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A new constraint qualification and conjugate duality for composed convex optimization problems

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Abstract. We give a new constraint qualification which guarantees strong duality between a cone constrained convex optimization problem and its Fenchel-Lagrange dual. This result is applied to a convex optimization problem having, for a given non-empty closed convex cone K, as objective function a K-convex function postcomposed with a K-increasing convex function. For this so-called composed convex optimization problem we present a strong duality assertion, too, under weaker conditions than the ones considered so far. As application we show that the formula of the conjugate of a postcomposition with a K-increasing convex function is valid under weaker conditions than the ones existing in the literature.

Keywords. Conjugate functions, Fenchel-Lagrange duality, composed convex optimization problems, cone constraint qualifications

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1 Introduction

A natural generalization of the optimization problems that consist in minimizing a function subject to the negativity of some other functions comes from considering the constraint functions as smaller than zero from the perspective of a partial ordering induced by a non-empty closed convex cone. Many mathematicians have considered such problems in various contexts and the authors of the present

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paper have some contributions to the literature regarding this special kind of optimization problems too, through to the papers [2], [3], [4], [17], [18] and [19].

The objective functions of the optimization problems may have different formulations, too. Many convex optimization problems arising from various directions may be formulated as minimizations of some compositions of functions subject to some constraints. We cite here [5], [6], [13], [14] and [16] as articles dealing with composed convex optimization problems. Duality assertions for this kind of problems may be delivered in different ways, one of the most common consisting in considering an equivalent problem to the primal one, whose dual is easier determinable. If the desired duality results are based on conjugate functions, sometimes even a more direct way is available by obtaining a dual problem based on the conjugate function of the composed objective function of the primal, which is writable, in some situations, by using only the conjugates of the functions involved and the dual variables. Depending on the framework, the formula of the conjugate of the composed functions is taken mainly from [6], [10], [11] or [12].

With this paper we bring weaker conditions under which the known formula of the conjugate of a composed function holds when one works in $\mathbb{R}^n$. No closedness or continuity concerning the functions composed is necessary, while the regularity condition saying that the image set of the postcomposed function should contain an interior point of the domain of the other function is weakened to a relation involving relative interiors which is actually implied by the first one. This important result is present in our paper as an application, followed by the concrete case of calculating the conjugate of $1/F$, when F is a concave strictly positive function defined over the set of strictly positive reals. The theoretical part of the paper consists in presenting duality assertions concerning a cone constrained convex optimization problem and its Fenchel-Lagrange dual problem. This problem has been introduced by Boţ and Wanka (see, for example, [1], [2], [3] or [4]) and is a combination of the widely-used Lagrange and Fenchel dual problems.

The conditions under which the strong duality takes place for the cone constrained problem are weaker than the ones considered usually for this kind of problems, i.e. instead of asking for a feasible point at which the value of the constraint function should be an interior point of the negative cone (the classical Slater constraint qualification), we need this value to belong only to the relative interior of the negative cone while the variable belongs to the relative interior of the constraint set. Therefore this strong duality assertion proves to be valid also for convex problems whose constraints involve cones with empty interiors. Frenk and Kassay in [8] and [9] and Boţ, Kassay and Wanka in [2] used this kind of constraint qualifications even under generalized convexity assumptions.

The ordinary convex program (cf. [7] and [15]) is given as a special case. Then we consider as objective function the postcomposition of a K-increasing convex function to a K-convex function for a given non-empty closed convex cone K. Strong duality holds here under weaker conditions than the ones taken so far in the literature. As a special case we eliminate the constraint functions. The strong duality statement obtained for this unconstrained problem is used later for the above mentioned application.

The structure of the paper arises quite naturally. The next section is dedicated to the general cone constrained convex optimization problem and the third to the composed optimization problem. The application consisting in the determination of the formula of the conjugate function of a composed function follows, while a short conclusive section and the list of references close the paper.

2 The cone constrained convex optimization problem

2.1 Preliminaries

We present here the notations and some not so widely-known denotations used within this paper. As usual, $\mathbb{R}^n$ denotes the n-dimensional real space for any positive integer n. Throughout this paper all the vectors are considered as column vectors belonging to $\mathbb{R}^n$, unless otherwise specified. An upper index ^{T} transposes a column vector to a row one and viceversa. The inner product of two vectors $x = \big(x_1, ..., x_k\big)^T$ and $y = \big(y_1, ..., y_k\big)^T$ in the k-dimensional real space is denoted by $x^T y = \sum_{i=1}^k x_i y_i$. To write the relative interior of a set we use the prefix ri, while for the effective domain of a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we use the notation $\mathrm{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$. For some set $X \subseteq \mathbb{R}^n$ we use the well-known indicator function $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X. \end{cases}$$

For $D \subseteq \mathbb{R}^n$ and a function $f : D \to \mathbb{R}$ we remind of the definition of the so-called conjugate function relative to the set D

$$f_D^* : \mathbb{R} \to \overline{\mathbb{R}},\ f_D^*(p) = \sup_{x \in D} \big\{p^T x - f(x)\big\}.$$

When $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $D = \mathrm{dom}(f)$ we obtain actually the classical (Legendre - Fenchel) conjugate function denoted by f^*.

Concerning the conjugate functions we have the following inequality known as the Fenchel-Young inequality

$$f_D^*(p) + f(x) \geq p^T x \ \forall x \in D \ \forall p \in \mathbb{R}^n.$$

Having now a given non-empty closed convex cone K in $\mathbb{R}^k$ we define two properties involving it that play an important role throughout this paper.

Definition 1. When $D \subseteq \mathbb{R}^k$ a function $f : D \to \mathbb{R}$ is called K-increasing if for $x, y \in D$ such that $x - y \in K$, follows $f(x) \geq f(y)$.

Definition 2. Given a subset $X \subseteq \mathbb{R}^n$, a function $F : X \to \mathbb{R}^k$ is called K-convex if for any x and $y \in X$ and $\lambda \in [0, 1]$ one has

$$\lambda F(x) + (1 - \lambda)F(y) - F\big(\lambda x + (1 - \lambda)y\big) \in K.$$

2.2 Duality for the cone constrained convex optimization problem

In order to formulate the general cone constrained convex optimization problem we want to deal with throughout this section the following sets and functions must be given. Let X be a non-empty convex subset of $\mathbb{R}^n$, C a non-empty closed convex cone in $\mathbb{R}^m$, $f : X \to \mathbb{R}$ a convex function and $g : X \to \mathbb{R}^m$ a C-convex function. The problem we consider is

$$(P) \qquad \inf_{\substack{x \in X, \\ g(x) \in -C}} f(x),$$

called further the primal problem. To (P) we attach a dual problem. This dual problem may be obtained by perturbations (cf. [1], [2] or [17]) or as we derive it within the proof of the strong duality theorem. It is called Fenchel-Lagrange dual problem because of the way it was obtained and it is

$$(D) \qquad \sup_{\substack{q \in C^*, \\ p \in \mathbb{R}^n}} \Big\{ - f_X^*(p) - \big(q^T g\big)_X^*(-p) \Big\},$$

where by $q^T g$ we denote the function defined on X whose value at any $x \in X$ is equal to $\sum_{j=1}^m q_j g_j(x)$, with $q = (q_1, ..., q_m)^T$. Denoting by $v(P)$ the optimal objective value of the problem (P), $v(D)$ is, obviously, the optimal objective value of (D).

