# An Identification of Convolution Operators on Cones

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ABSTRACT. In [6] Simonenko studied properties of convolution type operators on cones in $\mathbb{R}^n$. The purpose of this note is to show that every convolution operator on a suitable cone in $\mathbb{R}^n$ or $\mathbb{Z}^n$ can be identified with a standard Wiener-Hopf operator, i.e. a convolution operator on $\mathbb{R}^n_+$ or $\mathbb{Z}^n_+$, respectively. We demonstrate this identification and give explicit formulae for the convolution kernels and symbols of these Wiener-Hopf operators.

## 1 Introduction

To mention only one example, the study of the finite section method

$$P_{\tau\Omega} A P_{\tau\Omega}\, u_\tau \;=\; P_{\tau\Omega}\, b, \qquad \tau \to \infty \tag{1}$$

for the convolution (type) equation $Au = b$, where $\tau > 0$, $\Omega$ is a polytope in $\mathbb{R}^n$ and $P_{\tau\Omega}$ is the operator of multiplication by the characteristic function of $\tau\Omega = \{\tau\omega : \omega \in \Omega\}$, leads to the study of convolution operators on cones (see [2, 3, 4, 5]). Hereby, let $C_1, ..., C_k$ denote the collection of cones in $\mathbb{R}^n$ which $\Omega$ locally coincides with at its respective vertices $v_1, ..., v_k$. The operators to be studied in connection with (1) are the compressions of $A$ onto $C_1, ..., C_k$.

If the cone $C \subset \mathbb{R}^n$ has exactly $n$ facets (which is the minimum number for full-dimensional pointed cones), it can clearly be interpreted as an affine-linear deformation of the first orthant $\mathbb{R}^n_+ := [0, \infty)^n$. By means of this deformation, the compression of a convolution operator to $C$ can be identified with the compression of an associated convolution operator to $\mathbb{R}^n_+$, which is a standard Wiener-Hopf operator then. We will demonstrate this identification for convolution operators on $L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$. We will also discuss the discrete case $\ell^p(\mathbb{Z}^n)$ which is slightly more sophisticated! Here the convolution operators are so-called Laurent operators, and the Wiener-Hopf operators are also referred to as Toeplitz operators. In both cases, we give a full description of the associated Wiener-Hopf operator.

## 2 The Function Case

We first discuss the case $L^p := L^p(\mathbb{R}^n)$ with $1 \le p \le \infty$.

## 2.1 Convolution Operators

Given a function $k \in L^1$, let $Fk$ refer to its Fourier transform

$$\Big(Fk\Big)(z) = \int_{\mathbb{R}^n} k(x)\, e^{i(x,z)}\, dx, \qquad z \in \mathbb{R}^n,$$

and denote the set of functions $\{Fk : k \in L^1\}$ by $FL^1$. With every function $a = Fk$, one can associate a *convolution operator* $\mathring{W}_a$ acting on $L^p$ by

$$\Big(\mathring{W}_a u\Big)(t) \ := \ \int_{\mathbb{R}^n} k(t-s)u(s)\, ds, \qquad t \in \mathbb{R}^n,$$

and say that the function $a$ is the *symbol* of the operator $\mathring{W}_a$, while $k$ is referred to as the *convolution kernel* of $\mathring{W}_a$.

For every bounded and measurable set $U \subset \mathbb{R}^n$, let $P_U$ denote the operator of multiplication by the characteristic function of $U$. The operator $P_U A P_U$ is called *compression* of an operator $A$ to $U$. The compression of $\mathring{W}_a$ to the first orthant $\mathbb{R}^n_+$ is referred to as the *Wiener-Hopf operator* $W_a$.

**Remark 2.1** The operators $\mathring{W}_a$ and $W_a$ are labelled by their symbol $a$ – rather than by their kernel $k$ – because the function $a$ is the most convenient object in order to study their properties, including spectra and essential spectra (see [1], for instance). □

## 2.2 Cones

Given vectors $a_0, a_1, ..., a_n \in \mathbb{R}^n$, where $a_1, ..., a_n$ are linearly independent, we denote by $M \in \mathbb{R}^{n\times n}$ the matrix with columns $a_1, ..., a_n$. Note that $M$ is invertible. Clearly,

$$C \ := \ a_0 + \operatorname{cone}\{a_1, ..., a_n\} \ = \ a_0 + M\mathbb{R}^n_+ \tag{2}$$

is a full-dimensional pointed cone (with vertex $a_0$) with $n$ facets. Conversely, every such cone can be written in the form (2).

