{\sqrt{\alpha}}{2}\,R_0$$

and with (52) for the a priori parameter choice $\alpha(\delta)\sim\delta$ the convergence rate

$$\|x^\delta_{\alpha(\delta)}-x_0\| \;=\; \mathcal{O}\left(\sqrt{\delta}\right) \qquad \text{as} \qquad \delta\to 0. \tag{54}$$

The source condition (53) and resulting convergence rate (54) are considered as *canonical* in the sequel, since the compliance of a condition (53) seems to be a caesura in the variety of possible relative smoothness properties of x_0 with respect to A and modifications (see formula (89) in §8 below) play also an important role in the regularization theory of nonlinear ill-posed problems (see [10], [8, Chapter 10]), where the convergence rate (54) can be obtained if A^* is replaced by the adjoint $F'(x_0)^*$ of the Fréchet derivative of the forward operator F at the point x_0. Moreover, note that on the one hand the rate (54) can also be proven in [35] if a semi-norm generated by a closed linear operator forms the stabilizing term in Tikhonov regularization and that on the other hand weaker source conditions introduced in [11] also may yield the rate (54) for nonlinear inverse problems in a P.D.E context.

There are different reasons for an element x_0 not to satisfy a source condition (53). In the literature frequently the case is mentioned, where A is infinitely smoothing and x_0 has to be very smooth (e.g. analytic) to satisfy (53). But also for finitely smoothing A and/or smooth x_0 (53) can be injured if x_0 does not satisfy corresponding (e.g. boundary) conditions required for all elements that belong to the range $R(A^*)$ of A^*. It is rather natural for an element $x_0 \in X$ not to fulfill the canonical source condition (53). To obtain convergence rates, nevertheless, in the recent years *general source conditions*

$$x_0 \;=\varphi(A^*A)\,w \qquad\qquad (w\in X) \tag{55}$$

were used sometimes in combination with variable Hilbert scales (see, e.g., [18], [29], [39], [40], [41], [42], [43], [47], [51] and [52]).

Following [40] we call a function

$$\varphi(t)\ \ (0<t\le\bar{t}) \qquad \text{with} \qquad \bar{t}\ge\|A\|^2$$

index function in the context of (55) if φ is a *positive, continuous* and *increasing* function with $\lim_{t\to 0}\varphi(t)=0$.

Lemma 5.1 *Provided that the index function $\varphi(t)$ is concave for $0<t\le\hat{t}$ with some positive constant $\hat{t}\le\bar{t}$, then the profile function (50) satisfies an estimate from above of the form*

$$f(\alpha)\;=\;\|\alpha\,(A^*A+\alpha I)^{-1}\,\varphi(A^*A)\,w\|\;\le\;K\,\varphi(\alpha)\,\|w\| \qquad (0<\alpha\le\overline{\alpha}) \tag{56}$$

for some constants $\overline{\alpha}>0$ and $K\ge 1$. For $\hat{t}\ge\|A\|^2$ we even have $K=1$.

Proof: We set $\overline{\alpha}=\hat{t},\;K=\frac{\varphi(\bar{t})}{\varphi(\hat{t})}\ge 1$ and will show that

$$\frac{\alpha\,\varphi(\lambda)}{\lambda+\alpha}\le K\,\varphi(\alpha) \qquad (0<\lambda\le\bar{t},\;0<\alpha\le\overline{\alpha}=\hat{t}\,).$$

Taking into account that $\|A\|^2\le\bar{t}$ we therefore have from spectral theory ([8, p.45, formula (2.47)]) the estimate (56) to be proven. In the first case $0<\lambda\le\alpha\le\overline{\alpha}$ due to the monotonicity of φ we have

$$\frac{\alpha\,\varphi(\lambda)}{\lambda+\alpha}\le\varphi(\lambda)\le\varphi(\alpha)\le K\,\varphi(\alpha).$$

In the second case $0<\alpha<\lambda\le\hat{t}$ we have $\frac{\varphi(\lambda)}{\lambda}\le\frac{\varphi(\alpha)}{\alpha}$, since φ is concave for all α and λ under consideration in this case. This also implies $\frac{\alpha\,\varphi(\lambda)}{\lambda+\alpha}\le\varphi(\alpha)\le K\,\varphi(\alpha)$. In the third case $0<\alpha<\lambda$ with $\hat{t}<\lambda\le\bar{t}$, however, we really need the constant $K>1$. Namely, here we have

$$\frac{\alpha\,\varphi(\lambda)}{\lambda+\alpha}\le\frac{\alpha\,\varphi(\lambda)}{\lambda}\le\frac{\alpha\,\varphi(\lambda)}{\hat{t}}\le\frac{\alpha\,\varphi(\bar{t})}{\hat{t}}\le\frac{\alpha\,K\,\varphi(\hat{t})}{\hat{t}}\le K\,\varphi(\alpha),$$

since $\frac{\varphi(\hat{t})}{\hat{t}}\le\frac{\varphi(\alpha)}{\alpha}$. If we have $\hat{t}\ge\|A\|^2$, then we can reset $\hat{t}=\bar{t}=\|A\|^2$ and obtain $K=1$. This proves the lemma ∎

Under the assumption of lemma 5.1 concerning the concavity of the index function φ we can discuss convergence rates for the Tikhonov regularization based on general source conditions (55). For any index function the auxiliary function $\Theta(t):=\sqrt{t}\,\varphi(t)\;(0<t\le\bar{t})$ is positive and *strictly increasing*. Moreover, we have $\lim_{t\to 0}\Theta(t)=0$ and for sufficiently small $\overline{\delta}>0$ the *a priori* choice of the regularization parameter $\alpha=\alpha(\delta)\;(0<\delta\le\overline{\delta})$ based on the equation

$$\Theta(\alpha)\;=\;\begin{cases}\frac{\delta}{\|w\|} & \text{if } \|w\|>0 \text{ is available}\\[2ex] C\,\delta & \text{with a constant } C>0 \text{ otherwise}\end{cases} \tag{57}$$

is well-defined and yields with (52) a *convergence rate*

$$\|x^\delta_{\alpha(\delta)}-x_0\|\;=\;\mathcal{O}\left(\varphi\left(\Theta^{-1}(\delta)\right)\right) \quad \text{as} \quad \delta\to 0 \tag{58}$$

(cf. theorem 2, remark 5 and proposition 3 in [40] and example 2 in [41]). For $\hat{t} \geq \|A\|^2$ in lemma 5.1 we even obtain the result of corollary 5 in [40],

$$\|x^{\delta}_{\alpha(\delta)} - x_0\| \leq \frac{3}{2}\,\|w\|\,\varphi\left(\Theta^{-1}\left(\frac{\delta}{\|w\|}\right)\right) \qquad (0 < \delta \leq \overline{\delta})\,, \tag{59}$$

whenever $\|w\| > 0$ is available.

