Weakly Singular Integral Operators in Weighted $\mathbf{L}^\infty$–Spaces

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Abstract. We study integral operators on $(-1,1)$ with kernels $k(x,t)$ which may have weak singularities in (x,t) with $x \in N_1$, $t \in N_2$, or $x = t$, where N_1, N_2 are sets of measure zero. It is shown that such operators map weighted $\mathbf{L}^\infty$–spaces into certain weighted spaces of smooth functions, where the degree of smoothness is as higher as smoother the kernel $k(x,t)$ as a function in x. The spaces of smooth function are generalizations of the Ditzian-Totik spaces which are defined in terms of the errors of best weighted uniform approximation by algebraic polynomials.

1 Introduction

In all of what follows we consider an integral operator K on $(-1,1)$,

$$(Kf)(x) = \int_{-1}^{1} k(x,t)\, f(t)\, dt\,, \quad x \in (-1,1)\,,$$

where the kernel function $k(x,t)$ is defined and continuous on $[-1,1]^2 \setminus N$, N a set of measure zero. More precisely, we suppose that there are continuous weight functions

$$v : D(v) \to (0,\infty) \quad \text{and} \quad w : D(w) \to (0,\infty)$$

with $D(v), D(w) \subseteq [-1,1]$ and $\operatorname{meas} D(v) = \operatorname{meas} D(w) = 2$, such that

$$g(x,t) = (x-t)\, v(x)\, k(x,t)\, w(t) \in \mathbf{C}\big([-1,1]^2\big) \;\text{ and }\; g(t,t) = 0. \tag{1.1}$$

This means that $k(x,t)$ is defined and continuous on $[D(v)\times D(w)]\setminus\{(x,t) : x = t\}$ and that $g(x,t)$ can be continuously extended onto $[-1,1]^2$, where the extension vanishes on the diagonal $\{x = t\}$ of $[-1,1]^2$. We will show that, under some additional conditions on v, w and k (namely, $g(x,t)$ has to be smooth enough in x and $g(x_0,t) = 0$ for all zeros x_0 of v),

$$K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{C}^{\gamma,\delta}_{uvw}\big) \quad \text{for certain weights } u \text{ with } \frac{1}{uw} \in \mathbf{L}^1(-1,1)\,, \tag{1.2}$$

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where $\mathbf{C}_{uvw}^{\gamma,\delta}$ can be replaced by $\mathbf{C}_v^{\gamma,\delta}$ if even $v(x)\,k(x,t)\,w(t)$ is continuous on $[-1,1]^2$ and smooth enough in x. (By $\mathcal{L}(\mathbf{X},\mathbf{Y})$ we denote the space of all bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$.) Here, $\mathbf{L}_u^\infty = \{f : fu \in \mathbf{L}^\infty(-1,1)\}$ (endowed with $\|f\|_u = \|fu\|_\infty$) and $\mathbf{C}_v^{\gamma,\delta}$ belongs to a certain scale of subspaces of

$$\mathbf{C}_v = \{f : D(v) \to \mathbb{C} : \; fv \in \mathbf{C}[-1,1]\} \quad (\|f\|_{\mathbf{C}_v} = \|f\|_v = \|fv\|_\infty)$$

which are compactly embedded into $\mathbf{C}_v$. For the precise definition we need the weighted polynomial best approximation errors of $f \in \mathbf{C}_v$,

$$E_n^v(f) := \inf_{P_n \in \Pi_n \cap \mathbf{C}_v} \|f - P_n\|_v\,, \quad \Pi_n = \operatorname{span}\{x^k : k = 0,\dots,n-1\}$$

($E_0^v(f) := \|f\|_v$). Now, for $\gamma > 0$ and $\delta \in \mathbb{R}$,

$$\mathbf{C}_v^{\gamma,\delta} := \left\{ f \in \mathbf{C}_v : \; \|f\|_{v,\gamma,\delta} = \sup_{n=0,1,\dots} E_n^v(f)\,(n+1)^\gamma \ln^\delta(n+2) < \infty \right\}.$$

In the case $v \equiv 1$ we write shortly $\mathbf{C}^{\gamma,\delta}$ and $\|\,.\,\|_{\gamma,\delta}$ instead of $\mathbf{C}_v^{\gamma,\delta}$ and $\|\,.\,\|_{v,\gamma,\delta}$. Let us give some properties of these spaces (see [2] or [1] for the proofs).

