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Abstract. We present some Farkas-type results for inequality systems involving finitely many convex constraints as well as convex max-functions. Therefore we use the dual of a minmax optimization problem. The main theorem and its consequences allows us to establish, as particular instances, some set containment characterizations and to rediscover some famous theorems of the alternative.

Key Words. duality, Farkas-type results, minmax programming, set containment, theorems of the alternative

AMS subject classification. 49N15, 90C25, 90C46

1 Introduction

In the paper [9], Mangasarian introduced a new approach in order to give dual characterizations for different set containment problems. He succeeded to characterize the containment of a polyhedral set in another polyhedral set and in a reverse-convex set defined by convex quadratic constraints and the containment of a general closed convex set in a reverse-convex set defined by convex nonlinear constraints, respectively. By incorporating them as prior knowledge, these characterizations can be very useful in the determination of knowledge-based classifiers, the most famous example being here the so-called support vector machines classifiers.

Motivated by the paper [9], Jeyakumar has established in [7] dual characterizations for the containment of a closed convex set, defined by infinitely many convex constraints, in an arbitrary polyhedral set, in a reverse-convex set and in another convex set, respectively. The characterizations are given in terms of epigraphs of conjugate functions.

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Recently, Boţ and Wanka have presented in [3] some new Farkas-type results for inequality systems involving a finite as well as an infinite number of convex constraints. This approach bases on the theory of conjugate duality for convex optimization problems, namely by using the so-called Fenchel and Fenchel-Lagrange duality concepts (see also [10], [12], [1], [2]). Moreover the authors show how these new Farkas-type results generalize some of the results obtained by Jeyakumar in [7].

The aim of the present paper is to extend the results obtained in [3] by considering inequality systems involving finitely many convex constraints as well as convex max-functions. Then we particularize them in order to obtain set containment characterizations and, on the other hand, to rediscover some famous theorems of the alternative. Therefore we give an extended formulation of the Lagrange dual of a minmax optimization problem, which leads to new Farkas-type results employing the conjugates of the functions involved. Thus we succeed to underline the connections that exist between Farkas-type results and theorems of the alternative and, on the other hand, the theory of the duality.

The paper is organized as follows. In section 2 we present definitions and preliminary results that will be used later in the paper and we introduce the primal minmax optimization problem. In section 3 we construct its dual problem by using the Lagrange duality. After proving the strong duality we formulate and prove also the optimality conditions for these problems. Section 3 contains our main results. By using the duality developed in the previous section we give a Farkas-type theorem. Then we apply this theorem and its corollaries to three set containment characterization problems. In the last section we rediscover some famous theorems of the alternative by using the general results obtained in section 3.

2 Preliminaries

In this section we describe the notations we use throughout this paper and present some necessary preliminary results. All vectors will be column vectors. A column vector will be transposed to a raw vector by an upper index ^{T}. If A is a matrix, then A^T stands for its transpose. The inner product of two vectors $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in the n-dimensional real space $\mathbb{R}^n$ will be denoted by $x^T y = \sum\limits_{i=1}^{n} x_i y_i$.

The following convention for inequalities will be used. If $x, y \in \mathbb{R}^n, n \geq 2$, then

$$\begin{aligned} x \geqq y \quad &\Leftrightarrow \quad x_i \geq y_i, i = 1, ..., n, \\ x \geq y \quad &\Leftrightarrow \quad x \geqq y \ \text{ and } \ x \neq y, \\ x > y \quad &\Leftrightarrow \quad x_i > y_i, i = 1, ..., n. \end{aligned}$$

For $x, y \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we write, as usual, $x \geq y$ and $x > y$ if x is greater than or equal to y and if x is strictly greater than y, respectively.

For a set $X \subseteq \mathbb{R}^n$ we shall denote the *relative interior* of X by $ri(X)$. Furthermore, let the *indicator function* of X be defined by $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{otherwise.} \end{cases}$$

Considering now a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote by

$$dom(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$$

its *effective domain.* We say that f is proper if $dom(f) \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in \mathbb{R}^n$.

When X is a nonempty subset of $\mathbb{R}^n$ we define for f the so-called *conjugate relative to the set* X

$$f_X^* : \mathbb{R}^n \to \overline{\mathbb{R}},\ f_X^*(p) = \sup_{x \in X} \{p^T x - f(x)\}.$$

By taking X equal to the whole space $\mathbb{R}^n$, the conjugate relative to the set X becomes the classical *conjugate function of* f (the Fenchel-Moreau conjugate)

$$f^* : \mathbb{R}^n \to \overline{\mathbb{R}},\ f^*(p) = \sup_{x \in \mathbb{R}^n} \{p^T x - f(x)\}.$$

Throughout the present paper we assume that X is a nonempty convex subset of $\mathbb{R}^n$ and that $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, ..., k$ are proper convex functions such that $\bigcap_{i=1}^{k} ri(dom(f_i)) \bigcap ri(X) \neq \emptyset$. Furthermore, let $g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with g_j convex functions, for $j = 1, ..., m$. Using them we introduce the following minmax optimization problem

$$(P) \quad \inf_{x} \quad \max_{i=1,...,k} \{f_i(x)\}, \quad \text{s.t.} \quad x \in X, g(x) \leqq 0.$$

Let us notice that (P) is a convex optimization problem, its objective function being convex. To (P) we associate another optimization problem (P') with the property that $v(P) = v(P')$, where $v(P)$ and $v(P')$ represent the optimal objective values of the problems (P) and (P'), respectively. We formulate (P'), which is also a convex optimization problem, in the following way (see for instance [11] and [1])

$$(P') \quad \inf_{x,a} \quad a, \quad \text{s.t.} \quad x \in X, g(x) \leqq 0, a \in \mathbb{R}, \quad f_i(x) - a \leq 0, i = 1, ..., k.$$

Proposition 2.1 states the equality between the optimal objective values of the problems (P) and (P').

Proposition 2.1 *It holds* $v(P) = v(P')$.

Proof. Let x be feasible to (P). If $\max\limits_{i=1,\dots,k}\{f_i(x)\} = +\infty$, then $\max\limits_{i=1,\dots,k}\{f_i(x)\} \geq v(P')$. Assuming now that $\max\limits_{i=1,\dots,k}\{f_i(x)\} < +\infty$ and taking $a = \max\limits_{i=1,\dots,k}\{f_i(x)\}$, we have that (x,a) is feasible to (P') and so $\max\limits_{i=1,\dots,k}\{f_i(x)\} = a \geq v(P')$. In both cases the objective function of (P) is greater than or equal to $v(P')$ and this implies that $v(P) \geq v(P')$.