The so-called weak duality holds between (P) and (D), i.e. $v(P) \geq v(D)$. The proof arises straightforwardly from the construction of the dual problem. In order to formulate the strong duality statement, i.e. some sufficient conditions that imply the equality between the optimal objective values of the primal and dual problem, we introduce the following constraint qualification (cf. [9])

$$(CQ) \qquad 0 \in \operatorname{ri}\big(g(X) + C\big).$$

Theorem 1. *Consider the constraint qualification (CQ) fulfilled. Then there is strong duality between the problem (P) and its dual (D) and the latter has an optimal solution if $v(P) > -\infty$.*

Proof. The Lagrange dual problem to (P) is

$$(D^L) \qquad \sup_{q \in C^*} \inf_{x \in X} \big[f(x) + q^T g(x) \big].$$

According to [9], the fulfillment of (CQ) together with the convexity assumptions introduced before are sufficient to assure the coincidence of $v(P)$ and $v(D^L)$, moreover guaranteeing the existence of an optimal solution $\bar{q}$ to (D^L) when $v(P) > -\infty$. Now let us write the Fenchel dual problem to the inner infimum in (D^L). For $q \in C^*$, both f and $q^T g$ are real-valued convex functions defined on X, so in order to apply rigourously Fenchel's duality theorem (cf. [15]) we have to consider their convex extensions to $\mathbb{R}^n$, say $\tilde{f}$ and $\widetilde{q^T g}$, which take the value $+\infty$ outside X. As $\mathrm{dom}(\tilde{f}) = \mathrm{dom}\big(\widetilde{q^T g}\big) = X$ and $\mathrm{ri}(X) \neq \emptyset$ due to the convexity of the non-empty set X, we have (cf. Theorem 31.1 in [15])

$$\inf_{x \in X} \big[f(x) + q^T g(x) \big] = \inf_{x \in \mathbb{R}^n} \Big[\tilde{f}(x) + \widetilde{q^T g}(x) \Big] = \sup_{p \in \mathbb{R}^n} \Big\{ -\tilde{f}^*(p) - \widetilde{q^T g}^*(-p) \Big\},$$

with the existence of a $\bar{p}$ where the supremum on the right-hand side is attained granted. As it is not difficult at all to notice that $\tilde{f}^*(p) = \tilde{f}^*_X(p) = f^*_X(p)$ and $\widetilde{q^T g}^*(-p) = \widetilde{q^T g}^*_X(-p) = \big(q^T g\big)^*_X(-p)$ it is straightforward that

$$v(P) = \sup_{q \in C^*} \inf_{x \in X} \big[f(x) + q^T g(x) \big] = \sup_{\substack{q \in C^*,\\ p \in \mathbb{R}^n}} \Big\{ -f^*_X(p) - \big(q^T g\big)^*_X(-p) \Big\}.$$

In case $v(P)$ is finite, because of the existence of an optimal solution for the Lagrange dual and the Fenchel dual, we get

$$v(P) = \sup_{q \in C^*} \inf_{x \in X} \big[f(x) + q^T g(x) \big] = \inf_{x \in X} \big[f(x) + \bar{q}^T g(x) \big] = f^*_X\big(\bar{p}\big) - \big(\bar{q}^T g\big)^*_X(-\bar{p}),$$

which is exactly that (D) has an optimal solution $(\bar{p}, \bar{q})$. □

Remark 1. One may notice that the constraint qualification (CQ) is sufficient to assure strong duality for both Lagrange and Fenchel-Lagrange dual problems.

Now let us deliver necessary and sufficient optimality conditions regarding the problems (P) and (D). The proof of the following theorem is similar to those in the papers [1] and [17], that is why we omit it here.

Theorem 2.

(a) *If the constraint qualification (CQ) is fulfilled and the primal problem (P) has an optimal solution $\bar{x}$, then the dual problem has an optimal solution $(\bar{p}, \bar{q})$ and the following optimality conditions are satisfied* *(i)* $f_X^*(\bar{p}) + f(\bar{x}) = \bar{p}^T \bar{x}$,

(ii) $\left(\bar{q}^T g\right)_X^*(-\bar{p}) + \bar{q}^T g(\bar{x}) = -\bar{p}^T \bar{x}$,

(iii) $\bar{q}^T g(\bar{x}) = 0$.

(b) If $\bar{x}$ is a feasible point to the primal problem (P) and $(\bar{p}, \bar{q})$ is feasible to the dual problem (D) fulfilling the optimality conditions (i)-(iii), then there is strong duality between (P) and (D) and the mentioned feasible points turn out to be optimal solutions of the corresponding problems.

Remark 2. We need to mention that (b) applies without any convexity assumption as well as constraint qualification. So the sufficiency of the optimality conditions (i)-(iii) is true in the most general case.

The fulfilment of the constraint qualification (CQ) seems quite hard to be verified sometimes, that is why we provide the following equivalent formulation to it.

Theorem 3. *The constraint qualification (CQ) is equivalent to*

$$(CQ') \qquad 0 \in g\big(\mathrm{ri}(X)\big) + \mathrm{ri}(C),$$

where g, X and C are as introduced in the beginning of the section.

Proof. Consider the set

$$M := \big\{(x, y) : x \in X, y - g(x) \in C\big\},$$

which is easily provable to be convex. For each $x \in X$ consider now the set $M_x := \big\{y \in \mathbb{R}^m : (x, y) \in M\big\}$. When $x \notin X$ is is obvious that $M_x = \emptyset$, while in the opposite case we have $y \in M_x \Leftrightarrow y - g(x) \in C \Leftrightarrow y \in g(x) + C$, so we conclude

$$M_x = \begin{cases} g(x) + C, & \text{if } x \in X, \\ \emptyset, & \text{if } x \notin X. \end{cases}$$

Therefore M_x is also convex for any $x \in X$. Let us see now how we can characterize the relative interior of the set M. According to Theorem 6.8 in [15] we have $(x, y) \in \mathrm{ri}(M)$ if and only if $x \in \mathrm{ri}(X)$ and $y \in \mathrm{ri}(M_x)$. On the other hand, for any $x \in \mathrm{ri}(X) \subseteq X$, $y \in \mathrm{ri}(M_x)$ means actually $y \in \mathrm{ri}\,\big(g(x) + C\big) = g(x) + \mathrm{ri}(C)$, so we can write further

$$\mathrm{ri}(M) = \big\{(x, y) : x \in \mathrm{ri}(X), y - g(x) \in \mathrm{ri}(C)\big\}.$$

Consider now the linear transformation $A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ defined by $A(x, y) = y$. Let us prove that $A(M) = g(X) + C$. Take first an element $y \in A(M)$. It follows that there is an $x \in X$ such that $y - g(x) \in C$, which yields $y \in g(X) + C$.

