Removing the "Nasty Condition" of Limit Operator Business in the Two Extremal Lebesgue Spaces

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Dedicated to JEAN AIMÉ FAVARD on the occasion of his 100th birthday.

ABSTRACT. Limit operators are used when the behaviour of an operator at some singular, outstanding point like ∞ is of interest. Typical applications are the study of Fredholmness and invertibility at infinity of an operator, but also the applicability of approximation methods. All these properties can be characterized by the invertibility of several limit operators and the uniform boundedness of their inverses. We will show that the uniform boundedness condition is redundant in the cases $L^p(\mathbb{R}^n)$ and $\ell^p(\mathbb{Z}^n)$ for $p = 1$ and $p = \infty$.

1 Introduction

In many situations one has to describe an operator's behaviour at some singular, outstanding point θ. This is where limit operators enter the scene. They are supposed to be introduced by FAVARD (1902-1965) in the late 1920's [2] for studying ordinary differential equations with almost-periodic coefficients. Since that time limit operators have been used in context with partial differential and pseudo-differential operators and in many other fields of numerical analysis (see [3]). The Fredholmness of elliptic differential operators in spaces of functions on $\mathbb{R}^n$ has been studied by MUHAMADIEV in [8], [9], and by RABINOVICH in [10]. The first time limit operator techniques were applied to a class of band-dominated operators was in 1985 by LANGE and RABINOVICH in [4] and [5].

During the last years the set of limit operators at $\theta = \infty$ has been studied for the whole class of band-dominated operators in most of the spaces $\ell^p(\mathbb{Z}^n)$ and $L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Typical applications are, for instance, the study of Fredholmness by RABINOVICH/ROCH/SILBERMANN in [12]. Moreover, the latter three and the author study invertibility at infinity in [13], [6] and the applicability of approximation methods in [12], [13], [7].

In a somewhat different situation, BÖTTCHER, KARLOVICH and RABINOVICH in [1] use limit operators to study the local behaviour of singular integral operators at the endpoint θ of a Carleson curve in the plane. However, we will here focus our investigations on the case $\theta = \infty$:

Take an arbitrary operator A on $L^p(\mathbb{R}^n)$, where $1 \le p \le \infty$. Given a point $\theta \in \mathbb{R}^n$, one might ask what A looks like from another "point of view", for instance, the point θ (taking the role of the origin in $\mathbb{R}^n$). Well, for finite points θ the answer is just $V_{-\theta}AV_\theta$ (where V_θ is a simple shift operator acting by $(V_\theta u)(x+\theta) = u(x)\ \forall x \in \mathbb{R}^n$) which is still a very close relative of the operator A itself.

But what, if we ask for the point $\theta = \infty$? Then θ can only be approached as the limit of a sequence h of finite points $h_m,\ m = 1, 2, ...$, and by doing the above process for every one of these finite points h_m, the answer for θ can only be understood in the sense of some sort of limit of the sequence $V_{-h_m}AV_{h_m},\ m = 1, 2, ...$ as m goes to infinity. And this is what we will call a *limit operator*. We will denote this limit operator by A_h since it heavily depends on the choice of the sequence $h = (h_m)$.

So the behaviour of A at $\theta = \infty$ can be expressed by a whole bunch of limit operators $\{A_h\}$. We collect them in the so called *operator spectrum* which is denoted by $\sigma^{\mathsf{op}}(A)$. In [6] Silbermann and the author prove that if a band-dominated operator A has sufficiently many limit operators – we write $A \in \mathrm{BDO}^p_\$$ and regard A as a *rich* operator in this case – it is invertible at infinity if and only if

❶ all of its limit operators are invertible **and**
❷ their inverses are uniformly bounded.

This is the typical result which was proven for the characterization of the properties mentioned above in terms of limit operators. And as long as such criteria are known, there was always one question:

May we drop condition ❷?