Conversely, let (x,a) be feasible to (P'), namely $x \in X, g(x) \leqq 0, a \in \mathbb{R}$ and $f_i(x) \leq a, \forall i = 1,...,k$. This implies the feasibility of x to problem (P) and that $a \geq \max\limits_{i=1,\dots,k}\{f_i(x)\} \geq v(P)$. This assures that the opposite inequality $v(P') \geq v(P)$ also holds. In conclusion, $v(P) = v(P')$. $\square$

3 Duality for the minmax optimization problem

The aim of this section is to construct a dual problem to (P') and to give sufficient conditions in order to achieve strong duality, namely that the optimal objective values of the primal and the dual problems coincide and the dual problem has an optimal solution. After that, we formulate and prove also the optimality conditions for these problems.

Let us consider the well-known Lagrange dual problem to (P') with $q^1 \in \mathbb{R}^k, q^2 \in \mathbb{R}^m, q^1 \geqq 0, q^2 \geqq 0$ as dual variables

$$(D) \quad \sup_{\substack{q^1 \geqq 0,\\ q^2 \geqq 0}} \inf_{\substack{x\in X,\\ a\in\mathbb{R}}} \left\{ a + \sum_{i=1}^k q_i^1 [f_i(x) - a] + (q^2)^T g(x) \right\}.$$

We can separate the variables in parentheses, so it follows

$$(D) \quad \sup_{\substack{q^1 \geqq 0,\\ q^2 \geqq 0}} \left\{ \inf_{x\in X} \left[\sum_{i=1}^k q_i^1 f_i(x) + (q^2)^T g(x) \right] + \inf_{a\in\mathbb{R}} \left[a \left(1 - \sum_{i=1}^k q_i^1 \right) \right] \right\}.$$

Since

$$\inf_{a\in\mathbb{R}} \left[a \left(1 - \sum_{i=1}^k q_i^1 \right) \right] = \begin{cases} 0, & \text{if } \sum\limits_{i=1}^k q_i^1 = 1, \\ -\infty, & \text{otherwise,} \end{cases}$$

the dual follows to be

$$(D) \quad \sup_{\substack{q^1 \geqq 0, q^2 \geqq 0,\\ \sum\limits_{i=1}^k q_i^1 = 1}} \inf_{x\in X} \left[\sum_{i=1}^k q_i^1 f_i(x) + (q^2)^T g(x) \right].$$

The infimum concerning $x \in X$ is rewritable as

$$\inf_{x\in X}\left[\sum_{i=1}^{k} q_i^1 f_i(x) + (q^2)^T g(x)\right] = \inf_{x\in\mathbb{R}^n}\left[\sum_{i=1}^{k} q_i^1 f_i(x) + (q^2)^T g(x) + \delta_X(x)\right] =$$

$$-\sup_{x\in\mathbb{R}^n}\left[-\sum_{i=1}^{k} q_i^1 f_i(x) - (q^2)^T g(x) - \delta_X(x)\right] = -\left(\sum_{i=1}^{k} q_i^1 f_i + (q^2)^T g + \delta_X\right)^*(0),$$

where δ_X is the indicator function of the set X.

The functions $q_i^1 f_i, i = 1, .., k$ and $(q^2)^T g + \delta_X$ are proper and convex and the intersection of the relative interiors of their effective domains fulfills

$$\bigcap_{i=1}^{k} ri(dom(q_i^1 f_i)) \bigcap ri(dom((q^2)^T g + \delta_X)) \supseteq \bigcap_{i=1}^{k} ri(dom(f_i)) \bigcap ri(X),$$

which is a nonempty set. Therefore we can apply Theorem 16.4 in [10] and so

$$\begin{gathered}\left(\sum_{i=1}^{k} q_i^1 f_i + (q^2)^T g + \delta_X\right)^*(0) = \\ \inf\left\{\sum_{i=1}^{k}(q_i^1 f_i)^*(p_i) + \left((q^2)^T g + \delta_X\right)^*(u) : \sum_{i=1}^{k} p_i + u = 0\right\},\end{gathered} \tag{1}$$

where the infimum is attained. This leads us to the following formulation for the dual (D)

$$(D) \quad \sup_{\substack{q^1 \geqq 0, \sum_{i=1}^{k} q_i^1 = 1, q^2 \geqq 0, \\ p_i \in\mathbb{R}^n, i=1,..,k, u\in\mathbb{R}^n, \\ \sum_{i=1}^{k} p_i + u = 0}} \left\{-\sum_{i=1}^{k}(q_i^1 f_i)^*(p_i) - \left((q^2)^T g + \delta_X\right)^*(u)\right\}.$$

Finally, because of $\left((q^2)^T g + \delta_X\right)^*(u) = \left((q^2)^T g\right)_X^*(u)$, we get

$$(D) \quad \sup_{\substack{q^1 \geqq 0, \sum_{i=1}^{k} q_i^1 = 1, q^2 \geqq 0, \\ p_i \in\mathbb{R}^n, i=1,..,k}} \left\{-\sum_{i=1}^{k}(q_i^1 f_i)^*(p_i) - \left((q^2)^T g\right)_X^*\left(-\sum_{i=1}^{k} p_i\right)\right\}.$$

It is obvious from the construction of the dual that the weak duality assertion between (P') and (D), i. e. the value of the primal objective function at any feasible point is greater than or equal to the value of the dual objective function at any dual feasible point, always stands. This implies that $v(P') \geq v(D)$, where $v(D)$ is the optimal objective value of (D). Unlike weak duality, strong duality can fail in the general case. To avoid this undesired situation, we introduce a constraint qualification that guarantees the validity of strong duality in case it is fulfilled. First let us divide the index set $\{1,...,m\}$ into two subsets,

$$L := \left\{ j \in \{1,...,m\} : g_j : \mathbb{R}^n \to \mathbb{R} \text{ is an affine function} \right\}$$

and $N := \{1,...,m\} \backslash L$. The constraint qualification follows

$$(CQ)\ \exists x' \in \bigcap_{i=1}^{k} ri(dom(f_i)) \bigcap ri(X) : \begin{cases} g_j(x') \leq 0, & j \in L, \\ g_j(x') < 0, & j \in N. \end{cases}$$

We are ready now to formulate the strong duality assertion.