Reversely, for any $y \in g(X) + C$ there is an $x \in X$ such that $y \in g(x) + C$, so $y - g(x) \in C$. This means $(x, y) \in M$, followed by $y \in A(M)$.

Regarding the relative interior in (CQ) we have (applying Theorem 6.6 in [15])

$$\text{ri}\left(g(X) + C\right) = \text{ri}\left(A(M)\right) = A\big(\text{ri}(M)\big) = g\big(\text{ri}(X)\big) + \text{ri}(C),$$

so (CQ) and (CQ') are equivalent. □

We give now a concrete problem that shows that a relaxation of (CQ') by considering the whole set X instead of its relative interior does not guarantee strong duality.

Example 1. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f\big((x_1, x_2)\big) = x_2$ and $g\big((x_1, x_2)\big) = x_1$, respectively. It is obvious that both are convex functions. Consider also the set

$$X = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0, \\ 1 < x_2 \leq 4, & \text{if } x_1 > 0 \end{array} \right\}$$

and formulate the problem

$$(P_1) \qquad \inf_{\substack{x \in X, \\ g(x) = 0}} f(x).$$

This problem fits into our scheme for $C = \{0\}$. The constraint qualification (CQ) becomes in this case $0 \in \text{ri}([0, 2] + 0) = (0, 2)$, that is false. On the other hand the condition $0 \in g(X) + \text{ri}(C)$ is fulfilled, being in this case $0 \in [0, 2]$, that is true.

As in [7], where this example has been borrowed from, the optimal objective value of (P_1) turns out to be $v(P_1) = 3$, while the one of the Lagrange dual problem is 1. Because of the convexity of the functions and sets involved the optimal objective value of the Lagrange dual coincides with the one of (D) in this case (see the proof of Theorem 1).

Therefore we state that a relaxation of (CQ') by considering $g(X)$ instead of $g(\text{ri}(X))$ does not guarantee strong duality.

We would like also to mention that Frenk and Kassay have shown in [8] that if there is an $y_0 \in \text{aff}(g(X))$ such that $g(X) \subseteq y_0 + \text{aff}(C)$ then $0 \in g(X) + \text{ri}(C)$ becomes equivalent to (CQ). For any $M \subseteq \mathbb{R}^n$, $\text{aff}(M)$ means the affine hull of the set M.

2.3 The ordinary convex programs as special case

The ordinary convex programs (cf. [15]) may be included among the problems to which the duality assertions formulated earlier are applicable. Consider such an ordinary convex program

$$(P_o) \qquad \inf_{\substack{x\in X,\\ g_i(x)\leq 0, i=1,...,r,\\ g_j(x)=0, j=r+1,...,m}} f(x),$$

where $X \subseteq \mathbb{R}^n$ is a convex set, f and g_i, $i = 1, ..., r$, are convex real-valued functions defined on X and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = r+1, ..., m$, are affine functions. Denote $g = (g_1, ..., g_m)^T$. This problem is a special case of (P) when we consider the cone $C = \mathbb{R}^r_+ \times \{0\}^{m-r}$. The Fenchel-Lagrange dual problem to (P_o) is

$$(D_o) \qquad \sup_{\substack{q\in\mathbb{R}^r_+\times\{0\}^{m-r},\\ p\in\mathbb{R}^n}} \Big\{ - f^*_X(p) - \big(q^T g\big)^*_X(-p)\Big\}.$$

The constraint qualification that assures strong duality is in this case

$$(CQ_o) \qquad 0 \in \operatorname{ri}\big(g(X) + \mathbb{R}^r_+ \times \{0\}^{m-r}\big),$$

equivalent to $0 \in g\big(\operatorname{ri}(X)\big) + \operatorname{ri}\big(\mathbb{R}^r_+ \times \{0\}^{m-r}\big)$, i.e.

$$(CQ_o) \qquad \exists x' \in \operatorname{ri}(X) : \begin{cases} g_i(x') < 0, & \text{if } i = 1, ..., r,\\ g_j(x') = 0, & \text{if } j = r+1, ..., m,\end{cases}$$

which is exactly the sufficient condition given in [15] to state strong duality between (P_o) and its Lagrange dual problem

$$(D^L_o) \qquad \sup_{q\in\mathbb{R}^r_+\times\{0\}^{m-r}} \inf_{x\in X} \big[f(x) + q^T g(x)\big].$$

As $\operatorname{ri}(\operatorname{dom}(f)) = \operatorname{ri}\big(\operatorname{dom}\big(q^T g\big)\big) = X \neq \emptyset$, we have that the value of the inner infimum in (D^L_o), as a convex optimization problem, is equal to the optimal value of its Fenchel dual, which is exactly the objective function in (D_o). The strong duality statement concerning the problems (P_o) and (D_o) follows.

Theorem 4. *Consider the constraint qualification (CQ_o) fulfilled. Then there is strong duality between the problem (P_o) and its dual (D_o) and the latter has an optimal solution if $v(P_o) > -\infty$.*

A special case of (P_o) is the following problem, where W is a closed convex cone in $\mathbb{R}^n$

$$(P_g) \qquad \inf_{\substack{x\in X, x\in W,\\ g_i(x)\leq 0, i=1,...,r,\\ g_j(x)=0, j=r+1,...,m}} f(x),$$

with the functions and sets defined as above. This problem generalizes, as we have proved earlier ([1]) the geometric programming problem, for whose strong duality assertion also closedness is required in the literature alongside some regularity condition. Considering the cone $C = W \times \mathbb{R}^r_+ \times \{0\}^{m-r}$, (P_g) is rewritable as

$$(P_g) \qquad \inf_{\substack{x \in X, \\ (-x, g(x)) \in -C}} f(x).$$

The constraint qualification (CQ) is in this case

$$(CQ_g) \qquad \exists x' \in \mathrm{ri}(X) : (-x', g(x')) \in -ri(W) \times -ri(\mathbb{R}^r_+) \times \{0\}^{m-r},$$

equivalent to

$$(CQ_g) \qquad \exists x' \in \mathrm{ri}(X) \cap \mathrm{ri}(W) : \begin{cases} g_i(x') < 0, & \text{if } i = 1, ..., r, \\ g_j(x') = 0, & \text{if } j = r+1, ..., m. \end{cases}$$

As (P_g) is a special case of (P) its Fenchel-Lagrange dual problem is after a small transformation

$$(D_g) \qquad \sup_{\substack{q \in \mathbb{R}^r_+ \times \{0\}^{m-r}, \\ p \in \mathbb{R}^n, w \in W^*}} \Big\{ - f^*_X(p) - \big(q^T g\big)^*_X (w - p) \Big\}.$$

Strong duality stands whenever the constraint qualification (CQ_g) is fulfilled, no closedness assumptions being necessary as prerequisite regarding any of the functions involved.