Proposition 5.2 *The rate (58) obtained for Tikhonov regularization under the assumption of lemma 5.1 is an order optimal convergence rate provided that $\varphi(t)$ is twice differentiable and $\ln(\varphi(t))$ is convave for $0 < t \leq \|A\|^2$.*

Proof: Under our assumptions we obtain the concavity of the function $\varphi^2((\Theta^2)^{-1}(t))$ for $0 < t \leq \|A\|^2$ from proposition 1 in [40]. This concavity condition implies that (58) is an order optimal convergence rate (see [40, §3]). Note that this condition is equivalent to the convexity of the function $\Theta^2 \circ \varphi^{-2}$ on the corresponding domain as required in [51] ∎

Special case 1 (Hölder convergence rates) For all parameters r with $0 < r \leq 1$ the index functions

$$\varphi(t) = t^r \qquad (0 < t < \infty) \tag{60}$$

are concave, where $\Theta(t) = t^{r+\frac{1}{2}}$, and with the a priori parameter choice $\alpha(\delta) \sim \delta^{\frac{2}{2r+1}}$ according to (57) from (58) we have the order optimal convergence rate

$$\|x^{\delta}_{\alpha(\delta)} - x_0\| = \mathcal{O}\left(\delta^{\frac{2r}{2r+1}}\right) \quad \text{as} \quad \delta \to 0\,. \tag{61}$$

Due to the equation $R(A^*) = R((A^*A)^{\frac{1}{2}})$ ([8, p.48, proposition 2.18]) for $r = \frac{1}{2}$ a condition (55) always implies a condition (53) and vice versa. Hence, in this case the general source condition and the canonical source condition coincide. Higher convergence rates that are not of interest in this paper are obtained for $\frac{1}{2} < r \leq 1$, but as a consequence of the qualification one of Tikhonov regularization method only rates up to a saturation level $\mathcal{O}\left(\delta^{\frac{2}{3}}\right)$ can be reached. On the other hand, weaker assumptions compared to (53) correspond with the parameter interval $0 < r < \frac{1}{2}$ yielding in a natural manner also slower convergence rates compared to (54).

Special case 2 (Logarithmic convergence rates) For all parameters $p > 0$ the index functions

$$\varphi(t) = \frac{1}{\left(\ln\left(\frac{1}{t}\right)\right)^p} \qquad (0 < t \leq \overline{t} = 1/e) \tag{62}$$

are concave on the subinterval $0 < t \leq \hat{t} = e^{-p-1}$ (see example 1 in [40]). Moreover, straightforward calculations show that $\ln(\varphi(t))$ is even concave on the whole domain $0 < t \leq \frac{1}{e}$. If the operator A is scaled such that $\|A\|^2 \leq \frac{1}{e}$, then proposition 5.2 applies. Using the a priori parameter choice

$$\alpha(\delta) = c_0\,\delta^{\kappa} \;\; (0 < \kappa < 2) \tag{63}$$

we obtain for sufficiently small $\delta > 0$

$$\|x^\delta_{\alpha(\delta)} - x_0\| \le \frac{K\,\|w\|}{(\kappa\,\ln(1/\delta) - \ln c_0)^p} + \frac{\delta^{1-\frac{\kappa}{2}}}{2\sqrt{c_0}}$$

as a consequence of the estimates (52) and (56) and hence

$$\|x^\delta_{\alpha(\delta)} - x_0\| = \mathcal{O}\left(\frac{1}{\left(\ln\left(\frac{1}{\delta}\right)\right)^p}\right) \quad \text{as} \quad \delta \to 0\,. \tag{64}$$

Note that the simply structured parameter choice (63) asymptotically for $\delta \to 0$ overestimates the values $\alpha(\delta)$ selected by solving the equation (57). Nevertheless, the convergence rate (64) implies the optimal rate (58) (see, e.g., [39] and [29]). Hence, in this special case the convergence is not very sensitive with respect to overestimation of regularization parameters.

We remark that for given $x_0 \in X$ and A the index function φ satisfying a condition (55) is not uniquely determined. Therefore, we now present another approach that works with a distance function d. This function, however, measuring the violation of canonical source condition is uniquely determined for given x_0 and A and hence expresses the relative smoothness of x_0 with respect to A in a unique manner. This approach is based on the following lemma formulated and proved in BAUMEISTER's book [5, Theorem 6.8]:

Lemma 5.3 *If we introduce the distance function*

$$d(R) := \inf\{\|x_0 - A^*\,v\| \,:\, v \in Y,\ \|v\| \le R\}, \tag{65}$$

then we have

$$f(\alpha) = \|x_\alpha - x_0\| \le \sqrt{(d(R))^2 + \alpha R^2} \le d(R) + \sqrt{\alpha}\,R \tag{66}$$

for all $\alpha > 0$ and $R > 0$ as an estimate for the profile function of regularized solutions in Tikhonov regularization.

For the distance function $d(R)$ defined by formula (65) we have:

Lemma 5.4 *For given $x_0 \in X$ and injective linear operator A, the nonnegative function $d(R)$ $(0 < R < \infty)$ is well-defined and non-increasing with*

$$\lim_{R\to\infty} d(R) = 0. \tag{67}$$

Proof: By definition (65) the function $d(R)$ cannot increase and as an immediate consequence of the injectivity of A we have $\overline{R(A^*)} = X$ and hence (67) ■

As the extreme case, where the canonical source condition (53) implying $f(\alpha) = \mathcal{O}(\sqrt{\alpha})$ as $\alpha \to 0$ is satisfied, we have a situation such that $d(R) = 0$ for $R \ge R_0 > 0$. Otherwise, we have $d(R) > 0$ for all $R > 0$ and a condition (53) cannot hold. Then the *decay rate* of $d(R) \to 0$ as $R \to \infty$ characterizes the relative smoothness of x_0 with respect to A and determines the decay rate of $f(\alpha) \to 0$ as $\alpha \to 0$ which is responsible for the associated convergence rate of Tikhonov regularization in the case of noisy data. We are going to analyze three typical situations for the decay of the distance function d in the following:

Situation 1 (logarithmic type decay) If $d(R)$ decreases to zero very slowly as $R \to \infty$, the resulting rate for $f(\alpha) \to 0$ as $\alpha \to 0$ is also very slow. Here, we consider the family of distance functions

$$d(R) \le \frac{K}{(\ln R)^p} \qquad (\underline{R} \le R < \infty) \tag{68}$$

for some constants $\underline{R} > 0$, $K > 0$ and for parameters $p > 0$. By setting

$$R = \frac{1}{\alpha^\kappa} \qquad (0 < \kappa < \frac{1}{2})$$

and taking into account that $\alpha = \mathcal{O}\left(1/(\ln(1/\alpha)^p\right)$ as $\alpha \to 0$ we have from (66)

$$f(\alpha) \le \frac{K}{\kappa^p \left(\ln \frac{1}{\alpha}\right)^p} + \alpha^{\frac{1}{2}-\kappa} = \mathcal{O}\left(\frac{1}{\left(\ln \frac{1}{\alpha}\right)^p}\right) \quad \text{as} \quad \alpha \to 0.$$

Then by using the a priori parameter choice (63) we obtain the same convergence rate (64) as derived for general source conditions with index function (62) in special case 2.