Theorem 3.1 (strong duality) *Assume that $v(P) > -\infty$. Provided that the constraint qualification (CQ) is fulfilled, the dual problem (D) has an optimal solution and $v(P) = v(P') = v(D)$.*

Proof. The constraint qualification (CQ) being fulfilled, Proposition 2.1 states that $v(P) = v(P') \in \mathbb{R}$. On the other hand, we can write (P') equivalently as

$$\begin{aligned} (P') \quad & \inf_{x,a} \quad a, \\ & \text{s.t.} \quad x \in \bigcap_{i=1}^{k} dom(f_i) \bigcap X, g(x) \leqq 0, a \in \mathbb{R}, \\ & \qquad\ f_i(x) - a \leq 0, i = 1,...,k. \end{aligned}$$

By Theorem 6.5 in [10], (CQ) yields

$$x' \in \bigcap_{i=1}^{k} ri(dom(f_i)) \bigcap ri(X) = ri\left(\bigcap_{i=1}^{k} dom(f_i) \bigcap X \right),$$

and so there exists $\left(x', \max_{i=1,...,k}\{f_i(x')\} + 1\right) \in ri\left(\left(\bigcap_{i=1}^{k} dom(f_i) \bigcap X\right) \times \mathbb{R}\right)$ such that

$$\begin{cases} g_j(x') \leq 0, j \in L, \\ g_j(x') < 0, j \in N, \\ f_i(x') - \left(\max_{i=1,...,k} \{f_i(x')\} + 1 \right) < 0, i = 1,...,k. \end{cases}$$

Under the present hypotheses, Theorem 5.7 in [4] states the existence of $\bar{q}^1 \in \mathbb{R}^k, \bar{q}^1 \geqq 0, \sum_{i=1}^{k} \bar{q}_i^1 = 1$ and $\bar{q}^2 \in \mathbb{R}^m, \bar{q}^2 \geqq 0$ such that strong duality for the Lagrange dual holds, i. e.

$$\begin{aligned} v(P') &= \max_{\substack{q^1 \geqq 0, q^2 \geqq 0, \\ \sum_{i=1}^k q_i^1 = 1}} \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^k q_i^1 f_i(x) + (q^2)^T g(x) + \delta_X(x) \right] = \\ &= \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^k \bar{q}_i^1 f_i(x) + (\bar{q}^2)^T g(x) + \delta_X(x) \right] = \\ &= - \left(\sum_{i=1}^k \bar{q}_i^1 f_i + (\bar{q}^2)^T g + \delta_X \right)^* (0). \end{aligned}$$

Using the fact that the infimum which appears in relation (1) is always attained, there exist $\bar{p}_i \in \mathbb{R}^n, i = 1, ..., k$ such that

$$v(P') = -\sum_{i=1}^k (\bar{q}_i^1 f_i)^* (\bar{p}_i) - \left((\bar{q}^2)^T g \right)_X^* \left(- \sum_{i=1}^k \bar{p}_i \right). \tag{2}$$

In the right-hand term of (2) one may recognize the objective function of (D) at $(\bar{q}^1, \bar{q}^2, \bar{p}_1, ..., \bar{p}_k)$. From weak duality it follows that the supremum of (D) is attained, becoming maximum. The element $(\bar{q}^1, \bar{q}^2, \bar{p}_1, ..., \bar{p}_k)$ turns out to be an optimal solution to (D) and therefore $v(P) = v(P') = v(D)$. □

Next we derive necessary and sufficient optimality conditions regarding the problems (P) and (D).

Theorem 3.2 (optimality conditions)

(a) If the constraint qualification (CQ) is fulfilled and $\bar{x}$ is an optimal solution to (P), then there exists $(\bar{q}^1, \bar{q}^2, \bar{p}_1, ..., \bar{p}_k)$, an optimal solution to (D), satisfying the following optimality conditions

$$\begin{array}{ll} (i) & f_i(\bar{x}) = \max\limits_{i=1,\ldots,k} \{f_i(\bar{x})\}, \text{ if } \bar{q}_i^1 > 0, i = 1, ..., k, \\ (ii) & (\bar{q}^2)^T g(\bar{x}) = 0, \\ (iii) & (\bar{q}_i^1 f_i)^* (\bar{p}_i) + \bar{q}_i^1 f_i(\bar{x}) = \bar{p}_i^T \bar{x}, i = 1, ..., k, \\ (iv) & \left((\bar{q}^2)^T g \right)_X^* \left(- \sum\limits_{i=1}^k \bar{p}_i \right) + (\bar{q}^2)^T g(\bar{x}) = \left(- \sum\limits_{i=1}^k \bar{p}_i \right)^T \bar{x}. \end{array}$$

(b) Let $\bar{x}$ be feasible to (P) and $(\bar{q}^1, \bar{q}^2, \bar{p}_1, ..., \bar{p}_k)$ be feasible to (D) such that (i)–(iv) are satisfied. Then $\bar{x}$ is an optimal solution to (P), $(\bar{q}^1, \bar{q}^2, \bar{p}_1, ..., \bar{p}_k)$ is an optimal solution to (D) and $v(P) = v(D)$.

Proof. By Theorem 3.1 follows that there exists an optimal solution to (D) $(\bar{q}^1, \bar{q}^2, \bar{p}_1, ..., \bar{p}_k)$ such that

$$\max_{i=1,\ldots,k} \{f_i(\bar{x})\} = v(P) = v(P') = v(D) = -\sum_{i=1}^k (\bar{q}_i^1 f_i)^* (\bar{p}_i) - \left((\bar{q}^2)^T g \right)_X^* \left(- \sum_{i=1}^k \bar{p}_i \right)$$

or, equivalently,

$$0 = \max_{i=1,...,k}\{f_i(\bar{x})\} + \sum_{i=1}^{k}(\bar{q}_i^1 f_i)^*(\bar{p}_i) + \left((\bar{q}^2)^T g\right)_X^* \left(-\sum_{i=1}^{k}\bar{p}_i\right) =$$

$$\max_{i=1,...,k}\{f_i(\bar{x})\} - \sum_{i=1}^{k}\bar{q}_i^1 f_i(\bar{x}) + \sum_{i=1}^{k}\left[(\bar{q}_i^1 f_i)^*(\bar{p}_i) + \bar{q}_i^1 f_i(\bar{x}) - \bar{p}_i^T\bar{x}\right] +$$

$$\left((\bar{q}^2)^T g\right)_X^* \left(-\sum_{i=1}^{k}\bar{p}_i\right) + (\bar{q}^2)^T g(\bar{x}) - \left(-\sum_{i=1}^{k}\bar{p}_i\right)^T \bar{x} - (\bar{q}^2)^T g(\bar{x}). \qquad (3)$$

By the so-called Young inequalities we have

$$(\bar{q}_i^1 f_i)^*(\bar{p}_i) + \bar{q}_i^1 f_i(\bar{x}) - \bar{p}_i^T\bar{x} \geq 0, \forall i = 1, ..., k$$

and

$$\left((\bar{q}^2)^T g\right)_X^* \left(-\sum_{i=1}^{k}\bar{p}_i\right) + (\bar{q}^2)^T g(\bar{x}) - \left(-\sum_{i=1}^{k}\bar{p}_i\right)^T \bar{x} \geq 0.$$

In addition, $-(\bar{q}^2)^T g(\bar{x}) \geq 0$ and $\max\limits_{i=1,...,k}\{f_i(\bar{x})\} - \sum\limits_{i=1}^{k}\bar{q}_i^1 f_i(\bar{x}) \geq 0$. Therefore the terms of the sum in (3) are greater than or equal to zero. This implies that all of them must be equal to zero and, in conclusion, the optimality conditions $(i)-(iv)$ must be fulfilled.