The reason for asking this question is that firstly, ❷ makes the derived criteria somewhat difficult – whence it is also referred to as "the nasty condition" by some authors – and secondly, no example was known where condition ❷ was not redundant.

The aim of this paper is to prove that indeed, in the cases $p = 1$ and $p = \infty$ the answer is "Yes" – the more beautiful but less expected one of the two possible answers!

2 Preliminaries

2.1 Basic agreements

By ℓ^p and L^p we denote the usual spaces of complex-valued sequences on $\mathbb{Z}^n$ and functions on $\mathbb{R}^n$, respectively. The Lebesgue parameter p is in $[1, \infty]$, as usual, and the dimension n is some fixed positive integer.

For $\tau \in \mathbb{R}^n$, let V_τ denote the so called *shift operator* on L^p, acting by the rule $(V_\tau u)(x) = u(x - \tau)$ for every $u \in L^p$. Without introducing a new symbol, we will say that V_α is the *shift operator* on ℓ^p, shifting by $\alpha \in \mathbb{Z}^n$ components, i.e. acting by the rule $(V_\alpha u)_\beta = u_{\beta-\alpha}$ on every $u \in \ell^p$.

Let M_b denote the operator of multiplication by the bounded function b. For every measurable set $U \subset \mathbb{R}^n$, P_U is the the operator of multiplication by the characteristic function of U. Clearly, P_U is a projector. We will refer to its complementary projector $I - P_U$ by Q_U.

2.2 Pre-Adjoints

For some technical reasons it is sometimes necessary to pass to the adjoint operator of $A \in Ł(L^p)$. This is usually an operator in $Ł(L^q)$ with $1/p + 1/q = 1$. The only exception is $p = \infty$ since the dual space of L^∞ is strictly larger than L^1. In this case we will do the following:

We will restrict ourselves to operators $A \in Ł(L^\infty)$ whose adjoint operator A^* maps L^1-functions to L^1-functions. Let us denote this set of operators by $\mathcal{S}^\infty$,

$$\mathcal{S}^\infty := \Big\{ \ A \in Ł(L^\infty) \ : \ A^*(L^1) \subseteq L^1 \ \Big\}.$$

If $A \in \mathcal{S}^\infty$, the restriction $B := A^*|_{L^1}$, seen as operator in $Ł(L^1)$, has the property $B^* = A$, whence we regard B as the *pre-adjoint* operator of A – the operator whose adjoint equals A.

If $p < \infty$, we will think of $\mathcal{S}^p$ as the whole algebra $Ł(L^p)$. In either case, if $A \in \mathcal{S}^p$, we will write A^*, where we mean the adjoint if $p < \infty$, and the pre-adjoint if $p = \infty$.

Proposition 2.1 *$\mathcal{S}^p$ is an inverse closed Banach subalgebra of $Ł(L^p)$.*

For $p < \infty$ this is clear and for $p = \infty$ see [6, Section 2.4].

2.3 Band- and band-dominated operators

An operator $A \in Ł(\ell^p)$ is a *band operator* if its matrix representation with respect to the standard basis in ℓ^p is a band matrix.

The set of band operators clearly turns out to be an algebra – but it is not closed. Hence, it is a natural desire to compute its closure with respect to the norm in $Ł(\ell^p)$, which is a Banach algebra then. The elements of the latter are called *band dominated operators*.

Put $H := [0, 1)^n$ and $H_\alpha := \alpha + H$ for every $\alpha \in \mathbb{Z}^n$. Every function $f \in L^p$ can be identified with a $L^p(H)$-valued sequence in ℓ^p, the α-th component of which is just the restriction of f to H_α. Via this identification, also every operator A on L^p can be identified with an operator $\tilde{A}$ on $L^p(H)$-valued ℓ^p, and we will regard A as *band (dominated)* if and only if $\tilde{A}$ is so.

We will denote the Banach algebra of band-dominated operators on L^p by BDO^p and its intersection with $\mathcal{S}^p$ by $\mathrm{BDO}^p_{\mathcal{S}}$ (which is a Banach algebra as well by Proposition 2.1).