Situation 2 (power type decay) Here we assume that $d(R)$ behaves as a power of R, i.e.,

$$d(R) \le \frac{K}{R^{\frac{\gamma}{1-\gamma}}} \qquad (\underline{R} \le R < \infty) \tag{69}$$

with parameters $0 < \gamma < 1$ and constants $K > 0$. Note that the exponent $\frac{\gamma}{1-\gamma}$ attains all positive values when γ covers the open interval $(0, 1)$. Then by setting

$$R = \frac{1}{\alpha^{\frac{1-\gamma}{2}}}$$

we have from (66)

$$f(\alpha) \le (K+1)\, \alpha^{\frac{\gamma}{2}} = \mathcal{O}\left(\alpha^{\frac{\gamma}{2}}\right) \quad \text{as} \quad \alpha \to 0.$$

If the a priori parameter choice $\alpha \sim \delta^{\frac{2}{1+\gamma}}$ is used, we find from (52)

$$\|x^\delta_{\alpha(\delta)} - x_0\| = \mathcal{O}\left(\delta^{\frac{\gamma}{1+\gamma}}\right) \quad \text{as} \quad \delta \to 0 \tag{70}$$

including all Hölder convergence rates that are slower than the canonical rate $\mathcal{O}(\sqrt{\delta})$. This situation of power type decay rates for $d(R)$ as $R \to \infty$ covers the subcase $0 < r < \frac{1}{2}$ of special case 1 based on source conditions $x_0 = (A^*A)^r\, w$.

Situation 3 (exponential type decay) Even if $d(R)$ falls to zero exponentially, i.e.,

$$d(R) \le K \exp\left(-c\, R^q\right) \qquad (\underline{R} \le R < \infty) \tag{71}$$

for parameters $q > 0$ and constants $K > 0$ and $c \geq \frac{1}{2}$, the canonical convergence rate $\mathcal{O}(\sqrt{\delta})$ cannot be obtained on the basis of lemma 5.3. From (66) we have with

$$R = \left(\ln \frac{1}{\alpha}\right)^{1/q}$$

$$f(\alpha) \leq K\,\alpha^c + \left(\ln \frac{1}{\alpha}\right)^{1/q} \sqrt{\alpha} = \mathcal{O}\left(\left(\ln \frac{1}{\alpha}\right)^{1/q} \sqrt{\alpha}\right) \quad \text{as} \quad \alpha \to 0.$$

Hence with $\alpha \sim \delta$ we derive a convergence rate

$$\|x^{\delta}_{\alpha(\delta)} - x_0\| = \mathcal{O}\left(\left(\ln \frac{1}{\delta}\right)^{1/q} \sqrt{\delta}\right) \quad \text{as} \quad \delta \to 0 \tag{72}$$

which is only a little slower than $\mathcal{O}(\sqrt{\delta})$.

In the forthcoming paper [28], for compact linear operators A sufficient conditions for the situations 1 and 2 will be given formulated as range inclusions with respect to $R(A^*)$ using variable Hilbert scales. On the other hand, the following paragraph presents examples for the situations 2 and 3 in the context of non-compact multiplication operator.

6 A first case study on pure multiplication operators

In this paragraph, let us consider in the spaces $X = Y = L^2(0,1)$ the Tikhonov regularization method for *non-compact multiplication operators* $A = M$ with multiplier functions m defined by formula (4). Regularized solutions x_α and x_α^δ are calculated according to the formulae (48) and (49). We focus on the case where $m \in L^\infty(0,1)$ with $0 \leq m(t) \leq 1$ a.e. on $[0,1]$ and $\operatorname{essinf}_{t\in[0,1]} m(t) = 0$ implying $R(M) \neq \overline{R(M)}$ and the ill-posedness of the linear operator equation (1). As has been proved in [24] (even for the more general case of increasing rearrangements, cf. also [9]) a single essential zero of m at $t = 0$ with limited decay rate for $m(t) \to 0$ as $t \to 0$ of the form

$$m(t) \geq C\,t^\kappa \quad \text{a.e. on} \quad [0,1] \qquad (\kappa > \frac{1}{4}) \tag{73}$$

with a constant $C > 0$ provides an estimate of the profile function

$$f(\alpha) = \mathcal{O}\left(\alpha^{\frac{1}{4\kappa}}\right) \qquad \text{as} \quad \alpha \to 0 \tag{74}$$

whenever $x_0 \in L^\infty(0,1)$. The obtained rate (74) cannot be improved.

By two examples we try to make clear the cross-connections to lemma 5.3 by verifying the distance function $d(R)$ for the pure multiplication operator (4) explicitly. Namely, for computing the distance function $d(R)$ we solve the optimization problem

$$\|x_0 - M^*v\| \to \min, \quad \text{subject to} \quad \|v\| \leq R \tag{75}$$

using the Lagrange multiplier method with the Lagrangian functional

$$L(v,\lambda) = \int_0^1 [x_0(t) - m(t)v(t)]^2 dt + \lambda \left[\int_0^1 v^2(t)dt - R^2\right].$$

If and only if the constraint in (75) is not active, i.e., the quotient function $\frac{x_0(t)}{m(t)}$ is in $L^2(0,1)$ and has a norm not greater than R, then with $A^* = M$ the source condition (53) is satisfied. Otherwise the functions

$$v_\lambda(t) = \frac{m(t)\, x_0(t)}{m^2(t)+\lambda} \qquad (\lambda > 0)$$

obtained by setting the partial derivative of $L(v,\lambda)$ with respect to v to zero yield unique solutions to (75), where $\lambda > 0$ is to be determined as the unique solution of equation

$$R^2 = \int_0^1 \frac{m^2(t)}{(m^2(t)+\lambda)^2}\, x_0^2(t)dt. \tag{76}$$

Moreover, we have for that λ

$$(d(R))^2 = \int_0^1 [x_0 - \frac{m^2(t)\, x_0(t)}{m^2(t)+\lambda}]^2 dt = \int_0^1 \frac{\lambda^2}{(m^2(t)+\lambda)^2}\, x_0^2(t)dt. \tag{77}$$

For simplicity, let us assume

$$x_0(t) = 1 \qquad (0 \le t \le 1) \tag{78}$$

in the following two examples.

Example 1 Consider (78) and let

$$m(t) = t \qquad (0 \le t \le 1). \tag{79}$$

This corresponds with the case $\kappa = 1$ in condition (73) yielding $f(\alpha) = \mathcal{O}(\sqrt[4]{\alpha})$.