All the calculations done before carried out in the reverse direction prove that assertion (b) also holds. □

4 A new Farkas-type result and its applications in set containment characterization

In the following we give a new Farkas-type result for inequality systems involving finitely many convex constraints as well as convex max-functions. The main theorem yields a new dual characterization for this kind of inequality systems and bases on the duality concepts introduced in the previous section. In the last part of this section we give some applications of this new result and its consequences by the characterization of three set containment problems. Two of them allow us to rediscover some results proved by Mangasarian in [9] and the last one characterizes the containment of a polyhedral set in a reverse-polyhedral set.

We assume that all the hypotheses introduced in section 2 are fulfilled, so we can formulate the main result of this paper.

Theorem 4.1 *Let the constraint qualification* (CQ) *be fulfilled. Then the following statements are equivalent*

(i) $x \in X, g(x) \leqq 0 \Rightarrow \max\limits_{i=1,...,k}\{f_i(x)\} \geq 0 \ (> 0)$.

(ii) There exist $q^1 \in \mathbb{R}^k, q^1 \geqq 0, \sum\limits_{i=1}^{k} q_i^1 = 1,\ q^2 \in \mathbb{R}^m, q^2 \geqq 0$ *and* $p_i \in \mathbb{R}^n, i = 1, ..., k$ *such that*

$$\sum_{i=1}^{k}(q_i^1 f_i)^*(p_i) + \left((q^2)^T g\right)_X^* \left(-\sum_{i=1}^{k} p_i\right) \leq 0 \ (< 0).$$

Proof. $(ii) \Rightarrow (i)$. Choose $q^1 \in \mathbb{R}^k, q^1 \geqq 0, \sum\limits_{i=1}^{k} q_i^1 = 1,\ q^2 \in \mathbb{R}^m, q^2 \geqq 0$ and $p_i \in \mathbb{R}^n, i = 1, ..., k$ such that $\sum\limits_{i=1}^{k}(q_i^1 f_i)^*(p_i) + \left((q^2)^T g\right)_X^* \left(-\sum\limits_{i=1}^{k} p_i\right) \leq 0 \ (< 0)$ or, equivalently, $-\sum\limits_{i=1}^{k}(q_i^1 f_i)^*(p_i) - \left((q^2)^T g\right)_X^* \left(-\sum\limits_{i=1}^{k} p_i\right) \geq 0 \ (> 0)$. The optimal objective value $v(D)$ of the dual optimization problem (D) is greater than or equal to zero, respectively, strictly greater than zero. This implies that the optimal objective value $v(P)$ of the problem

$$(P) \quad \inf_{x} \max_{i=1,...,k}\{f_i(x)\}, \quad \text{s.t. } x \in X, g(x) \leqq 0$$

fulfills $v(P) = v(P') \geq v(D) \geq 0 \ (> 0)$. We recall that by weak duality the inequality $v(P') \geq v(D)$ is true. Therefore for all $x \in X, g(x) \leqq 0$, we have $\max\limits_{i=1,...,k}\{f_i(x)\} \geq 0 \ (> 0)$ and so (i) is fulfilled.

$(i) \Rightarrow (ii)$. Assuming now that (i) is true, it follows that the optimal objective value of the problem (P) is greater than or equal to zero, respectively, strictly greater than zero. On the other hand, the constraint qualification (CQ) being fulfilled, we obtain by Theorem 3.1 that there exists an optimal solution to (D) $(q^1, q^2, p_1, ..., p_k)$ such that

$$v(P) = v(P') = v(D) = -\sum_{i=1}^{k}(q_i^1 f_i)^*(p_i) - \left((q^2)^T g\right)_X^* \left(-\sum_{i=1}^{k} p_i\right) \geq 0 \ (> 0).$$

This proves the validity of (ii). □

Remark 4.1 For the implication $(ii) \Rightarrow (i)$ the constraint qualification (CQ) is not necessary.

As an immediate consequence of Theorem 4.1 we get the following theorem of the alternative.

Corollary 4.2 *Let the constraint qualification* (CQ) *be fulfilled. Then either the inequality system*

$$(I)\ x \in X, g(x) \leqq 0, \max_{i=1,...,k}\{f_i(x)\} < 0\ (\leq 0)$$

has a solution or the system

$$(II) \begin{cases} \sum\limits_{i=1}^{k}(q_i^1 f_i)^*(p_i) + \left((q^2)^T g\right)_X^* \left(-\sum\limits_{i=1}^{k} p_i\right) \leq 0\ (<0), \\ q^1 \geqq 0, \sum\limits_{i=1}^{k} q_i^1 = 1, q^2 \geqq 0, p_i \in \mathbb{R}^n, i = 1,...,k \end{cases}$$

has a solution, but never both.

For $k = 1$, Theorem 4.1 and Corollary 4.2 imply the following results.

Theorem 4.3 *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty convex set,* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper convex function and* $g = (g_1, ..., g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* g_j *convex, for* $j = 1, ..., m$*. If there exists* $x' \in ri(dom(f)) \cap ri(X)$ *such that* $g_j(x') \leq 0, \forall j \in L$ *and* $g_j(x') < 0, \forall j \in N$*, then the following statements are equivalent:*

(i) $x \in X, g(x) \leqq 0 \Rightarrow f(x) \geq 0\ (>0)$.

(ii) There exist $q \in \mathbb{R}^m, q \geqq 0$ *and* $p \in \mathbb{R}^n$ *such that*

$$f^*(p) + (q^T g)_X^*(-p) \leq 0\ (<0).$$

Corollary 4.4 *Let the assumptions of Theorem 4.3 be fulfilled. Then either the inequality system*

$$(I)\ x \in X, g(x) \leqq 0, f(x) < 0\ (\leq 0)$$

has a solution or the system

$$(II)\ f^*(p) + (q^T g)_X^*(-p) \leq 0\ (<0), p \in \mathbb{R}^n, q \geqq 0$$

has a solution, but never both.

Remark 4.2 Let us notice that Theorem 4.3 and Corollary 4.4 have been obtained by Boţ and Wanka in [3]. This article is devoted to the presentation of new Farkas-type results for inequality systems involving a finite as well as an infinite number of convex constraints. The approach used in [3] bases on the theory of conjugate duality for convex optimization problems, the so-called Fenchel and Fenchel-Lagrange dual problems playing an important role. The results formulated and proved in [3] generalize some recently published results due to Jeyakumar in [7].