Proposition 1.1 *Let $\gamma > 0$, $\delta \in \mathbb{R}$ be fixed. The following assertions hold true.*

(i) $\mathbf{C}_v^{\gamma,\delta}$ *is a Banach spaces which is compactly embedded into* $\mathbf{C}_v$.

(ii) If $\gamma > r > 0$ *and* $s \in \mathbb{R}$ *or* $\gamma = r$ *and* $s > \delta$, *then* $\mathbf{C}_v^{\gamma,\delta}$ *is compactly embedded into* $\mathbf{C}_v^{r,s}$.

(iii) $f \in \mathbf{C}_v$ *belongs to the closure of* $\bigcup_n \Pi_n$ *in* $\mathbf{C}_v^{\gamma,\delta}$ *iff* $\lim_{n\to\infty} E_n^v(f)\, n^\gamma \ln^\delta n = 0$.

The spaces $\mathbf{C}_v^{\gamma,\delta}$ play an important role in the numerical analysis of Cauchy singular integral equations on $(-1,1)$, if these equations are studied in weighted spaces of continuous functions. In case of Jacobi-weights v it is well-known that mapping properties of the type (1.2) are a powerful tool in the study of $\mathbf{C}_v$–convergence of polynomial approximation methods for Cauchy singular integral equations on $(-1,1)$ (see, e.g., [7]). Recently, the known mapping properties of Cauchy singular integral operators in spaces $\mathbf{C}_v^{\gamma,\delta}$ with Jacobi-weight v (see [9]) were generalized to the case of power weights v (see [8]). It is the aim of the author to develop a concept which allows to study weighted uniform convergence of polynomial approximation methods for Cauchy singular integral equations on $(-1,1)$ those right hand sides belong to some space $\mathbf{C}_v^{\gamma,\delta}$ with power weight v. (Thus, the right hand sides may have singularities inside $(-1,1)$.) The purpose of this paper is to go the next step in this direction, which is the generalization of results of the type (1.2), which are known for Jacobi-weights u and v (see [7]), to the case of power weights or even more general weights u and v.

Before we start, we should explain why we do not use weighted Hölder spaces to describe the mapping properties of K, although there exist nice continuity results for Cauchy singular integral operators in such spaces (see [6, Section 9.10],

[4], and [5]). There are several reasons: First of all, for the investigation of polynomial approximation methods for singular integral equations it is natural to use polynomial approximation spaces instead of Hölder-Zygmund spaces which cannot be described equivalently in terms of polynomial approximation errors if spaces on an open curve are considered. Secondly, the mapping properties of Cauchy singular integral operators in the spaces $\mathbf{C}_v^{\gamma,\delta}$ are similar to that in weighted Hölder-Zygmund spaces and can be formulated even under less assumptions on the weight v (see [8]). Last but not least, we will see that it is natural that the images of a weakly singular integral operator K on $(-1,1)$ lie in some space $\mathbf{C}_v^{\gamma,\delta}$, since usually its kernel $k(x,t)$ can be approximated by polynomials of degree less than n in x, which leads to approximations of K by Π_n–valued operators K_n.