Theorem 6.1 *For x_0 from (78) and multiplier function (79) we have with some constant $\underline{R} > 0$ an estimate of the form*

$$d(R) \le \frac{\sqrt{2}}{R} \qquad (\underline{R} \le R < \infty) \tag{80}$$

for the distance function d of the pure multiplication operator M.

Proof: For the multiplier function (79) we obtain from (76) by using a well-known integration formula (see, e.g, [61, p.157, formula (56)])

$$R^2 = \int_0^1 \frac{t^2}{(t^2+\lambda)^2}\, dt = -\frac{1}{2(1+\lambda)} + \frac{1}{2\sqrt{\lambda}} \arctan\left(\frac{1}{\sqrt{\lambda}}\right). \tag{81}$$

The last sum in (81) is a decreasing function for $\lambda \in (0,\infty)$ and tends to infinity as $\lambda \to 0$. Then, for sufficiently large $R \geq \underline{R} > 0$, there is a uniquely determined $\lambda = \lambda_R > 0$ satisfying the equation (81). Based on [61, p.157, formula (48)] we find for equation (77)

$$(d(R))^2 \;=\; \lambda_R^2 \int_0^1 \frac{1}{(t^2+\lambda_R)^2}\, dt \;=\; \frac{\lambda_R}{2(1+\lambda_R)} + \frac{\sqrt{\lambda_R}}{2} \arctan\left(\frac{1}{\sqrt{\lambda_R}}\right). \tag{82}$$

If $\underline{R}$ is large enough and hence λ_R is small enough, then we have $\frac{\lambda_R}{1+\lambda_R} \leq \sqrt{\lambda_R} \arctan\left(\frac{1}{\sqrt{\lambda_R}}\right)$ and can estimate

$$(d(R))^2 \;\leq\; \sqrt{\lambda_R} \arctan\left(\frac{1}{\sqrt{\lambda_R}}\right).$$

Evidently, $\lambda_R \leq \hat{\lambda}_R$ holds and $(d(R))^2 \leq \sqrt{\hat{\lambda}_R} \arctan\left(\frac{1}{\sqrt{\hat{\lambda}_R}}\right)$ if $\lambda = \hat{\lambda}_R$ is the uniquely determined solution of the equation

$$R^2 \;=\; \frac{1}{2\sqrt{\lambda}} \arctan\left(\frac{1}{\sqrt{\lambda}}\right).$$

Then we derive $\frac{\sqrt{\hat{\lambda}_R}}{2} \arctan\left(\frac{1}{\sqrt{\hat{\lambda}_R}}\right) = \hat{\lambda}_R\, R^2$ and

$$(d(R))^2 \;\leq 2\, \hat{\lambda}_R\, R^2.$$

Now we use the inequality

$$\frac{1}{\sqrt{\lambda}} \arctan\left(\frac{1}{\sqrt{\lambda}}\right) \;\leq\; \frac{\pi}{2}\frac{1}{\sqrt{\lambda}} \;\leq\; \frac{2}{\sqrt{\lambda}} \qquad (\lambda > 0)$$

and obtain for the uniquely determined solution $\lambda = \tilde{\lambda}$ of equation $\frac{1}{\sqrt{\lambda}} = R^2$ the estimates $\hat{\lambda}_R \leq \tilde{\lambda}_R$ and

$$(d(R))^2 \;\leq\; 2\, \tilde{\lambda}_R\, R^2 \;=\; \frac{2}{R^4}\, R^2 = \frac{1}{R^2}$$

and hence (80) $\blacksquare$

Theorem 6.1 shows that the situation 2 of power type decay rate (69) for $d(R) \to 0$ as $R \to \infty$ with $\gamma = \frac{1}{2}$ yielding $f(\alpha) = \mathcal{O}(\sqrt[4]{\alpha})$ occurs in this example. Lemma 1 provides here the optimal convergence rate. This is not the case in the next example.

Example 2 Consider (78) and let

$$m(t) \;=\; \sqrt{t} \qquad (0 \leq t \leq 1). \tag{83}$$

This corresponds with $\kappa = \frac{1}{2}$ in condition (73) yielding $f(\alpha) = \mathcal{O}(\sqrt{\alpha})$.

Theorem 6.2 *For x_0 from (78) and multiplier function (83) we have with some constant $\underline{R} > 0$ an estimate*

$$d(R) \;\leq\; \exp\left(-\frac{1}{2} R^2\right) \qquad (\underline{R} \leq R < \infty) \tag{84}$$

for the distance function d of the pure multiplication operator M.

Proof: Here, we have for (76)

$$R^2 \;=\; \int_0^1 \frac{t}{(t+\lambda)^2}\,dt \;=\; \ln\left(1+\frac{1}{\lambda}\right) - \frac{1}{1+\lambda}. \tag{85}$$

Again for sufficiently large $R \geq \underline{R} > 0$, there is a uniquely determined $\lambda = \lambda_R > 0$ solving the equation (85) and we obtain for (77)

$$(d(R))^2 \;=\; \int_0^1 \frac{\lambda_R^2}{(t+\lambda_R)^2}\,dt \;=\; \frac{\lambda_R}{1+\lambda_R} \;\leq\; \lambda_R.$$

In the same manner as in the proof of theorem 6.1 we can estimate

$$d(R) \;\leq\; \sqrt{\lambda_R} \;\leq\; \sqrt{\tilde{\lambda}_R}$$

for sufficiently large R, where $\lambda = \tilde{\lambda}_R$ solves the equation $\ln\frac{1}{\lambda} = R^2$. Then we have $\sqrt{\tilde{\lambda}_R} = \exp\left(-\frac{R^2}{2}\right)$ and hence (84) ∎

Theorem 6.2 indicates for that example the situation 3 of exponential type decay rate (71) for $d(R) \to 0$ as $R \to \infty$ with $c = \frac{1}{2}$ and $q = 2$ yielding $f(\alpha) = \mathcal{O}\left(\sqrt{\left(\ln\frac{1}{\alpha}\right)\,\alpha}\right)$. In this example, the rate result provided by lemma 5.3 is only almost optimal. We should note that (83) represents with $\kappa = \frac{1}{2}$ a limit case in the family of functions $m(t) = t^\kappa$, since for smaller κ and (78) the canonical source condition (53) is satisfied which attains the form $\frac{1}{m} \in L^2(0,1)$ here.

Certainly, only in some exceptional cases (see examples 1 and 2) majorants for distance functions $d(R)$ can be found in practice. From practical point of view it is in like manner difficult to verify or estimate the relative smoothness of an admissible solution $x_0 \in X$ with respect to A by finding appropriate index functions φ and by finding upper bounds of the function d indicating the violation level of canonical source condition for that element x_0.