Next we give some applications of Theorem 4.3 in order to characterize the containment of a nonempty polyhedral set in an arbitrary polyhedral set and in a reverse-convex set determined by convex quadratic constraints, respectively, in a different manner than Mangasarian in [9].

Proposition 4.1 (polyhedral set containment) *Let* $A \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^p, b \in \mathbb{R}^m$ *and the sets* $\mathcal{A} := \{x \in \mathbb{R}^n : Ax \geqq a\}$ *and* $\mathcal{B} := \{x \in \mathbb{R}^n : Bx \leqq b\}$ *be such that* $\mathcal{B}$ *is not empty. Then the following statements are equivalent:*

(i) $\mathcal{B} \subseteq \mathcal{A}$.

(ii) There exists $Q \in \mathbb{R}^{p\times m}$, $Q \geqq 0$ *such that* $a + Qb \leqq 0$ *and* $A + QB = 0$.

Proof. In order to apply Theorem 4.3, let be $X = \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = Bx - b$ and $f_i : \mathbb{R}^n \to \mathbb{R}, f_i(x) = A_i^T x - a_i$, for $i = 1, ..., p$. Let $A_i \in \mathbb{R}^n$ and $a_i \in \mathbb{R}$ be such that $A_i^T, i = 1, ..., p$ are the row vectors of the matrix $A \in \mathbb{R}^{p\times n}$ and $a_i, i = 1, ..., p$ are the components of the vector $a \in \mathbb{R}^p$, respectively.

The statement (i) can be equivalently written as

(i) $x \in \mathbb{R}^n, g(x) \leqq 0 \Rightarrow f_i(x) \geq 0, \ \forall i = 1, ..., p.$

The set $\mathcal{B}$ being nonempty, yields that the constraint qualification which appears in Theorem 4.3 is fulfilled. As a consequence of this theorem we have that $\mathcal{B} \subseteq \mathcal{A}$ if and only if

(ii) $\forall i = 1, ..., p$ there exist $q^i \in \mathbb{R}^m, q^i \geqq 0$ and $p^i \in \mathbb{R}^n$ such that

$$f_i^*(p^i) + \left((q^i)^T g\right)^* (-p^i) \leq 0$$

or, equivalently,

$$\sup_{x\in\mathbb{R}^n}\{(p^i)^T x - A_i^T x + a_i\} + \sup_{x\in\mathbb{R}^n}\{-(p^i)^T x - (q^i)^T Bx + (q^i)^T b\} =$$

$$\sup_{x\in\mathbb{R}^n}\{(p^i - A_i)^T x\} + a_i + \sup_{x\in\mathbb{R}^n}\{(-p^i - B^T q^i)^T x\} + (q^i)^T b \leq 0. \qquad (4)$$

It is obvious that (4) is true just if $p^i = A_i, -p^i = B^T q^i$ and $a_i + (q^i)^T b \leq 0$, for $i = 1, ..., p$. Therefore (ii) is rewritable as

(ii) $\forall i = 1, ..., p$ there exists $q^i \in \mathbb{R}^m, q^i \geqq 0$ such that $A_i + B^T q^i = 0$ and $a_i + (q^i)^T b \leq 0$.

Considering $Q \in \mathbb{R}^{p \times m}$, the matrix with the row vectors $(q^i)^T, i = 1, ..., p$ we get the desired result

(ii) There exists $Q \in \mathbb{R}^{p \times m}$, $Q \geqq 0$ such that $a + Qb \leqq 0$ and $A + QB = 0$.

This finishes the proof. □

Proposition 4.2 (polyhedral set containment in a reverse-convex quadratic set) *Let be $B \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, $A_i \in \mathbb{R}^n, a_i \in \mathbb{R}, i = 1, ..., p$ and the symmetric positive semidefinite matrices $U_i \in \mathbb{R}^{n \times n}, i = 1, ..., p$. We consider the sets $\mathcal{A} := \{x \in \mathbb{R}^n : \frac{1}{2}x^T U_i x + A_i^T x \geq a_i, i = 1, ..., p\}$ and $\mathcal{B} := \{x \in \mathbb{R}^n : Bx \leqq b\}$ such that $\mathcal{B}$ is not empty. Then the following statements are equivalent:*

(i) $\mathcal{B} \subseteq \mathcal{A}$.

(ii) For $i = 1, ..., p$ there exist $x^i \in \mathbb{R}^n$ and $q^i \in \mathbb{R}^m, q^i \geqq 0$ such that

$$A_i^T + (q^i)^T B + (x^i)^T U_i = 0 \text{ and } a_i + (q^i)^T b + \frac{1}{2}(x^i)^T U_i x^i \leq 0.$$

Proof. We apply again Theorem 4.3. Therefore let be $X = \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = Bx - b$ and $f_i : \mathbb{R}^n \to \mathbb{R}, f_i(x) = \frac{1}{2}x^T U_i x + A_i^T x - a_i$, $i = 1, ..., p$. The statement (i) can be equivalently written as

(i) $x \in \mathbb{R}^n, g(x) \leqq 0 \Rightarrow f_i(x) \geq 0, \ \forall i = 1, ..., p.$

The set $\mathcal{B}$ is nonempty and so the constraint qualification in Theorem 4.3 is fulfilled. For this reason, $\mathcal{B} \subseteq \mathcal{A}$ if and only if

(ii) $\forall i = 1, ..., p$ there exist $q^i \in \mathbb{R}^m, q^i \geqq 0$ and $p^i \in \mathbb{R}^n$ such that

$$f_i^*(p^i) + \left((q^i)^T g\right)^* (-p^i) \leq 0. \tag{5}$$

We can write now (5) equivalently as follows

$$f_i^*(p^i) + ((q^i)^T g)^*(-p^i) \leq 0 \Leftrightarrow f_i^*(p^i) + \sup_{x \in \mathbb{R}^n} \left\{-(p^i)^T x - (q^i)^T Bx + (q^i)^T b\right\} \leq 0 \Leftrightarrow$$

$$f_i^*(p^i) + (q^i)^T b + \sup_{x \in \mathbb{R}^n} \left\{\left(-p^i - B^T q^i\right)^T x\right\} \leq 0 \Leftrightarrow$$

$$f_i^*(p^i) + (q^i)^T b \leq 0 \text{ and } p^i + B^T q^i = 0.$$

In order to calculate the conjugate of f_i, let $h_i : \mathbb{R}^n \to \mathbb{R}$ be defined by $h_i(x) = \frac{1}{2}x^T U_i x + A_i^T x = f_i(x) + a_i, i = 1, ..., p$. We have $f_i^*(p^i) = h_i^*(p^i) + a_i$, for $i = 1, ..., p$. On the other hand, the conjugate of $h_i, i = 1, ..., p$ can be calculated by using the Moore-Penrose pseudo-inverse U_i^- (see [5], [6])

$$h_i^*(p^i) = \begin{cases} \frac{1}{2}(p^i - A_i)^T U_i^-(p^i - A_i), & \text{if } p^i \in A_i + ImU_i, \\ +\infty, & \text{otherwise.} \end{cases}$$