Of course, all these reasons are only theoretically of interest as long as we do not have practical criteria to check whether a function $f \in \mathbf{C}_v$ belongs to $\mathbf{C}_v^{\gamma,\delta}$ or not. Recently, such criteria have been found for the case when $v \in \mathbf{C}[-1,1]$ is a generalized Jacobi-weight. We only mention that, in this case, the elements of $\mathbf{C}_v^{\gamma,\delta}$ can be described equivalently with the help of certain weighted Ditzian-Totik type moduli of smoothness. The interested reader can find the details in [3]. Here we will only give the following result which shows that there is some connection between the spaces $\mathbf{C}_v^{\gamma,\delta}$ and Hölder spaces

$$\mathbf{H}^\alpha[-1,1] = \left\{ f \in \mathbf{C}[-1,1] : \|f\|_{\mathbf{H}^\alpha} = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty \right\}, \quad \alpha > 0 \,.$$

Proposition 1.2 ([8], Lemma 2.3, Corollary 2.5) *Let* $v(x) = \prod_{j=1}^{M} |x - x_j|^{\beta_j}$ *(*$M \in \mathbb{N} \cup \{0\}$*) with* $x_j \in [-1,1]$ *and* $\beta_j > 0$*. (Set* $v = 1$ *if* $M = 0$*.)*

(i) If $f \in \mathbf{C}_v^{\gamma,\delta}$*, then* $(fv)(x_j) = 0$ *for all* j *and* $fv \in \mathbf{H}^\alpha[-1,1]$ *for some* $\alpha = \alpha(\gamma,\delta) > 0$*, where* $\|fv\|_{\mathbf{H}^\alpha} \le c\,\|f\|_{v,\gamma,\delta}$ *with* c *independent of* f*.*

(ii) If $fv \in \mathbf{H}^\alpha[-1,1]$ *and* $(fv)(x_j) = 0$ *for all* j*, then* $f \in \mathbf{C}_v^{\gamma,0}$ *for some* $\gamma = \gamma(\alpha) > 0$*, where* $\|f\|_{v,\gamma,0} \le c\,\|fv\|_{\mathbf{H}^\alpha}$ *with* c *independent of* f*.*

The paper is divided in two parts. In Section 2 we study the case of operators K with kernels $k(x,t)$ which may only have singularities on axis-parallel lines, but not on the diagonal $\{x = t\}$. In Section 3 we treat operators those kernels $k(x,t)$ satisfy (1.1).

2 Kernels with singularities on axis-parallel lines

First we consider the case of a kernel $k \in \mathbf{C}(D(v) \times D(w))$. More precisely, instead of (1.1) we suppose that even

$$v(x)\,k(x,t)\,w(t) \in \mathbf{C}\big([-1,1]^2\big)\,. \tag{2.1}$$

Thus, if v has zeros in $x = x_i$ and w has zeros in $t = t_j$, then $k(x,t)$ may have singularities on the lines $\{x_i\} \times [-1,1]$ and $[-1,1] \times \{t_j\}$.

Let $\mathcal{K}(\mathbf{X},\mathbf{Y})$ denote the set of all compact linear operators from $\mathbf{X}$ into $\mathbf{Y}$.

Theorem 2.1 *If (2.1) is satisfied, then, for all*

$$u : D(w) \to (0,\infty) \quad \textit{with} \quad \frac{1}{u\,w} \in \mathbf{L}^1(-1,1)\,, \tag{2.2}$$

$K \in \mathcal{K}(\mathbf{L}^\infty_u, \mathbf{C}_v)$. *If, in addition,* $v \in \mathbf{C}[-1,1]$ *(continuous extension) and*

$$\sup_{t\in D(w)} \|k(\,.\,,t)\,w(t)\|_{v,\gamma,\delta} < \infty\,, \tag{2.3}$$

then $K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{C}^{\gamma,\delta}_v\big)$.