As (2) gives a bijection between $C$ and $\mathbb{R}^n_+$, we can – in the same manner – construct a linear bijection $T : L^p(C) \to L^p(\mathbb{R}^n_+)$ by $(Tu)(x) := u(a_0 + Mx),\ x \in \mathbb{R}^n_+$.

## 2.3 Convolutions on Cones

Take some cone $C$ as in (2) and some $k \in L^1$. The compression

$$A \ := \ P_C \mathring{W}_a P_C \in L^p(C), \qquad a = Fk$$

of $\mathring{W}_a$ to the cone $C$ can be identified with a Wiener-Hopf operator

$$\tilde{A} \ := \ P_{\mathbb{R}^n_+} \mathring{W}_{\tilde{a}} P_{\mathbb{R}^n_+} \ = \ W_{\tilde{a}} \in L^p(\mathbb{R}^n_+)$$

via the linear bijection $T$ for some $\tilde{a} \in FL^1$. The kernel $\tilde{k}$ and the symbol $\tilde{a} = F\tilde{k}$ of $\tilde{A}$ can be easily calculated from $k$, $a$ and the matrix $M$:

**Theorem 2.2** *$A = T^{-1}\tilde{A}T$, where $\tilde{k}(x) = (\det M)\, k(Mx)$ and $\tilde{a}(z) = a(M^{-\top}z)$.*

$$\begin{array}{ccc} L^p(C) & \xrightarrow{\quad A \quad} & L^p(C) \\ T \big\downarrow & & \big\uparrow T^{-1} \\ L^p(\mathbb{R}^n_+) & \xrightarrow{\quad \tilde{A} \quad} & L^p(\mathbb{R}^n_+) \end{array}$$

*Proof.* Let $u \in L^p(C)$, and write $s, t \in C$ as $a_0 + Mx$ and $a_0 + My$, respectively, with $x, y \in \mathbb{R}^n_+$. Then

$$\begin{aligned}
\Big(TAu\Big)(y) &= \Big(Au\Big)(a_0 + My) \\
&= \Big(Au\Big)(t) \\
&= \int_C k(t-s)\, u(s)\; ds \\
&= \int_{\mathbb{R}^n_+} k((a_0 + My) - (a_0 + Mx))\, u(a_0 + Mx)\; d(a_0 + Mx) \\
&= \int_{\mathbb{R}^n_+} (\det M)\, k(M(y-x))\, u(a_0 + Mx)\; dx \\
&=: \int_{\mathbb{R}^n_+} \tilde{k}(y-x)\, u(a_0 + Mx)\; dx \\
&= \Big(\tilde{A}Tu\Big)(y),
\end{aligned}$$

whence $TA = \tilde{A}T$, i.e. $A = T^{-1}\tilde{A}T$, where the kernel of $\tilde{A}$ is subject to $\tilde{k}(x) = (\det M)\, k(Mx)$ for every $x \in \mathbb{R}^n$. It remains to check the connection between $a$ and $\tilde{a}$:

$$\begin{aligned}
\tilde{a}(z) &= \Big(F\tilde{k}\Big)(z) \\
&= \int_{\mathbb{R}^n} \tilde{k}(x)\, e^{i(x,z)}\; dx \\
&= \int_{\mathbb{R}^n} (\det M)\, k(Mx)\, e^{i(x,z)}\; dx \\
&= \int_{\mathbb{R}^n} k(t)\, e^{i(M^{-1}t,z)}\; dt \\
&= \int_{\mathbb{R}^n} k(t)\, e^{i(t,M^{-\top}z)}\; dt \\
&= \Big(Fk\Big)(M^{-\top}z) \\
&= a(M^{-\top}z) \;\blacksquare
\end{aligned}$$

# 3 The Discrete Case

Now we pass to the case $\ell^p := \ell^p(\mathbb{Z}^n)$ with $1 \le p \le \infty$. In analogy to the function case, put $\mathbb{Z}^n_+ := \{0, 1, 2, ...\}^n$. Moreover, let $\mathbb{T}$ denote the complex unit circle. Although some details will turn out to be a bit more sophisticated, we will essentially be able to do the same things as in the function case.