Relation (ii) becomes

(ii) $\forall i = 1, ..., p$ there exist $p^i \in \mathbb{R}^n$ and $q^i \in \mathbb{R}^m, q^i \geqq 0$ such that

$$p^i \in A_i + ImU_i, a_i + \frac{1}{2}(p^i - A_i)^T U_i^-(p^i - A_i) + (q^i)^T b \leq 0, p^i + B^T q^i = 0.$$

By taking $p^i - A_i = U_i x^i, i = 1, ..., p$, we get the following assertion

(ii) $\forall i = 1, ..., p$ there exist $x^i \in \mathbb{R}^n$ and $q^i \in \mathbb{R}^m, q^i \geqq 0$ such that

$$a_i + \frac{1}{2}(x^i)^T U_i^T U_i^-(U_i x^i) + (q^i)^T b \leq 0, A_i + U_i x^i + B^T q^i = 0.$$

Because of the symmetry of U_i and the fact that $U_i U_i^-(y) = y, \forall y \in ImU_i$, we get

$$U_i^T U_i^-(U_i x^i) = U_i U_i^-(U_i x^i) = U_i x^i, i = 1, ..., p.$$

Finally, relation (ii) can be written as

(ii) $\forall i = 1, ..., p$ there exist $x^i \in \mathbb{R}^n$ and $q^i \in \mathbb{R}^m, q^i \geqq 0$ such that

$$A_i^T + (q^i)^T B + (x^i)^T U_i = 0 \text{ and } a_i + (q^i)^T b + \frac{1}{2}(x^i)^T U_i x^i \leq 0.$$

□

The last result of this section provides a characterization of the containment of a polyhedral set in a reverse-polyhedral set and can be obtained as a direct consequence of Theorem 4.1. One should notice that the methods used by Mangasarian in [9], which apply the duality theory for differentiable convex optimization problems, fail in case of Proposition 4.3.

Proposition 4.3 (polyhedral set containment in a reverse-polyhedral set) *Let $A \in \mathbb{R}^{k \times n}, B \in \mathbb{R}^{m \times n}$, $a \in \mathbb{R}^k, b \in \mathbb{R}^m$ and the sets $\mathcal{A} := \{x \in \mathbb{R}^n : Ax \geqq a\}$ and $\mathcal{B} := \{x \in \mathbb{R}^n : Bx \leqq b\}$ be such that $\mathcal{B}$ is not empty. Then the following statements are equivalent:*

(i) $\mathcal{B} \subseteq \mathbb{R}^n \setminus \mathcal{A}$.

(ii) There exist $q^1 \in \mathbb{R}^k, q^1 \geq 0$ *and* $q^2 \in \mathbb{R}^m, q^2 \geqq 0$ *such that* $B^T q^2 = A^T q^1$ *and* $b^T q^2 < a^T q^1$.

Proof. Let be $X = \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = Bx - b$ and $f_i : \mathbb{R}^n \to \mathbb{R}, f_i(x) = a_i - A_i^T x$, for $i = 1, ..., k$. Then the statement (i) is nothing else than

(i) $x \in \mathbb{R}^n, g(x) \leqq 0 \Rightarrow \max\limits_{i=1,\dots,k} \{f_i(x)\} > 0$.

Because $\mathcal{B}$ is a nonempty set it follows that the constraint qualification (CQ) is fulfilled. So, by Theorem 4.1, $\mathcal{B} \subseteq \mathbb{R}^n \setminus \mathcal{A}$ if and only if

(ii) There exist $q^1 \in \mathbb{R}^k, q^1 \geqq 0, \sum\limits_{i=1}^k q_i^1 = 1$, $q^2 \in \mathbb{R}^m, q^2 \geqq 0$ and $p_i \in \mathbb{R}^n, i = 1, ..., k$ such that

$$\sum_{i=1}^k (q_i^1 f_i)^*(p_i) + \left((q^2)^T g\right)^* \left(-\sum_{i=1}^k p_i\right) < 0$$

or, equivalently,

$$\sum_{i=1}^k \sup_{x \in \mathbb{R}^n} \{(p^i)^T x - q_i^1 a_i + q_i^1 A_i^T x\} + \sup_{x \in \mathbb{R}^n} \left\{ \left(-\sum_{i=1}^k p^i\right)^T x - (q^2)^T B x + (q^2)^T b \right\} =$$

$$\sum_{i=1}^k \sup_{x \in \mathbb{R}^n} \{(p^i + q_i^1 A_i)^T x\} - \sum_{i=1}^k q_i^1 a_i + \sup_{x \in \mathbb{R}^n} \left\{ \left(-\sum_{i=1}^k p^i - B^T q^2\right)^T x \right\} + (q^2)^T b < 0.$$

Therefore (i) is true if and only if

(ii) There exist $q^1 \in \mathbb{R}^k, q^1 \geqq 0, \sum\limits_{i=1}^k q_i^1 = 1$, $q^2 \in \mathbb{R}^m, q^2 \geqq 0$ and $p_i \in \mathbb{R}^n, i = 1, ..., k$ such that $p^i = -q_i^1 A_i, i = 1, .., k$, $-\sum\limits_{i=1}^k p^i = B^T q^2$ and $-a^T q^1 + b^T q^2 < 0$,

which is rewritable as

(ii) There exist $q^1 \in \mathbb{R}^k, q^1 \geqq 0, \sum\limits_{i=1}^k q_i^1 = 1$ and $q^2 \in \mathbb{R}^m, q^2 \geqq 0$ such that $B^T q^2 - A^T q^1 = 0$ and $b^T q^2 - a^T q^1 < 0$.

We conclude the proof by remarking that (ii) is true if and only if

(ii) There exist $q^1 \in \mathbb{R}^k, q^1 \geq 0$ and $q^2 \in \mathbb{R}^m, q^2 \geqq 0$ such that $B^T q^2 = A^T q^1$ and $b^T q^2 < a^T q^1$. □

5 Rediscovering some famous theorems of the alternative

In the last section of this paper we give other applications for the general results presented above, namely by getting some famous theorems of the alternative as consequences of the corollaries 4.2 and 4.4. The results we deal with are the non-homogeneous theorem of Farkas and the theorems of Gale, Tucker and Motzkin. Further theorems of the alternative, including those of Stiemke, Gordan and Slater, can be obtained from the results we mentioned above. For a detailed presentation of theorems of the alternative we invite the reader to consult Mangasarian's book [8].