Proof. The first assertion is only a reformulation of a classical result in which $\mathbf{C}[-1,1]$ appears as image space. Although the theorem of Arcela-Ascoli is used in the standard proof, it is worth to present the following approximation-theoretical derivation, since this will give an idea how to obtain the second assertion. Let $h(x,t)$ be the continuous extension of $v(x)\,k(x,t)\,w(t)$. We may write

$$(Kf)(x) = \frac{1}{v(x)} \int_{-1}^{1} h(x,t)\,f(t)\,\frac{dt}{w(t)}\,. \tag{2.4}$$

From (2.2) it follows that K is a bounded linear operator from $\mathbf{L}^\infty_u$ into

$$\mathbf{B}_v = \left\{ g : D(v) \to \mathbb{C} :\ \|g\|_{\mathbf{B}_v} := \sup_{x\in D(v)} |g(x)v(x)| < \infty \right\},$$

where the operator norm satisfies the estimate

$$\big\|K\big\|_{\mathbf{L}^\infty_u \to \mathbf{B}_v} \le c\,\|h\|_{\mathbf{C}([-1,1]^2)} \qquad \left(c = \int_{-1}^{1} \frac{dt}{u(t)w(t)} \right). \tag{2.5}$$

By a theorem of Weierstrass we can find polynomials $h_n(x,t)$ of degree less than n in both variables such that $h_n \to h$ uniformly on $[-1,1]^2$. Now we define K_n by

$$(K_n f)(x) := \frac{1}{v(x)} \int_{-1}^{1} h_n(x,t)\,f(t)\,\frac{dt}{w(t)}\,. \tag{2.6}$$

Also K_n is an operator of the type (2.4) and, consequently, $K_n \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_v\big)$. Moreover, K_n is a finite rank operator, since $K_n f \in v^{-1}\Pi_n$ for all $f \in \mathbf{L}^\infty_u$ (particularly, $K_n(\mathbf{L}^\infty_u) \subseteq \mathbf{C}_v$). It follows

$$K_n \in \mathcal{K}\big(\mathbf{L}^\infty_u, \mathbf{C}_v\big) \quad \text{for all } \ n\,. \tag{2.7}$$

Now we consider $K - K_n$. Again this an operator of the type (2.4) (replace $h(x,t)$ by $h(x,t) - h_n(x,t)$) and (2.5) shows that

$$\big\|K - K_n\big\|_{\mathbf{L}^\infty_u \to \mathbf{B}_v} \le c\,\|h - h_n\|_{\mathbf{C}([-1,1]^2)} \longrightarrow 0 \quad \text{for } \ n \to \infty\,. \tag{2.8}$$

Together with (2.7) we obtain $K \in \mathcal{K}\big(\mathbf{L}^\infty_u, \mathbf{C}_v\big)$. If we would know that h can be approximated by polynomials h_n with a certain order of convergence, for example $n^\gamma \|h - h_n\|_{\mathbf{C}([-1,1]^2)} \ln^\delta(n+1) \le \text{const}$, then the above proof would even imply that the images Kf multiplied by v could be uniformly approximated of the same order by polynomials, for example $vK \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{C}^{\gamma,\delta}\big)$. But this is not exactly what we need in order to prove the second assertion. However, now it is clear what to do: If $v \in \mathbf{C}[-1,1]$, then we are looking for functions

$$h_n(x,t) = v(x)\big[c_0^{(n)}(t) + c_1^{(n)}(t)\,x + \ldots + c_{n-1}^{(n)}(t)\,x^{n-1}\big] \ \text{ with } c_i^{(n)} \in \mathbf{C}[-1,1] \tag{2.9}$$

(which implies that K_n defined by (2.6) maps $\mathbf{L}^\infty_u$ into Π_n, particularly, $K_n(\mathbf{L}^\infty_u) \subseteq \mathbf{C}_v$) such that

$$\|h - h_n\|_{\mathbf{C}([-1,1]^2)} \le \frac{c}{(n+1)^\gamma \ln^\delta(n+2)} \quad \text{for all } \ n \in \mathbb{N} \cup \{0\}\,, \tag{2.10}$$

where c is independent of n. If we could find such h_n, then estimate (2.8) would imply the second assertion. In the next lemma we will see that, under the additional condition (2.3) on $k(x,t)$, the required h_n exist. ∎

Lemma 2.2 *Let $v \in \mathbf{C}[-1,1]$ (continuous extension). If $h \in \mathbf{C}([-1,1]^2)$ and*

$$C := \sup_{t \in D(w)} \big\|v^{-1}(\,.\,)h(\,.\,,t)\big\|_{v,\gamma,\delta} < \infty\,,$$

then there are functions $h_n(x,t)$ of the form (2.9) such that (2.10) holds true.