## 3.1 Discrete Convolution Operators

Suppose we are given a sequence $(a_\gamma)_{\gamma\in\mathbb{Z}^n}$ of complex numbers. The discrete convolution operator, alias *Laurent operator* $L(a)$, acts on $\ell^p$ by the rule

$$\Big(L(a)u\Big)_\alpha \;=\; \sum_{\beta\in\mathbb{Z}^n} a_{\alpha-\beta}\, u_\beta, \qquad \alpha \in \mathbb{Z}^n.$$

Its symbol is the function $a : \mathbb{T}^n \to \mathbb{C}$ acting by

$$a(t_1, ..., t_n) \;:=\; \sum_{\gamma\in\mathbb{Z}^n} a_\gamma\, t_1^{\gamma_1} \cdots t_n^{\gamma_n}, \qquad t_i \in \mathbb{T}.$$

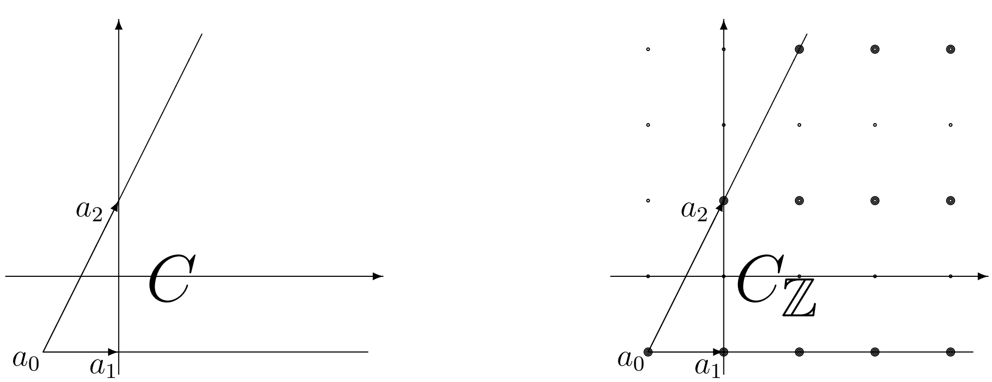
For brevity, we define $(t_1, ..., t_n)^{(\gamma_1,...,\gamma_n)} := t_1^{\gamma_1} \cdots t_n^{\gamma_n}$, to get $a(t) = \sum_{\gamma\in\mathbb{Z}^n} a_\gamma\, t^\gamma$, i.e. $(a_\gamma)_{\gamma\in\mathbb{Z}^n}$ are the Fourier coefficients of $a$. The classes of functions $a$ for which $L(a)$ is a bounded linear operator on $\ell^p$ are the so-called multiplicator algebras $M^p$ (for instance, see [1, §2.3ff]). The compression of $L(a)$ to $\mathbb{Z}^n_+$ is the discrete Wiener-Hopf operator, alias *Toeplitz operator* $T(a)$.

## 3.2 Discrete Cones

For the definition of a discrete cone, we essentially replace $\mathbb{R}$ by $\mathbb{Z}$ in Section 2.2. So we have integer entries in $a_0$ and $M$, and

$$C_\mathbb{Z} \;:=\; a_0 + \text{cone}_\mathbb{Z}\{a_1, ..., a_n\} \;=\; a_0 + M\mathbb{Z}^n_+. \tag{3}$$

We will say that $C_\mathbb{Z}$ from (3) is *fully occupied*, if $C_\mathbb{Z} = C \cap \mathbb{Z}^n$ with $C$ from (2).




An illustration of the cone $C$ and the discrete cone $C_\mathbb{Z}$ for $a_0 = \binom{-1}{-1}$, $a_1 = \binom{1}{0}$ and $a_2 = \binom{1}{2}$. Note that $C_\mathbb{Z}$ is not fully occupied! The parallelogram spanned by $a_1$ and $a_2$ is too large.

**Proposition 3.1** *The following conditions are equivalent:*

- *(i)* $C_{\mathbb{Z}}$ *is fully occupied,*
- *(ii)* $M\mathbb{Z}^n = \mathbb{Z}^n$,
- *(iii)* $M^{-1}$ *is an integer matrix,*
- *(iv)* $\det M = \pm 1$,
- *(v)* *the parallelotope spanned by* $a_1, ..., a_n$ *has volume* $1$.