Throughout this section the set X will be the whole space $\mathbb{R}^n$ and all the functions involved will be affine.

Theorem 5.1 (Gale's theorem for linear inequalities) *Let $A \in \mathbb{R}^{k\times n}$ and $c \in \mathbb{R}^k$ be given. Then either the inequality system*

(I) $Ax \leqq c$

has a solution $x \in \mathbb{R}^n$ or the system

(II) $A^T y = 0, c^T y < 0, y \geqq 0$

has a solution $y \in \mathbb{R}^k$, but never both.

Proof. Let be $g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = 0, \forall x \in \mathbb{R}^n$ and for $i = 1, ..., k$, $f_i : \mathbb{R}^n \to \mathbb{R}, f_i(x) = A_i^T x - c_i$, where A_i^T are the row vectors of the matrix $A \in \mathbb{R}^{k\times n}$ and c_i are the components of c, respectively. Then (I) is rewritable as

(I) $x \in \mathbb{R}^n, g(x) \leqq 0, \max\limits_{i=1,...,k} \{f_i(x)\} \leq 0.$

The constraint qualification (CQ) is fulfilled. By Corollary 4.2, (I) has a solution or the system

$$(II) \begin{cases} \sum\limits_{i=1}^k (q_i^1 f_i)^*(p_i) + \left((q^2)^T g\right)^* \left(-\sum\limits_{i=1}^k p_i\right) < 0, \\ q^1 \geqq 0, \sum\limits_{i=1}^k q_i^1 = 1, q^2 \geqq 0, p_i \in \mathbb{R}^n, i = 1, ..., k \end{cases}$$

has a solution, but never both. The system (II) becomes

$$(II)\begin{cases} \sum_{i=1}^{k} \sup_{x\in\mathbb{R}^n} \left\{(p_i - q_i^1 A_i)^T x + q_i^1 c_i\right\} + \sup_{x\in\mathbb{R}^n} \left\{\left(-\sum_{i=1}^{k} p_i\right)^T x\right\} < 0, \\ q^1 \geqq 0, \sum_{i=1}^{k} q_i^1 = 1, p_i \in \mathbb{R}^n, i = 1, ..., k \end{cases}$$

or, equivalently,

$$(II)\begin{cases} p_i = q_i^1 A_i, \sum_{i=1}^{k} p_i = 0, \sum_{i=1}^{k} q_i^1 c_i < 0, \\ q^1 \geqq 0, \sum_{i=1}^{k} q_i^1 = 1, p_i \in \mathbb{R}^n, i = 1, ..., k \end{cases}$$

which is nothing else than

$$(II)\ A^T q^1 = 0, c^T q^1 < 0, q^1 \geqq 0.$$

This concludes the proof. □

The next result we give is the nonhomogeneous Farkas' theorem.

Theorem 5.2 (nonhomogeneous Farkas' theorem) *Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ be given. Then either the inequality system*

$$(I)\ b^T x > \beta, Ax \leqq c$$

has a solution $x \in \mathbb{R}^n$ or the system

$$(II)\begin{cases} A^T y = 0, c^T y < 0, y \geqq 0 \\ \textit{or} \\ A^T y = b, c^T y \leq \beta, y \geqq 0 \end{cases}$$

has a solution $y \in \mathbb{R}^m$, but never both.

Proof. We show that $(\bar{I}) \Leftrightarrow (II)$, where by $(\bar{I})$ we denote the nonoccurrence of (I). Obviously, $(\bar{I})$ means that the system $Ax \leqq c$ has no solution or, on the other hand, $\{x \in \mathbb{R}^n : Ax \leqq c\} \neq \emptyset$ and $Ax \leqq c, b^T x > \beta$ has no solution. By Theorem 5.1, the system $Ax \leqq c$ has no solution $x \in \mathbb{R}^n$ if and only if $A^T y = 0, c^T y < 0, y \geqq 0$ has a solution $y \in \mathbb{R}^m$.

Let us treat the case when $\{x \in \mathbb{R}^n : Ax \leqq c\} \neq \emptyset$. By Corollary 4.4, $Ax \leqq c, b^T x > \beta$ has no solution if and only if the system

$$(II)\ f^*(p) + (q^T g)^*(-p) \leq 0, p \in \mathbb{R}^n, q \geqq 0 \tag{6}$$

has solution, where $g : \mathbb{R}^n \to \mathbb{R}^m, g(x) = Ax - c$ and $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \beta - b^T x$. Because the constraint qualification (CQ) is in this case fulfilled, we can apply the mentioned corollary.

Thus relation (6) can be written as

$$(II) \ \sup_{x\in\mathbb{R}^n}\{(p+b)^Tx-\beta\}+\sup_{x\in\mathbb{R}^n}\{(-p-A^Tq)^Tx+q^Tc\}\leq 0, p\in\mathbb{R}^n, q\geqq 0,$$

which becomes

$$(II) \ A^Tq=b, c^Tq\leq\beta, q\geqq 0.$$

This leads to the desired result. □

The last two theorems of this section, known as Tucker's and, respectively, Motzkin's theorem of the alternative, characterize the existence of solutions for homogeneous systems containing equalities as well as inequalities.

Theorem 5.3 (Tucker's theorem) *Let $B\in\mathbb{R}^{r\times n}$, $C\in\mathbb{R}^{s\times n}$ and $D\in\mathbb{R}^{t\times n}$ be given with $B\neq 0$. Then either the inequality system*

$$(I) \ Bx\geq 0, Cx\geqq 0, Dx=0$$

has a solution $x\in\mathbb{R}^n$ or the system

$$(II) \ B^Ty_2+C^Ty_3+D^Ty_4=0, y_2>0, y_3\geqq 0$$

has a solution $y_2\in\mathbb{R}^r$, $y_3\in\mathbb{R}^s$, $y_4\in\mathbb{R}^t$, but never both.