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and let $P_n(\,.\,,t) \in \Pi_n$ ($t \in D(w)$ fixed) denote a polynomial of best approximation to $v^{-1}(\,.\,)h(\,.\,,t)$ in the norm of $\mathbf{C}_v$. Then,

$$\|h(\,.\,,t) - v(\,.\,)P_n(\,.\,,t)\|_\infty = E_n^v\big(v^{-1}(\,.\,)h(\,.\,,t)\big) \le \frac{C}{(n+1)^\gamma \ln^\delta(n+2)}\,.$$

Further, choose $\delta_n > 0$ such that

$$|h(x,t) - h(x,t_0)| \le \frac{1}{(n+1)^\gamma \ln^\delta(n+2)} \quad \text{for } \ t, t_0, x \in [-1,1] \ \text{ with } \ |t - t_0| < \delta_n.$$

Then it follows

$$|h(x,t) - v(x)P_n(x,t_0)| \le \frac{1+C}{(n+1)^\gamma \ln^\delta(n+2)} \tag{2.11}$$

for all $(x,t) \in [-1,1]^2$ and $t_0 \in D(w)$ with $|t - t_0| < \delta_n$. Now we choose numbers $t_k \in D(w)$, $k = 1, \ldots, m$ (t_k and m depending on n), such that

$$\begin{aligned} &-1 < t_1 < t_2 < \ldots < t_m < 1 \quad \text{and} \\ &\max\{t_1 + 1,\, t_2 - t_1\,, \ldots\,,\, t_m - t_{m-1},\, 1 - t_m\} < \delta_n\,. \end{aligned}$$

Then we define

$$h_n(x,t) = v(x) \sum_{k=0}^{m+1} P_n(x, \widetilde{t}_k)\, B_k(t)\,, \quad \widetilde{t}_k = \begin{cases} t_k\,, & 1 \le k \le m\,, \\ t_1\,, & k = 0\,, \\ t_m\,, & k = m+1\,, \end{cases}$$

where B_k, $k = 0, \ldots, m+1$, are the linear B-splines with respect to the partition $t_0 = -1$, $t_1, \ldots, t_m$, $t_{m+1} = 1$, i.e.,

$$B_0(t) = \frac{\max\{0,\, t_1 - t\}}{t_1 + 1}\,, \quad B_{m+1}(t) = \frac{\max\{0,\, t - t_m\}}{1 - t_m}\,,$$

$$B_k(t) = \max\left\{0\,,\ \min\left\{\frac{t - t_{k-1}}{t_k - t_{k-1}}\,,\ \frac{t_{k+1} - t}{t_{k+1} - t_k}\right\}\right\}\,, \quad k = 1, \ldots, m\,.$$

Clearly, $h_n(x,t)$ is a function of the required form. Moreover, $\sum_{k=0}^{m+1} B_k = 1$ on $[-1,1]$ and, consequently,

$$h(x,t) - h_n(x,t) = \sum_{k=0}^{m+1} \left[h(x,t) - v(x) P_n(x, \widetilde{t}_k)\right] B_k(t)\,.$$

If we take into account that this sum has at most two non-zero addends for every t and that the distance between $\widetilde{t}_k$ and any point $t \in [-1,1]$ of the support of $B_k(t)$ is less than δ_n, then, in view of (2.11), we obtain (2.10) (with $c = 2 + 2C$).