7 A second case study on composite linear operators

Taking into account the considerations in the first part of §4 we consider in this paragraph again the Tikhonov regularization for the linear operator equation (1) with $X = Y = L^2(0,1)$ and $A = B$, where the injective composite linear operator $B = M \circ J$ is defined by formula (2). We are going to study once more the influence of an injective multiplication operator M with varying multiplier functions m having an essential zero on the error and convergence of regularized solutions. In a first step we assume that the canonical source condition (36) is satisfied for the simple integration operator J defined by formula (3), but we do not assume the stronger condition (37). Under the assumption (36) lemma 5.3 applies and helps to evaluate the profile function $f(\alpha) = \|x_\alpha - x_0\|$ with $x_\alpha = (B^*B + \alpha I)^{-1}B^*y$ and $y = Bx_0$. Namely, we have for

$$d(R) = \inf\{\|x_0 - B^*\,w\| \,:\, w \in L^2(0,1),\ \|w\| \leq R\}$$

the estimate

$$d(R_0) \leq \|J^* v - B^* v\| = \|J^*(v - Mv)\| \quad \text{with} \quad \|v\| \leq R_0$$

and hence

$$(d(R))^2 \leq \int_0^1 \left(\int_s^1 (1 - m(t)) v(t) dt \right)^2 ds \leq \left(\int_0^1 (1 - m(t))^2 dt \right) \|v\|^2$$

and from (66) the profile function estimate

$$f(\alpha) \leq R_0 \sqrt{\alpha + \frac{(d(R_0))^2}{R_0^2}} \leq R_0 \sqrt{\alpha + \int_0^1 (1 - m(t))^2 dt}. \tag{86}$$

If we compare the upper bound of the profile function in (86) with the function $R_0\sqrt{\alpha}$ that would occcur as a bound whenever (37) holds, we see that the former function is obtained by the latter by applying a *shift to the left* with value $\int_0^1 (1 - m(t))^2 dt \geq 0$. A positive shift destroys the convergence rate. However if for noisy data the shift is small and δ not too small, then the regularization error is nearly the same as in the case (37).

If, for example, values of the continuous non-decreasing multiplier function m as plotted in figure 1 deviate from one only on a small interval $[0, \varepsilon)$, i.e.,

$$\begin{aligned} m(t) &= 1 \quad (\varepsilon \leq t \leq 1), \\ m(t) &\to 0 \quad \text{as} \quad t \to 0, \end{aligned}$$

we have

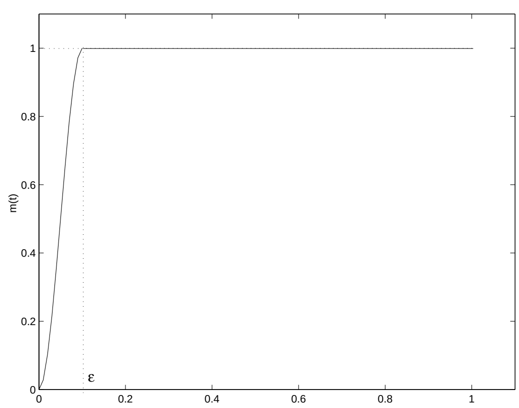

Figure 1: Function $m(t)$ deviating from 1 only on a small interval $0 \leq t \leq \varepsilon$.

$$f(\alpha) \leq R_0 \sqrt{\alpha + \varepsilon} \tag{87}$$

and the influence of the multiplier function disappears as ε tends to zero. As the consideration above and formula (87) show, the decay rate of $m(t) \to 0$ as $t \to 0$ as a power (cf. (6)) or exponentially (cf. (7)) in a neighbourhood of of $t = 0$ is in that case without meaning for the regularization error. Only the integrals $\int_0^1 (1-m(t))^2 dt$ play an important role. This is analogous to the character of the integral $\int_0^1 m(t)dt$ which is as an essential factor for the singular values of B (see theorem 4.1, corollary 4.2 and conjecture 4.3 in §4).

8 On approximate source conditions for nonlinear Tikhonov regularization

Finally, we are going to treat the nonlinear inverse problem (8) immediately by using Tikhonov regularization along the lines of the seminal paper [10] of Engl, Kunisch and Neubauer (see also [8]). Regularized solutions $x_\alpha^\delta \in D(F)$ are stable approximate solutions of (8) based on noisy data $y^\delta \in Y$, where $y = F(x_0)$ with $x_0 \in D(F)$ represents the exact right-hand side and $\delta > 0$ with $\|y - y^\delta\| \le \delta$ is the noise level. In this context, the elements x_α^δ are minimzers of the extremal problem

$$\|F(x) - y^\delta\|^2 + \alpha\, \|x - x^*\|^2 \to \min, \qquad \text{subject to} \qquad x \in D(F), \tag{88}$$

where $x^* \in X$ is an initial guess for the solution x_0 to be determined. If F is continuous and weakly closed, then for all $\alpha > 0$ regularized solutions x_α^δ exist (not necessarily unique) and depend stably on the data y^δ. Moreover, the theory of [10] on convergence and convergence rates applies. We focus again on a source condition which is considered as canonical, here

$$x_0 - x^* \;=\; F'(x_0)^*\, v \qquad\qquad (v \in Y), \tag{89}$$

but we assume that (89) is satisfied only in an approximate manner and the canonical convergence rate (54) proven in [10] cannot be expected. We present a proposition, which is a version of theorem 4.1 in [37] proven by Lukaschewitsch in his PhD thesis [36, theorem 2.2.3]:

Proposition 8.1 *Let $F : D(F) \subset X \to Y$ be a continuous nonlinear operator mapping between the between Hilbert spaces X and Y and let the following assumptions hold:*

(i) The domain $D(F)$ is convex.

(ii) The operator F is weakly (sequentially) closed.

(iii) The element $x_0 \in D(F)$ an x^*-minimum-norm solution, i.e.,

$$\|x_0 - x^*\| \;=\; \min\{\, \|x - x^*\| : \, F(x) = y,\;\, x \in D(F)\}.$$

(iv) For some radius $\rho > 0$ there is a ball $B_\rho(x_0)$ with centre x_0 such that the Fréchet derivative $F'(x)$ of F exists for all $x \in D(F) \cap B_\rho(x_0)$ and an estimate

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\| \;\le\; \frac{L}{2}\, \|x - x_0\|^2 \tag{90}$$

is satisfied for some constant $L > 0$ and all $x \in D(F) \cap B_\rho(x_0)$.