Proof. We prove that $(\bar{I})\Leftrightarrow(II)$. The fact that (I) has no solution means that

$$-Bx\leqq 0, -Cx\leqq 0, Dx=0, -B_i^Tx<0 \tag{7}$$

has no solution, for any $i=1,...,r$. By $B_i^T, i=1,...,r$ we denote the row vectors of the matrix B. Considering $g:\mathbb{R}^n\to\mathbb{R}^r\times\mathbb{R}^s\times\mathbb{R}^t\times\mathbb{R}^t, g(x)=(-Bx,-Cx,Dx,-Dx)^T$ and $f_i:\mathbb{R}^n\to\mathbb{R}, f_i(x)=-B_i^Tx, i=1,...,r$, relation (7) says that for all $i=1,...,r$ the system

$$(I) \ x\in\mathbb{R}^n, g(x)\leqq 0, f_i(x)<0$$

has no solution. The constraint qualification which appears in Corollary 4.4 is fulfilled for $x'=0$. The mentioned result implies that for all $i=1,...,r$ the system

$$f_i^*(p^i)+\left((q^i)^Tg\right)^*(-p^i)\leq 0, p^i\in\mathbb{R}^n, q^i\geqq 0 \tag{8}$$

has a solution. We can rewrite system (8) as

$$\begin{cases} \sup\limits_{x\in\mathbb{R}^n} \left\{(p^i + B_i)^T x\right\} + \sup\limits_{x\in\mathbb{R}^n} \left\{(-p^i + B^T q^{i1} + C^T q^{i2} + D^T q^{i3})^T x\right\} \leq 0, \\ q^{i1} \in \mathbb{R}^r, q^{i1} \geqq 0, q^{i2} \in \mathbb{R}^s, q^{i2} \geqq 0, q^{i3} \in \mathbb{R}^t, p \in \mathbb{R}^n, \end{cases}$$

which becomes, for $i = 1, ..., r$,

$$B^T q^{i1} + C^T q^{i2} + D^T q^{i3} = -B_i, q^{i1} \in \mathbb{R}^r, q^{i1} \geqq 0, q^{i2} \in \mathbb{R}^s, q^{i2} \geqq 0, q^{i3} \in \mathbb{R}^t.$$

The system above has a solution if and only if

$$B^T(q^{i1}+e_i)+C^T q^{i2}+D^T q^{i3} = 0, q^{i1} \in \mathbb{R}^r, q^{i1} \geqq 0, q^{i2} \in \mathbb{R}^s, q^{i2} \geqq 0, q^{i3} \in \mathbb{R}^t \qquad (9)$$

has a solution. By $e_i, i = 1, ..., r$, we denoted the unit vectors of $\mathbb{R}^r$.

By taking $y_2 = \sum\limits_{i=1}^{r}(q^{i1}+e_i)$, $y_3 = \sum\limits_{i=1}^{r} q^{i2}$ and $y_4 = \sum\limits_{i=1}^{r} q^{i3}$ follows that the system

$$(II)\ B^T y_2 + C^T y_3 + D^T y_4 = 0, y_2 > 0, y_3 \geqq 0$$

has a solution $y_2 \in \mathbb{R}^r$, $y_3 \in \mathbb{R}^s$, $y_4 \in \mathbb{R}^t$. Moreover, it is quite obvious, that if (II) has a solution, then the system (9) has a solution, too. □

Theorem 5.4 (Motzkin's theorem) *Let $A \in \mathbb{R}^{k\times n}$, $C \in \mathbb{R}^{s\times n}$ and $D \in \mathbb{R}^{t\times n}$ be given with $A \neq 0$. Then either the inequality system*

$$(I)\ Ax > 0, Cx \geqq 0, Dx = 0$$

has a solution $x \in \mathbb{R}^n$ or the system

$$(II)\ A^T y_1 + C^T y_3 + D^T y_4 = 0, y_1 \geq 0, y_3 \geqq 0$$

has a solution $y_1 \in \mathbb{R}^k$, $y_3 \in \mathbb{R}^s$, $y_4 \in \mathbb{R}^t$, but never both.

Proof. The system (I) can be rewritten as

$$(I)\ -Cx \leqq 0, Dx = 0, \max_{i=1,...,k}\{-A_i^T x\} < 0,$$

$A_i^T, i = 1, ..., k$, being the row vectors of the matrix A. If $g : \mathbb{R}^n \to \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^t, g(x) = (-Cx, Dx, -Dx)^T$ and $f_i : \mathbb{R}^n \to \mathbb{R}, f_i(x) = -A_i^T x, i = 1, .., k$, then (I) is nothing else than

$$(I)\ x \in \mathbb{R}^n, g(x) \leqq 0, \max_{i=1,...,k}\{f_i(x)\} < 0.$$

By Corollary 4.2, using the fact that the constraint qualification (CQ) is fulfilled for $x' = 0$, we get that either (I) has a solution or the system

$$(II)\begin{cases}\sum\limits_{i=1}^{k}(q_i^1 f_i)^*(p_i)+\left((q^2)^T g\right)^*\left(-\sum\limits_{i=1}^{k}p_i\right)\leq 0,\\ q^1\geqq 0,\sum\limits_{i=1}^{k}q_i^1=1, q^2\geqq 0, p_i\in\mathbb{R}^n, i=1,...,k\end{cases}$$

has a solution, but never both. The last inequality system becomes

$$(II)\begin{cases}\sum\limits_{i=1}^{k}\sup\limits_{x\in\mathbb{R}^n}\left\{(p_i+q_i^1A_i)^Tx\right\}+\sup\limits_{x\in\mathbb{R}^n}\left\{\left(-\sum\limits_{i=1}^{k}p_i+C^Tq^{2'}+D^Tq^{2''}\right)^T x\right\}\leq 0,\\ q^1\geqq 0,\sum\limits_{i=1}^{k}q_i^1=1, q^{2'}\in\mathbb{R}^s, q^{2'}\geqq 0, q^{2''}\in\mathbb{R}^t, p_i\in\mathbb{R}^n, i=1,...,k,\end{cases}$$

which is the same as

$$(II)\begin{cases}p_i=-q_i^1A_i, C^Tq^{2'}+D^Tq^{2''}=\sum\limits_{i=1}^{k}p_i,\\ q^1\geqq 0,\sum\limits_{i=1}^{k}q_i^1=1, q^{2'}\in\mathbb{R}^s, q^{2'}\geqq 0, q^{2''}\in\mathbb{R}^t, p_i\in\mathbb{R}^n, i=1,...,k.\end{cases}\tag{10}$$

We conclude the proof by remarking that (10) has a solution if and only if

$$(II)\ A^Tq^1+C^Tq^{2'}+D^Tq^{2''}=0, q^1\geq 0, q^{2'}\geqq 0$$

has a solution $q^1\in\mathbb{R}^k$, $q^{2'}\in\mathbb{R}^s$, $q^{2''}\in\mathbb{R}^t$. □

6 Conclusion

In this paper we present some Farkas-type results for inequality systems involving finitely many convex constraints as well as convex max-functions. Therefore an important role is played by the dual of a minmax optimization problem. The approach we use here leads to Farkas-type formulations by employing the conjugates of the functions involved. The main theorem is a generalization of a another recent Farkas-type theorem formulated by Boţ and Wanka in [3]. Moreover, it allows us to establish some results concerning set containment characterization and to rediscover some famous theorems of the alternative.

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