Remark 3. Some authors take as ordinary convex program the following problem, where f, g and X are defined as before and "$\leqq$" is the partial ordering introduced by the m-dimensional positive orthant considered as cone,

$$(P_o') \qquad \inf_{\substack{x \in X, \\ g(x) \leqq 0}} f(x),$$

for which the strong duality is attained provided the fulfillment of the constraint qualification (cf. [1], [7])

$$(CQ_o') \qquad \exists x' \in \mathrm{ri}(X) : \begin{cases} g_i(x') < 0, & \text{if } i = 1, ..., r, \\ g_j(x') \leq 0, & \text{if } j = r+1, ..., m. \end{cases}$$

A first look would make someone think that (P_o') is a special case of (P) by taking $C = \mathbb{R}^m_+$ and (CQ) requires in this case the existence of an $x' \in \mathrm{ri}(X)$ such that $g(x') \in -\mathrm{ri}\left(\mathbb{R}^m_+\right)$, i.e. for any $i = 1, ..., m$, $g_i(x') < 0$, condition that is stronger than (CQ_o'). Let us prove that there is another possible choice of the cone C such that (CQ_o') implies the fulfilment of (CQ), namely $0 \in g(\mathrm{ri}(X)) + \mathrm{ri}(C)$ for the primal problem rewritten as

$$(P_o') \qquad \inf_{\substack{x \in X, \\ g(x) \in -C}} f(x).$$

Consider (CQ'_o) fulfilled and take the set

$$I := \left\{i \in \{r+1, ..., m\} : x \in X \text{ such that } g(x) \leqq 0 \Rightarrow g_i(x) = 0\right\}.$$

When $I = \emptyset$ then for each $i \in \{r+1, ..., m\}$ there is an $x^i \in X$ feasible to (P'_o) such that $g_i(x^i) < 0$. Take the cone $C = \mathbb{R}^m_+$. Introducing

$$x^0 := \sum_{i=r+1}^{m} \frac{1}{m-r+1} x^i + \frac{1}{m-r+1} x',$$

we show that it belongs to $\text{ri}(X)$. First,

$$\sum_{i=r+1}^{m} \frac{1}{m-r+1} x^i = \frac{m-r}{m-r+1} \sum_{i=r+1}^{m} \frac{1}{m-r} x^i$$

and $\sum_{i=r+1}^{m} \big(1/(m-r)\big) x^i \in X$. Applying Theorem 6.1 in [15] it follows that $x^0 \in \text{ri}(X)$. For any $j \in \{1, ..., m\}$ we have

$$g_j(x^0) \leq \sum_{i=r+1}^{m} \frac{1}{m-r+1} g_j(x^i) + \frac{1}{m-r+1} g_j(x') < 0.$$

Therefore there exists $x^0 \in \text{ri}(X)$ such that $0 \in g(x^0) + \text{ri}(C)$, which is the desired result.

When $I \neq \emptyset$, without loss of generality as we perform at most a reindexing of the functions g_j, $r+1 \leq j \leq m$, let $I = \{r+l, ..., m\}$, where l is a positive integer smaller than $m-r$. This means that for $j \in \{r+l, ..., m\}$ follows $g_j(x) = 0$ if $x \in X$ and $g(x) \leqq 0$. Then (P'_0) is a special case of (P) for $C = \mathbb{R}^{r+l-1}_+ \times \{0\}^{m-r-l+1}$. For each $j \in \{r+1, ..., r+l-1\}$ there is an x^j feasible to (P'_o) such that $g_j(x^j) < 0$. Taking

$$x^0 := \sum_{i=r+1}^{r+l-1} \frac{1}{l} x^i + \frac{1}{l} x'$$

we have as above that $x^0 \in \text{ri}(X)$ and $g_j(x^0) < 0$ for any $j \in \{1, ..., r+l-1\}$ and $g_j(x^0) = 0$ for $j \in I$ (because of the affinity of the functions g_j, $r+1 \leq j \leq m$), which is exactly what (CQ) asserts.

Therefore there is always a choice of the cone C which guarantees that for the reformulated problem (CQ) stands.

3 The convex optimization problem with composed objective function

3.1 The cone constrained case

Let K and C be non-empty closed convex cones in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, and X a convex subset of $\mathbb{R}^n$. Take $f : D \to \mathbb{R}$ to be a K-increasing convex function, with D a convex subset of $\mathbb{R}^k$, $F : X \to \mathbb{R}^k$ a K-convex function and $g : X \to \mathbb{R}^m$ a C-convex function. Moreover, we impose the condition $F(X) \subseteq D$. The problem we consider within this section is

$$(P_c) \qquad \inf_{\substack{x\in X,\\ g(x)\in -C}} f(F(x)).$$

We could formulate its dual as a special case of (P) since $f\circ F$ is a convex function, but the existing formulae which allow to separate the conjugate of $f \circ F$ into a combination of the conjugates of f and F ask the functions to be closed even in some particular cases (cf. [10], [11]). To avoid this too strong requirement we formulate the following problem equivalent to (P_c) in the sense that their optimal objective values coincide,

$$(P_c') \qquad \inf_{\substack{x\in X, y\in D,\\ g(x)\in -C,\\ F(x)-y\in -K}} f(y).$$

Proposition 1. $v(P_c) = v(P_c')$.

Proof. Let x be feasible to (P_c). Take $y = F(x)$. As $F(X) \subseteq D$, y belongs to D, too, while $F(x) - y = 0 \in K$ (remember that K is non-empty closed). Thus (x, y) is feasible to (P_c') and $f(F(x)) = f(y)$ $\forall x \in X$ such that $g(x) \in -C$. As the objective function of (P_c') may take some other values besides covering all the values of the one of (P_c) as shown above, we conclude that $v(P_c) \geq v(P_c')$.