(v) Let exist a number $\theta \geq 0$ such that there is an element $w \in Y$ with

$$c = L\|w\| < 1 \tag{91}$$

and

$$\|x_0 - x^* - F'(x_0)^* w\| \leq \theta. \tag{92}$$

Then, for an a priori choice $\alpha = K\delta$ with some constant $K > 0$ and $0 < \delta \leq \delta_0$ and if $\rho > 2\|x_0 - x^*\| + \sqrt{\frac{\delta_0}{K}}$, we have the estimate

$$\|x_\alpha^\delta - x_0\| \leq \frac{1}{\sqrt{1-c}} \left(\left(\frac{1}{\sqrt{K}} + \frac{\sqrt{K}c}{L} \right) \sqrt{\delta} + \sqrt{2\rho}\sqrt{\theta} \right). \tag{93}$$

Now we consider the specific class (9) of nonlinear problems and apply proposition 8.1, where we assume that the generator function k is smooth enough such that the nonlinear operator F is continuous, weakly (sequentially) closed and $F'(x_0)$ from (12) defines a bounded linear operator in $L^2(0,1)$ which is a Lipschitz continuous Fréchet derivative of F satisfying the inequality (90) for some $L > 0$. An example for estimating θ already formulated in [13, Chapter 7] can be given if we assume a weaker source condition

$$(x_0 - x^*)(t) = \int_t^1 w(s)ds \qquad (0 \leq t \leq 1)$$

for some $w \in L^2(0,1)$ and $c = L\|w\| < 1$ instead of (89). Then we have with $F'(x_0)^* = J^* \circ M^* = J^* \circ M$

$$\|x_0 - x^* - F'(x_0)^* w\| = \|J^* w - J^* M w\| \leq \theta = \left(\sqrt{\int_0^1 (1 - m(t))^2 \, dt} \right) \|w\|.$$

For a non-decreasing multiplier function $m(t)$ with zero at $t = 0$ as shown in figure 1 deviating from one only on an interval $0 \leq t \leq \varepsilon$ we get

$$\|x_\alpha^\delta - x_0\| \leq \frac{1}{\sqrt{1-c}} \left(\left(\frac{1}{\sqrt{K}} + \frac{\sqrt{K}c}{L} \right) \sqrt{\delta} + \sqrt{\frac{2\rho}{L}} \sqrt[4]{\varepsilon} \right)$$

and therefore a small influence on the convergence rate for small $\varepsilon > 0$. Again only an integral $\int_0^1 (1 - m(t))^2 ds$ and not the decay rate of $m(t) \to 0$ as $t \to 0$ influences the regularization properties. This is the same qualitative result as observed for the Tikhonov regularization applied to the linearized problem in §7.

In the nonlinear case, unfortunately, a construction of convergence rates based on proposition 8.1 on the one hand and on the decay rate of the distance function

$$d(R) := \inf \{ \|x_0 - x^* - F'(x_0)^* w\| \, : \, w \in Y, \; \|w\| \leq R \}$$

to zero as $R \to \infty$ on the other hand as presented above for the linear case seems to fail completely, since the smallness condition (91) will be injured for large R. We conjecture that such an approach could be successful under stronger assumptions on the nonlinear operator F in a neighbourhood of x_0 (see [49]) if an estimate

$$\|x_\alpha^\delta - x_0\| \;\le\; \|x_\alpha - x_0\| + K\,\frac{\delta}{\sqrt{\alpha}}$$

similar to (52) in the linear case holds and smallness requirements of the form (91) could be omitted.

Appendix

Proof of theorem 4.1: Let $\sigma = \sigma(B) > 0$ be a singular value of the compact linear operator B defined by (40). Then $\lambda = \frac{1}{\sigma^2} > 0$ satisfies the eigenvalue equation $u - \lambda\, B^* B u = 0$ for some non-trivial function $u \in L^2(0,1)$. Taking into account the explicitly given structures of $[B^*\, y](t) = \int\limits_t^1 m(s) y(s) ds$ and $[B^* B\, x](t) = \int\limits_t^1 m^2(s) \left(\int\limits_0^s x(\tau) d\tau \right) ds\,,$ for $m(t) = t^r$ $(r > -1)$ the eigenvalue equation can be rewritten as

$$u(t) \;-\; \lambda \int\limits_t^1 s^{2r} \left(\int\limits_0^s u(\tau) d\tau \right) ds \;=\; 0 \qquad\qquad (0 < t < 1) \tag{94}$$

implying the first boundary condition

$$u(1) \;=\; 0. \tag{95}$$

Differentiation of (94) leads to the equation $u'(t) + \lambda\, t^{2r} \int\limits_0^t u(\tau) d\tau \;=\; 0$ which can be rewritten as

$$\frac{u'(t)}{t^{2r}} \;+\; \lambda \int\limits_0^t u(\tau) d\tau \;=\; 0 \qquad\qquad (0 < t < 1) \tag{96}$$

and hence to the second boundary condition

$$\lim_{t \to 0} \frac{u'(t)}{t^{2r}} \;=\; 0. \tag{97}$$

Finally, differentiating (96) and multiplying by the factor t^{2r+2} yields the second order O.D.E.

$$t^2\, u''(t) \;-\; 2r\, t\, u'(t) \;+\; \lambda\, t^{2r+2}\, u(t) \;=\; 0 \qquad\qquad (0 < t < 1). \tag{98}$$

Conversely, from the differential equation (98) and the boundary conditions (95) and (97) the integral equation (94) follows such that the boundary value problem for equation (98) determines the singular values $\sigma = \frac{1}{\sqrt{\lambda}} > 0$ under consideration with associated eigenfunctions u.

For all $r > -1$, the differential equation (98) possesses the explicit solution (see, e.g., formula (1a) on p. 440 in [31])

$$u(t) \;=\; t^{r+\frac{1}{2}}\, \mathcal{Z}_\nu \left(\frac{t^{r+1}}{(r+1)\,\sigma}\right), \qquad\qquad \nu \;=\; \frac{2r+1}{2r+2},$$

exploiting the general Bessel function $\mathcal{Z}_\nu$ of order $\nu \in (-\infty, 1)$. Then the general solution of (98) is of the form

$$u(t) \;=\; C_1\, u_1(t) + C_2\, u_2(t) \tag{99}$$

where

$$u_1(t) \;=\; t^{r+\frac{1}{2}}\, \mathcal{J}_{-\nu} \left(\frac{t^{r+1}}{(r+1)\,\sigma}\right)$$

for all $r > -1$, i.e. for all $\nu \in (-\infty, 1)$, with Bessel function of the first kind $\mathcal{J}_\nu$ (cf. [12, §7.2]) and

$$u_2(t) \;=\; t^{r+\frac{1}{2}}\, \mathcal{J}_{\nu} \left(\frac{t^{r+1}}{(r+1)\,\sigma}\right) \qquad (\nu \in (-\infty, 1),\ \nu \neq 0, -1, -2, ...),$$

respectively,

$$u_2(t) \;=\; t^{r+\frac{1}{2}}\, \mathcal{Y}_{\nu} \left(\frac{t^{r+1}}{(r+1)\,\sigma}\right) \qquad (\nu = 0, -1, -2, ...)$$

with Bessel function of the second kind $\mathcal{Y}_\nu$. The constants C_1 and C_2 are to be selected such that the boundary conditions (95) and (97) are satisfied.