*Proof.*

$(i) \Leftrightarrow (ii)$: Since $a_0 \in \mathbb{Z}^n$, $(i)$ is equivalent to

$$M\mathbb{Z}_+^n \;=\; (M\mathbb{R}_+^n) \cap \mathbb{Z}^n. \tag{4}$$

Set-subtracting (4) from itself, we get

$$M\mathbb{Z}^n \;=\; (M\mathbb{R}^n) \cap \mathbb{Z}^n. \tag{5}$$

On the other hand, (5) implies (4) as we see by taking intersection with $M\mathbb{R}_+^n$ on both sides of (5). But since $M$ is invertible, we have $M\mathbb{R}^n = \mathbb{R}^n$, and hence, (5) is the same as $(ii)$.

$(ii) \Rightarrow (iii)$: The (unique) solutions $s_1, ..., s_n$ of $Ms_i = e_i$ (the $i$-th unit vector) are the columns of $M^{-1}$. By $(ii)$, these are integer vectors.

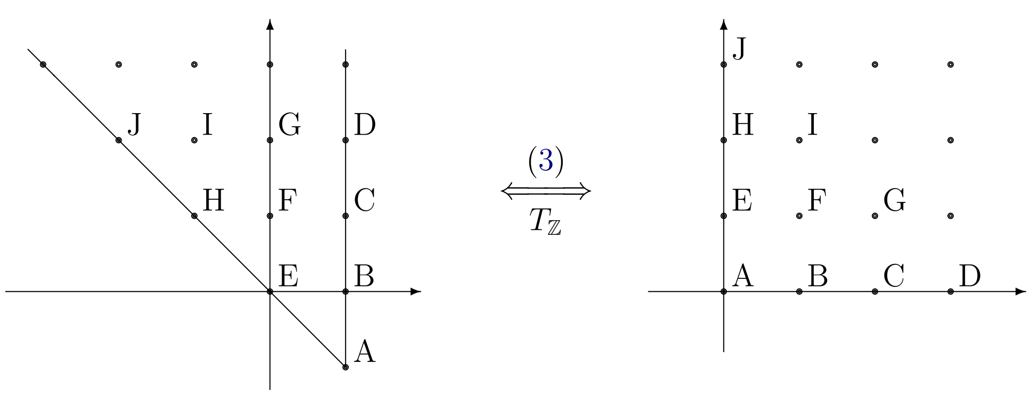
$(iii) \Rightarrow (ii)$: trivial

$(iii) \Rightarrow (iv)$: By $(iii)$, $\det M^{-1}$ is an integer. But $\det M$ is an integer as well, and since their product is 1, both have to be $+1$ or $-1$.

$(iv) \Rightarrow (iii)$: $M^{-1} = (1/\det M)[M_{ji}]_{i,j=1}^n$ shows that $M^{-1}$ is an integer matrix.

$(iv) \Leftrightarrow (v)$: This is trivial since the volume of this parallelotope is $|\det M|$. ■

Again, (3) gives a bijection between $C_{\mathbb{Z}}$ and $\mathbb{Z}_+^n$. So we will construct a linear bijection $T_{\mathbb{Z}} : \ell^p(C_{\mathbb{Z}}) \to \ell^p(\mathbb{Z}_+^n)$ by $(T_{\mathbb{Z}}u)_\alpha := u_{a_0+M\alpha},\ \alpha \in \mathbb{Z}_+^n$.




An illustration of the bijection (3) between the discrete cone $C_{\mathbb{Z}} = \binom{1}{-1} + \mathrm{cone}_{\mathbb{Z}} \left\{ \binom{0}{1}, \binom{-1}{1} \right\}$ and the discrete quarter plane $\mathbb{Z}_+^2$. This identification yields to the bijection $T_{\mathbb{Z}}$ between $\ell^p(C_{\mathbb{Z}})$ and $\ell^p(\mathbb{Z}_+^2)$.