On the other hand, for (x, y) feasible to (P_c') we have $x \in X$ and $g(x) \in -C$, so x is feasible to (P_c). Since f is K-increasing we get $v(P_c) \leq f(F(x)) \leq f(y)$. Taking the infimum on the right-hand side after (x, y) feasible to (P_c') we get $v(P_c) \leq v(P_c')$. Therefore $v(P_c) = v(P_c')$. □

The problem (P_c') is a special case of (P) with the variable $(x, y) \in X \times D$, the objective function $A : X \times D \to \mathbb{R}$ defined by $A(x, y) = f(y)$, the constraint function $B : X \times D \to \mathbb{R}^m \times \mathbb{R}^k$ defined by $B(x, y) = \big(g(x), F(x) - y\big)$ and the cone $C \times K$, which is a non-empty closed convex cone in $\mathbb{R}^{m+k}$. We also use that $(C \times K)^* = C^* \times K^*$. The Fenchel-Lagrange dual problem to (P_c') is

$$(D_c') \qquad \sup_{\substack{\alpha\in C^*, \beta\in K^*,\\ (p,s)\in\mathbb{R}^n\times\mathbb{R}^k}} \Big\{ - A^*_{X\times D}(p, s) - \big((\alpha, \beta)^T B\big)^*_{X\times D}(-p, -s)\Big\}.$$

Let us determine the values of these conjugates. We have

$$A^*_{X\times D}(p, s) = \sup_{x\in X, y\in D} \big\{p^T x + s^T y - f(y)\big\} = f^*_D(s) + \delta^*_X(p)$$

and

$$\begin{aligned}\big((\alpha, \beta)^T B\big)^*_{X\times D}(-p, -s) &= \sup_{x\in X, y\in D} \big\{ - p^T x - s^T y - \alpha^T g(x) - \beta^T (F(x) - y)\big\}\\ &= \big(\alpha^T g + \beta^T F\big)^*_X(-p) + \delta^*_D(\beta - s).\end{aligned}$$

Pasting these formulae into the objective function of the dual problem we get

$$-A^*_{X\times D}(p,s) - \big((\alpha,\beta)^T B\big)^*_{X\times D}(-p,-s) = -f^*_D(s) - \delta^*_X(p)$$
$$-\big(\alpha^T g + \beta F\big)^*_X(-p) - \delta^*_D(\beta - s).$$

By the properties of the conjugate functions we have that (supremum is attained for $s = \beta$)

$$\sup_{s\in\mathbb{R}^k} \big\{ - f^*_D(s) - \delta^*_D(\beta - s)\big\} = -f^*_D(\beta)$$

and (supremum is attained for $p = 0$)

$$\sup_{p\in\mathbb{R}^n} \big\{ - \delta^*_X(p) - \big(\alpha^T g + \beta F\big)^*_X(-p)\big\} = -\big(\alpha^T g + \beta^T F\big)^*_X(0).$$

So the dual problem turns into

$$(D_c) \qquad \sup_{\alpha\in C^*,\beta\in K^*} \Big\{ - f^*_D(\beta) - \big(\alpha^T g + \beta^T F\big)^*_X(0)\Big\}.$$

Applying now Theorem 16.4 in [15] we get

$$\big(\alpha^T g + \beta^T F\big)^*_X(0) = \inf_{u\in\mathbb{R}^n} \big\{ \big(\beta^T F\big)^*_X(u) + \big(\alpha^T g\big)^*_X(-u)\big\}$$

and this leads to the following formulation of the dual problem

$$(D_c) \qquad \sup_{\substack{\alpha\in C^*,\beta\in K^*,\\ u\in\mathbb{R}^n}} \Big\{ - f^*_D(\beta) - \big(\beta^T F\big)^*_X(u) - \big(\alpha^T g\big)^*_X(-u)\Big\}.$$

Thanks to Proposition 1 we may call (D_c) a dual problem to (P_c), too.

Now let us write what becomes the constraint qualification (CQ') (equivalent to (CQ) by Theorem 3) in this case. We have

$$(CQ_c) \qquad 0 \in B\big(\operatorname{ri}(X\times D)\big) + \operatorname{ri}(C\times K),$$

equivalent to

$$(CQ_c) \qquad \exists x' \in \operatorname{ri}(X) : \begin{cases} g(x') \in -\operatorname{ri}(C), \\ F(x') \in \operatorname{ri}(D) - \operatorname{ri}(K). \end{cases}$$

The strong duality statement follows accompanied by the necessary and sufficient optimality conditions.

Theorem 5. *Consider the constraint qualification (CQ_c) fulfilled. Then there is strong duality between the problem (P_c) and its dual (D_c) and the latter has an optimal solution if $v(P_c) > -\infty$.*

Proof. According to Theorem 1, the fulfilment of (CQ_c) is sufficient to guarantee that $v(P'_c) = v(D_c)$ and, if $v(P'_c) > -\infty$, the existence of an optimal solution to the dual problem. Applying now Proposition 1 it follows $v(P_c) = v(D_c)$ and (D_c) must have an optimal solution if $v(P_c) > -\infty$. $\square$

Theorem 6.

(a) If the constraint qualification (CQ_c) is fulfilled and the primal problem (P_c) has an optimal solution $\bar{x}$, then the dual problem has an optimal solution $(\bar{u}, \bar{\alpha}, \bar{\beta})$ and the following optimality conditions are satisfied

(i) $f_D^*(\bar{\beta}) + f(F(\bar{x})) = \bar{\beta}^T F(\bar{x})$,

(ii) $\left(\bar{\beta}^T F\right)_X^*(\bar{u}) + \bar{\beta}^T F(\bar{x}) = \bar{u}^T \bar{x}$,

(iii) $\left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) + \bar{\alpha}^T g(\bar{x}) = -\bar{u}^T \bar{x}$,

(iv) $\bar{\alpha}^T g(\bar{x}) = 0$.

(b) If $\bar{x}$ is a feasible point to the primal problem (P_c) and $(\bar{u}, \bar{\alpha}, \bar{\beta})$ is feasible to the dual problem (D_c) fulfilling the optimality conditions (i)-(iv), then there is strong duality between (P_c) and (D_c) and the mentioned feasible points turn out to be optimal solutions of the corresponding problems.