■

Remark 2.3 *The above proof shows that the assertions of Theorem 2.1 remain true if $\mathbf{L}^\infty_u$ is replaced by $\mathbf{L}^1_{w^{-1}} := \{f : f w^{-1} \in \mathbf{L}^1(-1,1)\}$ and that more general approximation spaces of the type $\mathbf{C}^{\{a_n\}}_v = \{f \in \mathbf{C}_v : \sup_n a_n E^v_n(f) < \infty\}$ can be considered instead of $\mathbf{C}^{\gamma,\delta}_v$. We have restricted ourselves to $\mathbf{L}^\infty_u$ and $\mathbf{C}^{\gamma,\delta}_v$ only since these are the spaces of main interest if one wants to study singular integral equations in weighted spaces of continuous functions (see, e.g., [7]).*

3 Kernels with additional singularities on the diagonal

Now we consider kernels $k(x,t)$ for which

$$h(x,t) = (t - x)\, v(x)\, k(x,t)\, w(t) \in \mathbf{C}\big([-1,1]^2\big) \ \text{ and } \ h(t,t) = 0,\ t \in [-1,1]. \quad (3.1)$$

Here we assume that $v \in \mathbf{C}[-1,1]$ is a power weight with $v^{-1} \in \mathbf{L}^1(-1,1)$, i.e.,

$$v(x) = \prod_{j=1}^{M} |x - x_j|^{\beta_j} \quad \text{with} \quad x_j \in [-1,1] \ \text{ and } \ \beta_j \in (0,1)\,. \quad (3.2)$$

(We set $D(v) = [-1,1] \setminus \{x_j\}_{j=1}^M$. If $M = 0$, then $v := 1$.) We further assume that there is a second power weight $\sigma(t) = \prod_{i=1}^{N} |t - t_i|^{\alpha_i}$ ($t_i \in [-1,1]$, $N \in \mathbb{N} \cup \{0\}$) such that, with some constant $c > 0$,

$$w(t) \ge c\, \sigma(t) \ \text{ for all } \ t \in D(w)\,, \quad \alpha_i > 0 \ \text{ for all } \ i\,, \quad \frac{1}{\sigma\, v} \in \mathbf{L}^1(-1,1)\,. \quad (3.3)$$

In the sequel we shall denote by c positive constants that may have different values at different places. By $c \neq c(f, x, \dots)$ we will indicate that c is independent of $f, x \dots$.

Lemma 3.1 ([8], Lemma 3.4 and its proof) *Let v be a weight of the form (3.2). There is a constant $c \neq c(g, x)$ such that*

$$\int_{-1}^{1} \left| \frac{g(x,t)}{t-x} \right| \frac{dt}{v(t)} \leq \frac{c}{v(x)} \left(\|g(x, \,.\,)\|_\infty + \int_{-1}^{1} \left| \frac{g(x,t)}{t-x} \right| dt \right)$$

for all $x \in D(v)$ and all $g : [-1,1]^2 \to \mathbb{C}$ with $g(x, \,.\,) \in \mathbf{L}^\infty(-1,1)$, $x \in D(v)$.

Theorem 3.2 *Let (3.1)–(3.3) be satisfied and suppose that*

$$\sup_{t \in D(w)} \|(t - \,.\,)\, k(\,.\,, t)\, w(t)\|_{v,\gamma,\delta} < \infty \,.$$

Then, $K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{C}^{\gamma,\delta-1}_{v\sigma}\big)$ for all

$$u : D(w) \to (0,\infty) \quad \textit{with} \quad u\,w \geq c\,\sigma \;\; \textit{a.e. on } (-1,1) \,. \tag{3.4}$$

Particularly, $K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{C}^{\gamma,\delta-1}_{uvw}\big)$ for all bounded $u : D(w) \to (0,\infty)$ for which uw is a power weight with nonnegative exponents and $(uvw)^{-1} \in \mathbf{L}^1(-1,1)$. (Set $\sigma = uw$ in this case.)