### 3.3 Discrete Convolutions on Discrete Cones

Fix a discrete cone $C_{\mathbb{Z}}$ and a function $a \in M^p$ with Fourier coefficients $(a_\gamma)_{\gamma\in\mathbb{Z}^n}$. The compression

$$A \;:=\; P_{C_{\mathbb{Z}}} L(a) P_{C_{\mathbb{Z}}}$$

of $L(a)$ to $C_{\mathbb{Z}}$ corresponds to a Toeplitz operator

$$\tilde{A} \;:=\; P_{\mathbb{Z}^n_+} L(\tilde{a}) P_{\mathbb{Z}^n_+} \;= T(\tilde{a})$$

via the bijection $T_{\mathbb{Z}}$:

**Theorem 3.2** **a)** *$A = T_{\mathbb{Z}}^{-1} \tilde{A} T_{\mathbb{Z}}$, where $\tilde{a}_\gamma = a_{M\gamma}$.*

**b)** *If $\mathbb{C}_{\mathbb{Z}}$ is fully occupied, then $\tilde{a}(t) = a(t^{s_1}, ..., t^{s_n})$, where* $\begin{bmatrix} | & & | \\ s_1 & \cdots & s_n \\ | & & | \end{bmatrix} = M^{-1}$.

*Proof.* **a)** is almost identic to the proof of the first part of Theorem 2.2.

**b)** By Proposition 3.1, $M\mathbb{Z}^n = \mathbb{Z}^n$. Then for arbitrary $t \in \mathbb{T}^n$,

$$\begin{aligned} \tilde{a}(t) \;=\; \sum_{\gamma\in\mathbb{Z}^n} \tilde{a}_\gamma\, t^\gamma \;=\; \sum_{\gamma\in\mathbb{Z}^n} a_{M\gamma}\, t^\gamma \;&=\; \sum_{\delta\in\mathbb{Z}^n} a_\delta\, t^{M^{-1}\delta} \;=\; \sum_{\delta\in\mathbb{Z}^n} a_\delta\, t^{\delta_1 s_1 + ... + \delta_n s_n} \\ &=\; \sum_{\delta\in\mathbb{Z}^n} a_\delta\, (t^{s_1})^{\delta_1} \cdots (t^{s_n})^{\delta_n} \;=\; a(t^{s_1}, ..., t^{s_n}). \; \blacksquare \end{aligned}$$

# 4 An Example

We will briefly illustrate the results of Theorems 2.2 and 3.2 by an example in the plane, $n = 2$. Therefore, let $a_0$ be an arbitrary integer vector,

$$a_1 \;=\; \begin{pmatrix} 3 \\ 7 \end{pmatrix} \qquad \text{and} \qquad a_2 \;=\; \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Consequently, $M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, $\det M = 1$ and $M^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$. As in Sections 2.2 and 3.2, put

$$C \;:=\; a_0 \,+\, M\mathbb{R}^n_+ \qquad \text{and} \qquad C_{\mathbb{Z}} \;:=\; a_0 \,+\, M\mathbb{Z}^n_+.$$

By Theorem 2.2, the compression of a convolution operator $\mathring{W}_a$ with kernel $k \in L^1$ and symbol $a = Fk$ to the cone $C$ can be identified with a Wiener-Hopf operator $W_{\tilde{a}}$ (on the quarter plane) with kernel $\tilde{k} \in L^1$ and symbol $\tilde{a} = F\tilde{k}$:

$$\tilde{k}(x) \;=\; \tilde{k}(x_1, x_2) \;=\; k(3x_1 + 2x_2\,,\, 7x_1 + 5x_2), \tag{6}$$

$$\tilde{a}(x) \;=\; \tilde{a}(x_1, x_2) \;=\; a(5x_1 - 7x_2\,,\, -2x_1 + 3x_2) \tag{7}$$

By Proposition 3.1, $\mathbb{C}_\mathbb{Z}$ is fully occupied. So both parts of Theorem 3.2 are applicable, and the compression of the Laurent operator $L(a)$ with symbol $a \in M^p$ and Fourier coefficients $(a_\gamma)_{\gamma\in\mathbb{Z}^n}$ to the discrete cone $C_\mathbb{Z}$ can be identified with the Toeplitz operator $T(\tilde{a})$ (on the quarter plane) with symbol $\tilde{a} \in M^p$ and Fourier coefficients $(\tilde{a}_\gamma)_{\gamma\in\mathbb{Z}^n}$:

$$\tilde{a}_\gamma = \tilde{a}_{(\gamma_1,\gamma_2)} = a_{(3\gamma_1+2\gamma_2\,,\,7\gamma_1+5\gamma_2)}, \tag{8}$$

$$\tilde{a}(t) = \tilde{a}(u,v) = a(u^5v^{-7}\,,\,u^{-2}v^3) \tag{9}$$

Note the incidence between (6) and (8), and that between (7) and (9).

# References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