To fit the boundary condition (97) at $t = 0$ we consider

$$\frac{u_1'(t)}{t^{2r}} = \frac{t^{\frac{1}{2}}}{\sigma}\, \mathcal{J}_{-\nu}' \left(\frac{t^{r+1}}{(r+1)\,\sigma}\right) + \left(r + \frac{1}{2}\right) t^{-r-\frac{1}{2}}\, \mathcal{J}_{-\nu} \left(\frac{t^{r+1}}{(r+1)\,\sigma}\right).$$

Taking into account the well-known asymptotics (cf. [12, §7.2]) of the Bessel functions of the first kind and their derivatives for $t \to 0$,

$$\mathcal{J}_{-\nu}(t) \sim \frac{1}{\Gamma(1-\nu)} \left\{ \left(\frac{t}{2}\right)^{-\nu} - \frac{1}{1-\nu} \left(\frac{t}{2}\right)^{2-\nu} \right\}$$

and

$$\mathcal{J}_{-\nu}'(t) \sim \frac{1}{2\,\Gamma(-\nu)} \left\{ \left(\frac{t}{2}\right)^{-\nu-1} - \frac{1}{1-\nu} \left(\frac{t}{2}\right)^{1-\nu} \right\},$$

we obtain after some algebra

$$\frac{u_1'(t)}{t^{2r}} \;=\; \mathcal{O}(t) \qquad \text{as} \qquad t \to 0.$$

Hence u_1 satisfies (97). Analogously, one can show that u_2 does not fulfill (97). This implies $C_2 = 0$ in formula (99) as a consequence of the boundary condition (97). Without loss of generality we set $C_1 = 1$ and find from the boundary condition (95) the eigenvalue equation

$$\mathcal{J}_{-\nu} \left(\frac{1}{(r+1)\,\sigma}\right) \;=\; 0 \qquad\qquad (r > -1) \tag{100}$$

for determining the eigenvalues $\sigma > 0$ to which $u = u_1$ are the corresponding eigenfunctions. The well-known asymptotic behaviour of the n-th zero of Bessel functions $\mathcal{J}_{-\nu}$ (cf. [30, VIII (Zeros 1., p.146)]) provides

$$\frac{1}{(r+1)\,\sigma_n} \sim \left(n - \frac{1}{2}\right)\pi - \frac{\nu\,\pi}{2} + \frac{\pi}{4} \qquad \text{as} \qquad n \to \infty$$

and hence the asymptotics of the sequence $\{\sigma_n\}_{n=1}^{\infty}$ of solutions to (100) tending to zero as $n \to \infty$ expressed by

$$\sigma_n \sim [(r+1)\,\pi\,n]^{-1} \qquad \text{as} \qquad n \to \infty.$$

This yields the relation (41) and proves the theorem ∎

Main part of this proof is basically due to ideas of L. von Wolfersdorf.

Acknowlegdgements

The author would like to express his sincere thanks to Masahiro Yamamoto (The University of Tokyo) for fruitful disussions concerning the consequences of used distance functions in the linear case. Moreover the author is grateful to Lothar von Wolfersdorf (BA TU Freiberg) for helpful comment on section 2, to Sergei Pereverzyev (RICAM Linz), Peter Mathé (WIAS Berlin), Ulrich Tautenhahn (UAS Zittau/Görlitz) for important hints to results on generalized source conditions in the linear case and to Ronny Ramlau (Bremen) for an information concerning the used result of M.Lukaschewitsch on Tikhonov regularization and approximate source conditions in the nonlinear case. Last but not least the author thanks Melina Freitag (University of Bath) for fruitful studies during her work in Chemnitz last semester and valuable comments on the results presented in this paper.

References

[1] Ambrosetti, A.; Prodi, G. (1993): *A Primer of Nonlinear Analysis.* Cambridge: Cambridge University Press.

[2] Anger, G. (1990): *Inverse Problems in Differential Equations.* Berlin: Akademie Verlag.

[3] Appell, J.; Zabrejko, P.P. (1990): *Nonlinear superposition operators.* Cambridge: Cambridge University Press.

[4] Bakushinsky, A.; Goncharsky, A. (1994): *Ill-Posed Problems: Theory and Applications.* Dordrecht: Kluwer Academic Publsihers.

[5] Baumeister, J. (1987): *Stable Solution of Inverse Problems.* Braunschweig: Vieweg.

[6] Bertero, M.; Boccacci, P. (1993): An approach to inverse problems based on ε-entropy. In: *Inverse Problems: Principles and Applications in Geophysics, Technology, and Medicine* (Eds.: G. Anger et al.). Berlin: Akademie-Verlag, 76–86.

[7] Colton, D.; Kress, R. (1992): *Inverse Acoustic and Electromagnetic Scattering Theory*. Berlin: Springer.

[8] Engl, H.W.; Hanke, M.; Neubauer, A. (1996): *Regularization of Inverse Problems*. Dordrecht: Kluwer.

[9] Engl, H.W.; Hofmann, B.; Zeisel, H. (1993): A decreasing rearrangement approach for a class of ill-posed nonlinear integral equations. *Journal of Integral Equations and Applications* **5**, 443–463.

[10] Engl, H.W.; Kunisch, K.; Neubauer, A. (1989): Convergence rates for Tikhonov regularisation of non-linear ill-posed problems. *Inverse Problems* **5**, 523–540.

[11] Engl, H.W.; Zou, J. (2000): A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction. *Inverse Problems* **16**, 1907–1923.

[12] Erdélyi, A. (1981): *Higher transcendental functions*. Vol. II. Based, in part, on notes left by Harry Bateman. Malabar (Florida): R. E. Krieger Publ. Company.

[13] Freitag, M. (2004): *On the influence of multiplication operators on the ill-posedness of inverse problems*. Diploma thesis. Chemnitz: University of Technology, Faculty of Mathematics.

[14] Gorenflo, R.; Hofmann, B. (1994): On autoconvolution and regularization. *Inverse Problems* **10**, 353–373.

[15] Groetsch, C.W. (1984): *The Theory of Tikhonov Regularization for Fredholm Integral Equations of the First Kind*. Boston: Pitman.

[16] Groetsch, C.W. (1993): *Inverse Problems in the Mathematical Sciences*. Braunschweig: Vieweg.

[17] Hanke, M.; Neubauer, A.; Scherzer, O. (1995): A convergence analysis of the Landweber iteration for nonlinear ill-posed problems. *Numer. Math.* **72**, 21–37.

[18] Hegland, M. (1995): Variable Hilbert scales and their interpolation inequalities with applications to Tikhonov regularization. *Appl. Anal.* **59**, 207–223.

[19] Hein, T.; Hofmann, B. (2003): On the nature of ill-posedness of an inverse problem arising in option pricing. *Inverse Problems* **19**, 1319–1338.

[20] Hofmann, B. (1986): *Regularization for Applied Inverse and Ill-Poosed Problems*. Leipzig: Teubner.

[21] Hofmann, B. (1994): On the degree of ill-posedness for nonlinear problems. *J. Inv. Ill-Posed Problems* **2**, 61–76.