Proof. In view of Proposition 1.2, there exists some $\eta > 0$ such that $h(\,.\,,t) \in \mathbf{H}^\eta[-1,1]$ for all $t \in D(w)$, where $\sup_{t \in D(w)} \|h(\,.\,,t)\|_{\mathbf{H}^\eta} < \infty$. It follows

$$|h(x,t)| = |h(x,t) - h(t,t)| \leq c\,|x-t|^\eta \quad \text{for all} \;\; (x,t) \in [-1,1] \times D(w) \,. \tag{3.5}$$

Together with $|f(t)| w^{-1}(t) \leq c\,\|f\|_u\, \sigma^{-1}(t)$ a.e. on $(-1,1)$ and Lemma 3.1 (applied with σ instead of v) this shows that the absolute value of

$$(Kf)(x) = \frac{1}{v(x)} \int_{-1}^{1} \frac{h(x,t)}{t-x}\, f(t)\, \frac{dt}{w(t)} \tag{3.6}$$

can be estimated by $c\,\|f\|_u\, v^{-1}(x) \sigma^{-1}(x)$. Thus, $K \in \mathcal{L}\big(\mathbf{L}^\infty_u, \mathbf{B}_{v\sigma}\big)$. Now we approximate $h(x,t)$ by

$$g_n(x,t) = h_n(x,t) - \frac{v(x)}{v(t)}\, h_n(t,t) \,, \quad n \in \mathbb{N} \,,$$

where $h_n(x,t)$ is the function from Lemma 2.2. If we replace $h(x,t)$ by $g_n(x,t)$ in (3.6), then we obtain an operator K_n which maps $\mathbf{L}^\infty_u$ into Π_{n-1}, since its kernel

$$\frac{v^{-1}(x)\, h_n(x,t) - v^{-1}(t)\, h_n(t,t)}{(t-x)\, w(t)} = - \sum_{k=1}^{n-1} \frac{\left[v^{-1}(\,.\,) h_n(\,.\,,t)\right]^{(k)}_{|x=t}}{k!} \, \frac{(x-t)^{k-1}}{w(t)}$$

is a polynomial of degree less than $n-1$ in x the coefficients of which are $\mathbf{L}_w^\infty$–functions in t. We have to estimate the norm of $K - K_n$. For this aim, we introduce the intervals

$$I_{n,x} = \left[x - \frac{1+x}{n^s} \, , \, x + \frac{1-x}{n^s} \right] ,$$

where $s > 0$ is some sufficiently large constant. (The following considerations will show how big s must be.) Let $\chi_{n,x}(t)$ be the characteristic function of $I_{n,x}$ and let $f \in \mathbf{L}_u^\infty$. Then, for all $x \in D(v)$,

$$\begin{aligned}
\big|[(K-K_n)f](x)\big| \le c\,\|f\|_u \Bigg[& \frac{1}{v(x)} \int_{-1}^{1} \left| \frac{\chi_{n,x}(t)\, h(x,t)}{t-x} \right| \frac{dt}{\sigma(t)} \\
& + \int_{-1}^{1} \left| \frac{\chi_{n,x}(t)\, [v^{-1}(x)\, h_n(x,t) - v^{-1}(t)\, h_n(t,t)]}{t-x} \right| \frac{dt}{\sigma(t)} \\
& + \frac{1}{v(x)} \int_{-1}^{1} \left| \frac{[1-\chi_{n,x}(t)]\,[h(x,t) - h_n(x,t)]}{t-x} \right| \frac{dt}{\sigma(t)} \\
& + \int_{-1}^{1} \left| \frac{1-\chi_{n,x}(t)}{t-x} \right| \frac{|h_n(t,t)|}{v(t)\sigma(t)}\, dt \quad \Bigg] \\
=: c\,\|f\|_u & \left[\frac{1}{v(x)}\, I_1 + I_2 + \frac{1}{v(x)}\, I_3 + I_4 \right] .
\end{aligned}$$