Proof. The previous theorem yields the existence of an optimal solution $(\bar{u}, \bar{\alpha}, \bar{\beta})$ to the dual problem. Strong duality is also attained, i.e.

$$f(F(\bar{x})) = -f_D^*(\bar{\beta}) - \left(\bar{\beta}^T F\right)_X^*(\bar{u}) - \left(\bar{\alpha}^T g\right)_X^*(-\bar{u}),$$

which is equivalent to

$$f(F(\bar{x})) + f_D^*(\bar{\beta}) + \left(\bar{\beta}^T F\right)_X^*(\bar{u}) + \left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) = 0.$$

The Fenchel-Young inequality asserts for the functions involved in the latter equality

$$f(F(\bar{x})) + f_D^*(\bar{\beta}) \geq \bar{\beta}^T F(\bar{x}), \tag{1}$$

$$\bar{\beta}^T F(\bar{x}) + \left(\bar{\beta}^T F\right)_X^*(\bar{u}) \geq \bar{u}^T \bar{x} \tag{2}$$

and

$$\bar{\alpha}^T g(\bar{x}) + \left(\bar{\alpha}^T g\right)_X^*(-\bar{u}) \geq -\bar{u}^T \bar{x}. \tag{3}$$

The last four relations lead to

$$0 \geq \bar{\beta}^T F(\bar{x}) + \bar{u}^T \bar{x} - \bar{\beta}^T F(\bar{x}) - \bar{u}^T \bar{x} - \bar{\alpha}^T g(\bar{x}) = -\bar{\alpha}^T g(\bar{x}) \geq 0,$$

as $\bar{\alpha} \in C^*$ and $g(\bar{x}) \in -C$. Therefore the inequalities above must be fulfilled as equalities. The last one implies the optimality condition (iv), while (i) arises from (1), (ii) from (2) and (iii) from (3).

The reverse assertion follows immediately, even without the fulfilment of (CQ_c) and of any convexity assumption we made concerning the involved functions and sets. $\square$

3.2 The unconstrained case

We find it useful to give here also the duality assertions regarding the unconstrained problem having as objective function the postcomposition of a K-increasing convex function to a K-convex function, where K is a non-empty closed convex cone, problem treated within different conditions in [5]. Another argument in favor of the inclusion of these results in our paper is that they will be applied within the next section. So let us consider the primal problem, where all the notations come from the preceding subsection,

$$(P_u) \qquad \inf_{x \in X} f(F(x)).$$

It may be obtained from (P_c) by taking g the zero function and $C = \{0\}^m$. So its Fenchel-Lagrange dual problem becomes

$$(D_u) \qquad \sup_{\substack{\alpha \in C^*, \beta \in K^*, \\ u \in \mathbb{R}^n}} \Big\{ -f_D^*(\beta) - \big(\beta^T F\big)_X^*(u) - \big(\alpha^T 0\big)_X^*(-u) \Big\},$$

equivalent, as $\alpha \in C^*$ is no longer necessary, to

$$(D_u) \qquad \sup_{\beta \in K^*, u \in \mathbb{R}^n} \Big\{ -f_D^*(\beta) - \big(\beta^T F\big)_X^*(u) - \sup_{x \in X}\{-u^T x\} \Big\}.$$

It is easy to see that (supremum is attained for $u = 0$)

$$\sup_{u \in \mathbb{R}^n} \Big\{ -\big(\beta^T F\big)_X^*(u) - \sup_{x \in X}\{-u^T x\} \Big\} = -\big(\beta^T F\big)_X^*(0).$$

Thus the dual problem becomes

$$(D_u) \qquad \sup_{\beta \in K^*} \Big\{ -f_D^*(\beta) - \big(\beta^T F\big)_X^*(0) \Big\},$$

while the constraint qualification that is sufficient to guarantee strong duality between this dual and the primal problem (P_u) is

$$(CQ_u) \qquad \exists x' \in \mathrm{ri}(X) : F(x') \in \mathrm{ri}(D) - \mathrm{ri}(K).$$

The strong duality statement follows.

Theorem 7. *Consider the constraint qualification (CQ_u) fulfilled. Then there is strong duality between the problem (P_u) and its dual (D_u) and the latter has an optimal solution if $v(P_u) > -\infty$.*

The necessary and sufficient optimality conditions concerning this pair of problems follow immediately from Theorem 6, so we skip them here.

4 The conjugate function of a postcomposition with a K-increasing convex function

This part of our paper is dedicated to an interesting and important application of the duality assertions presented so far. We calculate the conjugate function of a postcomposition of a K-convex function with a K-increasing convex function, for K non-empty closed convex cone, and we obtain the same formula as in some other works dealing with the same subject, [6], [10], [11] or [12]. But let us mention that we obtain this formula under weaker conditions than known so far or deducible from the ones given in more general contexts (for instance in the book [10], where the authors also work in finite-dimensional spaces, the two functions are required to be also closed).

The description of the context we consider follows now. Let K be a non-empty closed convex cone in $\mathbb{R}^k$, E a non-empty convex subset of $\mathbb{R}^n$, $f : \mathbb{R}^k \to \overline{\mathbb{R}}$ a K-increasing convex function and $F : E \to \mathbb{R}^k$ a K-convex function. We want to determine the formula of the conjugate function $(f \circ F)^*_E$ as a function of f^* and F^*_E. We have for some $p \in \mathbb{R}^n$

$$(f \circ F)^*_E(p) = \sup_{x \in E} \left\{ p^T x - f(F(x)) \right\} = - \inf_{x \in E} \left\{ f(F(x)) - p^T x \right\}.$$

By considering the set $X = \{x \in E : F(x) \in \operatorname{dom}(f)\}$, which is a convex set, the last infimum becomes

$$\inf_{x \in E} \left\{ f(F(x)) - p^T x \right\} = \inf_{x \in X} \left\{ f(F(x)) - p^T x \right\}.$$

We are interested in writing the minimization problem above in the form of (P_u). Consider the functions

$$A : \operatorname{dom}(f) \times \mathbb{R}^n \to \mathbb{R},\ A(z, y) = f(z) - p^T y$$

and

$$B : X \to \mathbb{R}^{k+n},\ B(x) = (F(x), x).$$

After a standard verification A turns out to be convex and $(K \times \{0\}^n)$-increasing, while B is $(K \times \{0\}^n)$-convex. Moreover, $B(X) \subseteq F(X) \times X \subseteq \operatorname{dom}(f) \times \mathbb{R}^n$. It is not difficult to notice that

$$\inf_{x \in X} \left\{ f(F(x)) - p^T x \right\} = \inf_{x \in X} A(B(x)).$$

According to Theorem 7, the values of these infima coincide with the optimal value of the Fenchel-Lagrange dual problem to the minimization problem in the right-hand side, let us call the latter (P_a), when the constraint qualification (CQ_u) is fulfilled for B and the corresponding sets. Let us formulate the dual problem and the constraint qualification needed here. The first one arises from (D_u), being

$$(D_a) \qquad \sup_{\beta \in K^*, \gamma \in \mathbb{R}^n} \Big\{ - A^*_{\operatorname{dom}(f) \times \mathbb{R}^n}(\beta, \gamma) - \big((\beta, \gamma)^T B \big)^*_X (0) \Big\},$$

while the constraint qualification is

$$(CQ_a) \qquad \exists x' \in \mathrm{ri}(X) : B(x') \in \mathrm{ri}(\mathrm{dom}(f) \times \mathbb{R}^n) - \mathrm{ri}(K \times \{0\}^n),$$

simplifiable to

$$(CQ_a) \qquad \exists x' \in \mathrm{ri}(X) : F(x') \in \mathrm{ri}(\mathrm{dom}(f)) - \mathrm{ri}(K)$$

or, equivalently,

$$(CQ_a) \qquad 0 \in F(\mathrm{ri}(X)) - \mathrm{ri}(\mathrm{dom}(f)) + \mathrm{ri}(K).$$