2.4 Invertibility at infinity

As a very close relative of the Fredholm property, we will introduce the notion of *invertibility at infinity* of an operator, which turns out to be a very useful property in [6], [7] and [13] for instance.

Definition 2.2 *$A \in \mathrm{BDO}^p$ is said to be invertible at infinity if and only if there exist operators $B_1, B_2 \in \mathrm{BDO}^p$ and a bounded and measurable set $U \subset \mathbb{R}^n$ such that*

$$Q_U A B_1 \;=\; Q_U \;=\; B_2 A Q_U.$$

By [6, Prop. 2.22], this definition is compatible with that given in the articles cited above.

2.5 $\mathcal{P}$-convergence

By $\mathcal{P}$ we will denote the collection of all projectors P_U with U running through all bounded and measurable subsets of $\mathbb{R}^n$.

Definition 2.3 *We say that a sequence $(A_\tau) \subset Ł(L^p)$ $\mathcal{P}-$converges to $A \in Ł(L^p)$ if*

$$\|P_U(A_\tau - A)\| \to 0, \qquad \|(A_\tau - A)P_U\| \to 0 \quad \text{as } \tau \to \infty$$

for all $P_U \in \mathcal{P}$.

It is readily seen that a $\mathcal{P}$-limit is unique if it exists. Moreover, the properties of $\mathcal{P}$-limits are very similar to those of strong limits:

Proposition 2.4 *Let (A_τ), $(B_\tau) \subset \mathrm{BDO}^p$ be bounded sequences with $\mathcal{P}$-limits $A, B \in \mathrm{BDO}^p$, respectively. Then*

a) $\|A\| \;\le\; \liminf \|A_\tau\| \;\le\; \sup \|A_\tau\| < \infty,$

b) $\mathcal{P}-\lim(A_\tau + B_\tau) \;=\; A + B,$

c) $\mathcal{P}-\lim(A_\tau B_\tau) \;=\; AB.$

For a proof see [6, Sec. 2.7] for instance.

3 Limit operators

Definition 3.1 *Let $A \in Ł(L^p)$ and $h = (h_m)$ be some sequence in $\mathbb{Z}^n$, tending to infinity. If the sequence*

$$V_{-h_m} A V_{h_m}, \qquad m = 1, 2, \ldots$$

$\mathcal{P}$-converges to some operator $B \in Ł(L^p)$ as $m \to \infty$, then we call B the limit operator *of A with respect to the sequence h, and denote it by A_h. In this case, we will also say that the sequence h leads to a limit operator of A.*

If $h \subset \mathbb{Z}^n$ is a sequence tending to infinity such that A_h exists, and if g is an infinite subsequence of h, then also g leads to a limit operator of A, and $A_g = A_h$.

Limit operators are compatible with addition, composition, passing to the norm-limit and to A^*:

Proposition 3.2 *Let A, B and $A^{(m)}$ $(m = 1, 2, ...)$ be arbitrary operators in* BDO^p, *and let $h \subset \mathbb{Z}^n$ be some sequence tending to infinity.*

- **a)** *If A_h exists, then $\|A_h\| \le \|A\|$.*
- **b)** *If A_h and B_h exist, then $(A + B)_h$ exists and is equal to $A_h + B_h$.*
- **c)** *If A_h and B_h exist, then $(AB)_h$ exists and is equal to $A_h B_h$.*
- **d)** *If $\|A^{(m)} - A\| \to 0$ as $m \to \infty$, and the limit operators $(A^{(m)})_h$ exist for sufficiently large m, then A_h exists, and $\|(A^{(m)})_h - A_h\| \to 0$ as $m \to \infty$.*

This proposition is taken from [12]. It is a simple consequence of Proposition 2.4. The following proposition is proven in [6, Sec. 2.8].