For $t \in I_{n,x}$ we have, by (3.5), $|h(x,t)| \le c\,|t-x|^\eta \le c\,n^{-s\eta} \le c\,n^{-\gamma} \ln^{-\delta}(n+1)$ (supposed that $s > \gamma/\eta$). Together with Lemma 3.1 it follows

$$\begin{aligned}
I_1 \;&\le\; \frac{c}{\sigma(x)} \left(\frac{1}{n^\gamma \ln^\delta(n+1)} + \int_{I_{n,x}} |t-x|^{\eta-1} dt \right) \\
&\le\; \frac{c}{\sigma(x)} \left(\frac{1}{n^\gamma \ln^\delta(n+1)} + \frac{1}{n^{s\eta}} \right) \le \frac{c}{\sigma(x)}\, \frac{1}{n^\gamma \ln^\delta(n+1)} \, .
\end{aligned}$$

To estimate I_2 we use that

$$\begin{aligned}
\chi_{n,x}(t)\, |v^{-1}(x)\, h_n(x,t) - v^{-1}(t)\, h_n(t,t)| \;&\le\; \big\| [v^{-1}(\,.\,)\, h_n(\,.\,,t)]' \big\|_\infty\, \chi_{n,x}(t)\, |x-t| \\
&\le\; \frac{2 \big\| [v^{-1}(\,.\,)\, h_n(\,.\,,t)]' \big\|_\infty}{n^s} \, .
\end{aligned}$$

If we take into account that $v^{-1}(\,.\,)\, h_n(\,.\,,t)$ is a polynomial of degree less than n and that Markov's inequality $\|P_n'\|_\infty \le n^2 \|P_n\|_\infty$ and Schur's inequality $\|P_n\|_\infty \le c\,n^\mu \|P_n\|_v$, $\mu = \mu(v) > 0$ some constant (see [10, (7.33)]), hold true for all $P_n \in \Pi_n$, then we obtain

$$\begin{aligned}
\big\| [v^{-1}(\,.\,)\, h_n(\,.\,,t)]' \big\|_\infty \;&\le\; c\,n^{2+\mu} \|h_n(\,.\,,t)\|_\infty \\
&\le\; c\,n^{2+\mu} \big(\|h_n - h\|_{\mathbf{C}([-1,1]^2)} + \|h\|_{\mathbf{C}([-1,1]^2)} \big) \le c\,n^{2+\mu} \, .
\end{aligned}$$

Together with Lemma 3.1 it follows

$$I_2 \leq \frac{c}{\sigma(x)} \left(\frac{n^{2+\mu}}{n^s} + n^{2+\mu} \int_{I_{n,x}} dt \right) \leq \frac{c}{\sigma(x)} \, \frac{1}{n^\gamma \ln^\delta(n+1)}$$

(supposed that $s > \gamma + \mu + 2$). In I_3 and I_4 we estimate $|h(x,t) - h_n(x,t)|$ and $|h_n(t,t)| = |h_n(t,t) - h(t,t)|$, respectively, by

$$\|h - h_n\|_{\mathbf{C}([-1,1]^2)} \leq \frac{c}{n^\gamma \ln^\delta(n+1)}$$

(see Lemma 2.2). By Lemma 3.1, the remaining integrals are bounded by

$$\frac{c}{\sigma(x)} \left[1 + \int_{[-1,1]\setminus I_{n,x}} \frac{dt}{|t-x|} \right] \quad \text{and} \quad \frac{c}{v(x)\sigma(x)} \left[1 + \int_{[-1,1]\setminus I_{n,x}} \frac{dt}{|t-x|} \right],$$

respectively. The last integral behaves like $\ln n$ and we obtain

$$I_3 \leq \frac{c}{\sigma(x)} \, \frac{1}{n^\gamma \ln^{\delta-1}(n+1)}, \quad I_4 \leq \frac{c}{v(x)\sigma(x)} \, \frac{1}{n^\gamma \ln^{\delta-1}(n+1)}.$$

Thus, $\|K - K_n\|_{\mathbf{L}^\infty_u \to \mathbf{B}_{v\sigma}} \leq c\, n^{-\gamma} \ln^{1-\delta}(n+1)$, $n \in \mathbb{N}$. Together with $K_n(\mathbf{L}^\infty_u) \subseteq \Pi_{n-1}$ we obtain the assertion. ■

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