Because $F(X) = F(E) \cap \mathrm{dom}(f)$ the last formula is rewritable as (cf. Theorem 3)

$$(CQ_a) \qquad 0 \in \mathrm{ri}\big(F(E) \cap \mathrm{dom}(f) + K\big) - \mathrm{ri}(\mathrm{dom}(f)),$$

which means the same as

$$(CQ_a) \qquad \mathrm{ri}\big(F(E) \cap \mathrm{dom}(f) + K\big) \cap \mathrm{ri}(\mathrm{dom}(f)) \neq \emptyset.$$

To determine a formulation of the dual problem that contains only the conjugates of f and F relative to E, we have to determine the conjugate functions of A and B regarding their domains, respectively. We have

$$\begin{aligned} A^*_{\mathrm{dom}(f)\times\mathbb{R}^n}(\beta,\gamma) &= \sup_{(z,y)\in\mathrm{dom}(f)\times\mathbb{R}^n} \left\{\beta^T z + \gamma^T y - A(z,y)\right\} \\ &= \sup_{(z,y)\in\mathrm{dom}(f)\times\mathbb{R}^n} \left\{\beta^T z + \gamma^T y - f(z) + p^T y\right\} \\ &= \sup_{z\in\mathrm{dom}(f)} \left\{\beta^T z - f(z)\right\} + \sup_{y\in\mathbb{R}^n} \left\{\gamma^T y + p^T y\right\} \\ &= f^*(\beta) + \begin{cases} 0, & \text{if } \gamma = -p, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$\big((\beta,\gamma)^T B\big)^*_X(0) = \sup_{x\in X} \left\{0 - \beta^T F(x) - \gamma^T x\right\} = \big(\beta^T F\big)^*_X(-\gamma).$$

As the plus infinite value is not relevant for $A^*_{\mathrm{dom}(f)\times\mathbb{R}^n}$ in (D_a) which is a maximization problem where this function appears with a leading minus in front of, we take further $\gamma = -p$ and the dual problem becomes

$$(D_a) \qquad \sup_{\beta\in K^*} \Big\{ - f^*(\beta) - \big(\beta^T F\big)^*_X(p)\Big\}.$$

When the constraint qualification is satisfied, i.e. $\mathrm{ri}\big(F(E) \cap \mathrm{dom}(f) + K\big) \cap \mathrm{ri}(\mathrm{dom}(f)) \neq \emptyset$, there is strong duality between (P_a) and (D_a), so we have

$$(f \circ F)^*_E(p) = -\inf_{x\in E} \left\{f(F(x)) - p^T x\right\} = -\inf_{x\in X} \left\{f(F(x)) - p^T x\right\}$$

$$= -\sup_{\beta\in K^*}\left\{-f^*(\beta)-\left(\beta^T F\right)^*_X(p)\right\} = \inf_{\beta\in K^*}\left\{f^*(\beta)+\left(\beta^T F\right)^*_X(p)\right\}, \tag{4}$$

where the minimal value of the infimum in the right-hand side being attained at some $\bar{\beta}\in K^*$ if it is finite.

Unlike [10] or [11] no closedness assumption regarding f or F is necessary for the validity of formula (4). Let us prove now that the condition (CQ_a) is weaker than the one required in the works cited above, which is in our case

$$F(E)\cap \operatorname{int}(\operatorname{dom}(f)) \neq \emptyset. \tag{5}$$

Assuming (5) true let z' belong to the both sets involved there. It follows that $\operatorname{int}(\operatorname{dom}(f)) \neq \emptyset$, so $\operatorname{ri}(\operatorname{dom}(f)) = \operatorname{int}(\operatorname{dom}(f))$ which is an open set. We have also

$$z' \in F(E)\cap \operatorname{int}(\operatorname{dom}(f)) \subseteq F(E)\cap \operatorname{dom}(f) \subseteq F(E)\cap \operatorname{dom}(f) + K,$$

as $0\in K$. On the other hand $F(E)\cap \operatorname{dom}(f)+K = F(X)+K$ which is a convex set, so it has non-empty relative interior. Take $z''\in \operatorname{ri}\big(F(X)+K\big)$.

According to Theorem 6.1 in [15], for any $\lambda\in(0,1]$ one has $(1-\lambda)z'+\lambda z''\in \operatorname{ri}\big(F(X)+K\big)$. As $z'\in \operatorname{int}(\operatorname{dom}(f))$ which is an open set, there is a $\bar{\lambda}\in(0,1]$ such that $\bar{z}=(1-\bar{\lambda})z'+\bar{\lambda}z''\in \operatorname{int}(\operatorname{dom}(f)) = \operatorname{ri}(\operatorname{dom}(f))$. Therefore

$$\bar{z}\in \operatorname{ri}\big(F(E)\cap \operatorname{dom}(f)+K\big)\cap \operatorname{ri}(\operatorname{dom}(f)),$$

i.e. (CQ_a) is fulfilled.

The formula of the conjugate of the postcomposition with an increasing convex function becomes for an appropriate choice of the functions and for $K=[0,+\infty)$ the result given in Theorem 2.5.1 by Hiriart-Urruty and Lemaréchal in [10]. As shown above, there is no need to impose closedness for the functions and a so strong constraint qualification as there.

We conclude this section with a concrete problem where the results given in this paper find a good application.

Example 2. Let $F: E\to\mathbb{R}$ be a concave function with strictly positive values, where E is a non-empty convex subset of R^n. We want to determine the value of the conjugate function of $1/F$ at some $p\in\mathbb{R}^n$. According to the preceding results, we write $\big(1/F\big)^*_E(p)$ as a unconstrained composed convex problem by taking $K=(-\infty,0]$, which is a non-empty closed convex cone and $f:\mathbb{R}\to\overline{\mathbb{R}}$ with $f(y)=1/y$ for $y\in(0,+\infty)$ and $+\infty$ otherwise. It is interesting to notice that the concave function F is actually K-convex for this K while f is K-increasing. Now let us see when the constrained qualification (CQ_a) specialized for this problem is valid. It is in this case

$$(CQ_e) \qquad \operatorname{ri}\big(F(E)\cap(0,+\infty)+(-\infty,0]\big)\cap \operatorname{ri}\big((0,+\infty)\big)\neq\emptyset,$$

equivalent to

$$(CQ_e) \qquad \mathrm{ri}\big(F(E) + (-\infty, 0]\big) \cap (0, +\infty) \neq \emptyset.$$

By Theorem 3 this is nothing else than

$$(CQ_e) \qquad \Big(F\big(\mathrm{ri}(E)\big) + (-\infty, 0)\Big) \cap (0, +\infty) \neq \emptyset,$$

which is always fulfilled since F has only strictly positive values.