[22] Hofmann, B. (1998): A local stability analysis of nonlinear inverse problems. In: *Inverse Problems in Engineering - Theory and Practice* (Eds.: D. Delaunay et al.). New York: The American Society of Mechanical Engineers, 313–320.

[23] HOFMANN, B. (1999): *Mathematik inverser Probleme*. Stuttgart: Teubner.

[24] HOFMANN, B.; FLEISCHER, G. (1999): Stability rates for linear ill-posed problems with compact and non-compact operators. *Z. Anal. Anw.* **18**, 267–286.

[25] HOFMANN, B.; SCHERZER, O. (1994): Factors influencing the ill-posedness of nonlinear problems. *Inverse Problems* **10**, 1277–1297.

[26] HOFMANN, B.; TAUTENHAHN, U. (1997) On ill-posedness measures and space change in Sobolev scales. Journal for Analysis and its Applications (ZAA) **16**, 979–1000.

[27] HOFMANN, B.; VON WOLFERSDORF, L. (2004): Some results and a conjecture on the degree of ill-posedness for integration operators with weights. Paper submitted.

[28] HOFMANN, B.; YAMAMOTO, M. (2004): Convergence rates for Tikhonov regularization based on range inclusions. Paper to be submitted.

[29] HOHAGE, T. (2000): Regularization of exponentially ill-posed problems. *Numer. Funct. Anal. & Optimiz.* **21**, 439–464.

[30] JAHNKE–ENDE (1960): *Tables of Higher Functions*. Leipzig: Teubner.

[31] KAMKE, E. (1961): *Differentialgleichungen, Lösungsmethoden und Lösungen – I. Gewöhliche Differentialgleichungen*. Leipzig: Akad. Verlagsges. Geest & Portig.

[32] KIRSCH, A. (1996): *An Introduction to the Mathematical Theory of Inverse Problems*. New York: Springer.

[33] KRESS, R. (1989): *Linear integral equations*. Berlin: Springer.

[34] KWOK, Y. K. (1998): *Mathematical Models of Financial Derivatives*. Singapore: Springer.

[35] KUNISCH, K.; RING, W. (1993): Regularization of nonlinear illposed problems with closed operators. *Numer. Funct. Anal. Optim.* **14**, 389–404.

[36] LUKASCHEWITSCH, M. (1999): *Inversion of geoelectric boundary data, a nonlinear ill-posed problem*. PhD thesis. Potsdam: University of Potsdam, Math.-Nat. Faculty.

[37] LUKASCHEWITSCH, M.; MAASS, P.; PIDCOCK, M. (2003): Tikhonov regularization for electrical impedance tomography on unbounded domains. *Inverse Problems* **19**, 585–610.

[38] LOUIS, A.K. (1989): *Inverse und schlecht gestellte Probleme*. Stuttgart: Teubner.

[39] MAIR, B.A. (1994): Tikhonov regularization for finitely and infinitely smoothing smoothing operators. *SIAM Journal Mathematical Analysis* **25**, 135–147.

[40] MATHÉ, P.; PEREVERZEV, S.V. (2003): Geometry of linear ill-posed problems in variable Hilbert scales. *Inverse Problems* **19**, 789–803.

[41] Mathé, P.; Pereverzev, S.V. (2003): Discretization strategy for linear ill-posed problems in variable Hilbert scales. *Inverse Problems* **19**, 1263–1277.

[42] Nair, M.T.; Schock, E.; Tautenhahn, U. (2003): Morozov's discrepancy principle under general source conditions. *Z. Anal. Anwendungen* **22**, 199-214.

[43] Nair, M.T.; Tautenhahn, U. (2004): Lavrentiev regularization for linear ill-posed problems under general source conditions. *Z. Anal. Anwendungen* **23**, 167–185.

[44] Natterer, F. (1984): Error bounds for Tikhonov regularization in Hilbert scales. *Applicable Analysis* **18**, 29–37.

[45] Neubauer, A. (1997): On converse and saturation results for Tikhonov regularization of linear ill-posed problems. *SIAM J. Numer. Anal.* **34**, 517–527.

[46] Nussbaum, M.; Pereverzev, S.V. (1999): *The degree of ill-posedness in stochastic and deterministic noise models*. Preprint 509. Berlin: WIAS.

[47] Pereverzev, S.V.; Schock, E. (2000): Morozov's discrepancy principle for Tikhonov regularization of severely ill-posed problems. *Numer. Funct. Anal. & Optimiz.* **21**, 901–920.

[48] Rieder, A. (2003): *Keine Probleme mit Inversen Problemen*. Vieweg: Wiesbaden.

[49] Scherzer, O.; Engl, H.W.; Kunisch, K. (1993): Optimal a posteriori parameter choice for Tikhonov regularization for solving nonlinear ill-posed problems. *SIAM J. Numer. Anal.* **30**, 1796–1838.

[50] Schock, E. (1985): Approximate solution of ill-posed equations: arbitrarily slow convergence vs. superconvergence. In: *Constructive Methods for the Practical Treatment of Integral Equations* (Eds.: G. Hämmerlin and K. H. Hoffmann). Basel: Birkhäuser, 234–243.

[51] Tautenhahn, U. (1998): Optimality for ill-posed problems under general source conditions. *Numer. Funct. Anal. & Optimiz.* **19**, 377–398.

[52] Tautenhahn, U. (2004): On order optimal regularization under general source conditions. Proc. Estonian Acad. Sci. Phys. Math. **53**, 116–123.

[53] Tikhonov, A.N.; Arsenin, V.Y. (1977): *Solutions of Ill-Posed Problems*. New York: Wiley.

[54] Tikhonov, A.N.; Leonov, A.S.; Yagola, A.G. (1998): *Nonlinear Ill-Posed Problems. Vol. 1 and 2.* London: Chapman & Hall.

[55] Vainikko, G.M.; Vertennikov, A.Y. (1986): *Iteration Procedures in Ill-Posed Problems* (in Russian). Moscow: Nauka.

[56] Vu Kim Tuan; Gorenflo, R. (1994): Asymptotics of singular values of fractional integral operators. *Inverse Problems* **10**, 949–955.

[57] Wahba, G. (1977): Practical approximate solutions to linear operator equations when the data are noisy. *SIAM J. Numer. Anal.* **14**, 651–667.

[58] WAHBA, G. (1980): *Ill-posed problems: Numerical and statistical methods for mildly, moderately and severely ill-posed problems with noisy data*. Technical Report No. 595. Madison (USA): University of Wisconsin.

[59] WING, G.M. (1985): Condition numbers of matrices arising from the numerical solution of linear integral equations of the first kind. *J. Integral Equations* **9** (Suppl.), 191–204.

[60] ZABREJKO, P.P. et al. (1975): *Integral Equations - a Reference Text*. Leyden: Noordhoff.

[61] ZEIDLER, E. (Ed.) (1996): *Teubner-Taschenbuch der Mathematik*. Leipzig: Teubner.