Proposition 3.3 *If $A \in \mathrm{BDO}^p$, then all limit operators of A are in BDO^p as well.*

The set of all limit operators of A is denoted by $\sigma^{\mathsf{op}}(A)$, and we refer to it as the *operator spectrum* of A.

This set contains all limit operators A_h of A, regardless of the direction in which h tends to infinity. But sometimes this information is significant, and so we will split $\sigma^{\mathsf{op}}(A)$ into many sets – the so called *local operator spectra*:

Definition 3.4 *Let S^{n-1} denote the unit sphere (w.r.t. the Euclidian norm $|.|_E$) of $\mathbb{R}^n$. Let $s \in S^{n-1}$. Then we say that a sequence $h = (h_m) \subset \mathbb{Z}^n$ tends to infinity in the direction s if for every $R > 0$ and every neighborhood $U \subset S^{n-1}$ of s there is a m_0 such that*

$$|h_m|_E > R \qquad \text{and} \qquad h_m/|h_m|_E \in U \qquad \text{for all } m > m_0.$$

The local operator spectrum $\sigma_s^{\mathsf{op}}(A)$ is defined as the set of all limit operators A_h with h tending to infinity in the direction s.

Then it is not surprising that

Proposition 3.5 *For every operator A, the identity*

$$\sigma^{\mathsf{op}}(A) = \bigcup_{s \in S^{n-1}} \sigma_s^{\mathsf{op}}(A)$$

holds.

For a proof see [12] or [6].

4 Removing the "Nasty Condition"

4.1 Invertibility at Infinity vs. Limit Operators

It is not very hard to show that if A is invertible at infinity, then all its limit operators are invertible. The reverse is not true in general because an operator might possess "too few" limit operators to cover all its behaviour at infinity. Therefore we have to restrict ourselves to a subclass of operators which possess enough limit operators for this purpose:

Definition 4.1 *By* $\mathrm{BDO}^p_{\$}$ *we denote the set of all operators* $A \in \mathrm{BDO}^p_{\mathcal{S}}$ *with the following property: Every sequence* $h = (h_m) \subset \mathbb{Z}^n$ *with* $h_m \to \infty$ *possesses a subsequence* g *which leads to a limit operator of* A. *We will regard operators in* $\mathrm{BDO}^p_{\$}$ *as rich operators.*

$\mathrm{BDO}^p_{\$}$ is a Banach subalgebra of $\text{Ł}(L^p)$, as is shown in [12]. This property is indeed strong enough to link the concepts of invertibility at infinity and of limit operators to each other by the following theorem:

Theorem 4.2 *An operator* $A \in \mathrm{BDO}^p_{\$}$ *is invertible at infinity if and only if the operator spectrum of* A *is uniformly invertible, that is,*

❶ *all of* A*'s limit operators are invertible and*
❷ *their inverses are uniformly bounded.*

Theorem 4.2 was proven for ℓ^p with $1 < p < \infty$ in [12], for L^2 in [13] and for the remaining cases ℓ^p and L^p with $1 \le p \le \infty$ in [6].

As promised in the introduction, we will show in the rest of this article that the "nasty condition" ❷ is redundant if $p \in \{1, \infty\}$. To see this we first have to take some closer look at the topological properties of the set of limit operators of a rich operator.

4.2 Topological Properties of the Operator Spectrum

In the following, C stands as an abbreviation for the hypercube $[-1, 1]^n$.

In [12] the operator spectrum is shown to be closed with respect to the operator norm. By a slight modification of the proof even the following can be shown.

Proposition 4.3 *For every* $A \in \mathrm{BDO}^p$ *every local as well as the global operator spectrum of* A *is closed with respect to* $\mathcal{P}$*-convergence.*

Proof. We will give the proof for the global operator spectrum. It is completely analogous for local operator spectra.