So the formula (4) obtained before can be applied without any additional assumption. We have

$$\left(\frac{1}{F}\right)^*_E (p) = \inf_{\beta \leq 0} \big\{ f^*(\beta) + (\beta F)^*_X(p) \big\}.$$

As f is known, we may calculate its conjugate function which is actually

$$f^*(\beta) = \sup_{y>0} \left\{ \beta y - \frac{1}{y} \right\} = \begin{cases} -2\sqrt{-\beta}, & \text{if } \beta < 0, \\ 0, & \text{if } \beta = 0. \end{cases}$$

According to its definition $X = \{x \in E : F(x) \in (0, +\infty)\} = E$ and for $(\beta F)^*_X$ we have

$$(\beta F)^*_X(p) = \begin{cases} -\beta(-F)^*_E\Big(\frac{p}{-\beta}\Big), & \text{if } \beta < 0, \\ \delta^*_E(p), & \text{if } \beta = 0, \end{cases}$$

We conclude after changing the sign of β that the formula of the conjugate of $1/F$ is

$$\left(\frac{1}{F}\right)^*_E (p) = \min\left\{ \inf_{\beta > 0} \left\{ \beta(-F)^*_E \left(\frac{p}{\beta}\right) - 2\sqrt{\beta} \right\}, \delta^*_E(p) \right\}.$$

When the value of the conjugate is finite either it is equal to $\delta^*_E(p)$ or there is a $\bar{\beta} > 0$ for which the infimum in the right-hand side is attained. The value of the infimum gives in this latter case actually the formula of the conjugate.

5 Conclusions

We have started with a convex optimization problem with a convex objective function and cone-convex constraints. To this problem we have attached the Fenchel-Lagrange dual problem. To achieve strong duality between these two problems we have used a new constraint qualification due to Frenk and Kassay ([9]), weaker than the ones used so far in the literature. An equivalent formulation of this constraint qualification is given, too. The ordinary convex programming problem (see [15], among many others) is a special case of this problem and its weakest constraint qualification known to us is actually the one mentioned above specialized for this problem. The convex optimization problem that consists in the minimization of the postcomposition of a K-increasing convex function to a K-convex function, where K is a non-empty closed convex cone, subject to cone-convex constraints follows. Strong duality and optimality conditions are derived also for this problem, as well as for its special case when the cone constraints are removed. On this last problem is based the application we deliver. We rediscover the formula of the conjugate of the composition of two functions, giving weaker conditions under which it holds than in the literature. A concrete example where our theoretical results are applicable is the calculation of the conjugate function of $1/F$ for any strictly positive concave function defined on a convex set $F : E \to \mathbb{R}$, which ends the presentation.

References

[1] Boţ, R.I., Grad, S.M., Wanka, G. (2003): *Fenchel-Lagrange versus geometric duality in convex optimization*, submitted.

[2] Boţ, R. I., Kassay, G., Wanka, G. (2003): *Strong duality for generalized convex optimization problems*, to appear in Journal of Optimization Theory and Applications.

[3] Boţ, R. I., Wanka, G. (2004): *An analysis of some dual problems in multiobjective optimization I*, Optimization 53(3), pp. 281–300.

[4] Boţ, R. I., Wanka, G. (2004): *An analysis of some dual problems in multiobjective optimization II*, Optimization 53(3), pp. 301–324.

[5] Boţ, R. I., Wanka, G. (2003): *Duality for composed convex functions with applications in location theory*, in: Multi-Criteria- und Fuzzy-Systeme in Theorie und Praxis, edited by W. Habenicht, B. Scheubrein, R. Scheubrein, Deutscher Universitäts-Verlag, Wiesbaden, pp. 1–18.

[6] Combari, C., Laghdir, M., Thibault, L. (1994): *Sous-différentiels de fonctions convexes composées*, Annales des Sciences Mathématiques du Québec 18(2), pp. 119–148.

[7] Elster, K.-H., Reinhardt, R., Schäuble, M., Donath, G. (1977): *Einführung in die nichtlineare Optimierung*, Mathematisch-Naturwissenschaftliche Bibliothek, BSB B.G. Teubner Verlagsgesellschaft, Leipzig.

[8] Frenk, J. B. G., Kassay, G. (2003): *Lagrangian duality and convexlike functions*, to appear in Journal of Optimization Theory and Applications.

[9] Frenk, J. B. G., Kassay, G. (1999): *On classes of generalized convex functions, Gordan-Farkas type theorems, and Lagrangian duality*, Journal of Optimization Theory and Applications 102(2), pp. 315–343.

[10] Hiriart-Urruty, J.-B., Lemaréchal, C. (1993): *Convex analysis and minimization algorithms. II. Advanced theory and bundle methods*, Springer-Verlag, Berlin.

[11] Hiriart-Urruty, J.-B., Martínez-Legaz, J.-E. (2003): *New formulas for the Legendre-Fenchel transform*, Journal of Mathematical Analysis and Applications 288(2), pp. 544–555.

[12] Kutateladže, S. S. (1977): *Changes of variables in the Young transformation*, Soviet Mathematics Doklady 18(2), pp. 1039–1041.

[13] Lemaire, B. (1985): *Application of a subdifferential of a convex composite functional to optimal control in variational inequalities*, in: Lecture Notes in Economics and Mathematical Systems 255, Springer Verlag, Berlin, pp. 103–117.

[14] Levin, V. L. (1970): *Sur le Sous-Différentiel de Fonctions Composées*, Doklady Akademia Nauk 194, pp. 28–29.

[15] Rockafellar, R.T. (1970): *Convex analysis*, Princeton University Press, Princeton.

[16] Volle, M. (2002): *Duality principles for optimization problems dealing with the difference of vector-valued convex mappings*, Journal of Optimization Theory and Application 114(1), pp. 223–241.

[17] Wanka, G., Boţ, R. I. (2002): *A new duality approach for multiobjective convex optimization problems*, Journal of Nonlinear and Convex Analysis 3(1), pp. 41–57.

[18] Wanka, G., Boţ, R. I. (2001): *Multiobjective duality for convex-linear problems II*, Mathematical methods of operations research 53(3), pp. 419–433.

[19] Wanka, G., Boţ, R. I., Grad, S. M. (2003): *Multiobjective duality for convex semidefinite programming problems*, Zeitschrift für Analysis und ihre Anwendungen (Journal for Analysis and its Applications) 22(3), pp. 711–728.