Given a sequence $A^{(1)}, A^{(2)}, \ldots \subset \sigma^{\mathsf{op}}(A)$ which $\mathcal{P}$-converges to B, we will show that also $B \in \sigma^{\mathsf{op}}(A)$. For $k = 1, 2, \ldots$, let $h^{(k)} = (h^{(k)}_m) \subset \mathbb{Z}^n$ denote a sequence such that

$A^{(k)} = A_{h^{(k)}}$, and put $P_m := P_{mC}$. We clearly can choose a subsequence $g^{(k)}$ out of $h^{(k)}$ for every $k = 1, 2, ...$ such that

$$\|P_m(V_{-g_m^{(k)}} A V_{g_m^{(k)}} - A^{(k)})\| < \frac{1}{m} \qquad \forall m = 1, 2, ...$$

holds. Now we put $g = (g_m) := (g_m^{(m)})$ and show that $B = A_g$. Therefore take some arbitrary bounded and measurable subset $U \subset \mathbb{R}^n$ and some $\varepsilon > 0$. If m_0 is large enough that $U \subset m_0 C$, then $P_U = P_U P_m \; \forall m \geq m_0$, and so

$$\begin{aligned} \|P_U(V_{-g_m} A V_{g_m} - B)\| &\leq \|P_U P_m(V_{-g_m} A V_{g_m} - A^{(m)})\| + \|P_U(A^{(m)} - B)\| \\ &\leq \|P_U\| \, \|P_m(V_{-g_m^{(m)}} A V_{g_m^{(m)}} - A^{(m)})\| \\ &\qquad\qquad + \|P_U(A^{(m)} - B)\| \\ &\leq 1 \cdot 1/m \quad + \quad \|P_U(A^{(m)} - B)\| \qquad \forall m \geq m_0 \end{aligned}$$

clearly tends to zero as $m \to \infty$. The same is true for P_U coming from the right, and so we have $B = A_g \in \sigma^{\mathsf{op}}(A)$. ■

Definition 4.4 *Fix an arbitrary operator $A \in Ł(L^p)$. Operators of the form $V_{-c} A V_c$ with $c \in \mathbb{Z}^n$ will be called shifts of A, and the set of all shifts of A will be denoted by $\mathcal{V}_A$,*

$$\mathcal{V}_A := \{ V_{-c} A V_c \}_{c \in \mathbb{Z}^n}.$$

A very easy but important proposition is the following.

Proposition 4.5 *If A_h is a limit operator of A, then for every $c \in \mathbb{Z}^n$, also $V_{-c} A_h V_c$ is a limit operator of A which is contained in the same local operator spectrum of A as A_h is.*

Proof. Let $h = (h_m) \subset \mathbb{Z}^n$ denote the sequence leading to A_h, and take some arbitrary constant $c \in \mathbb{Z}^n$. Then the sequence $h + c := (h_m + c) \subset \mathbb{Z}^n$ leads to

$$A_{h+c} = \mathcal{P}\text{–}\lim V_{-c} V_{-h_m} A V_{h_m} V_c = V_{-c}(\mathcal{P}\text{–}\lim V_{-h_m} A V_{h_m}) V_c = V_{-c} A_h V_c.$$

Clearly, $h + c$ tends to infinity in the same direction as h does. ■

So the operator spectrum is closed under shifting and under passing to $\mathcal{P}$-limits. The same is true for local operator spectra. As a very nice by-product, we conclude:

Corollary 4.6 *Every limit operator of some limit operator of A is already a limit operator of A itself.*

In other words: No further operators occur when repeatedly passing to the set of limit operators.

We will continue with a slight – but very natural – reformulation of an operator's "rich" property. As usual, we will say that a set L of operators is *(relatively) sequentially compact* with respect to some operator topology T if every infinite sequence from L has a subsequence with T-limit (which is not necessarily) in L.

Proposition 4.7 *An operator $A \in \mathrm{BDO}^p_{\mathcal{S}}$ is rich if and only if $\mathcal{V}_A$ – the set of all shifts of A – is relatively $\mathcal{P}$-sequentially compact.*

Proof. If $\mathcal{V}_A$ is relatively $\mathcal{P}$-sequentially compact, then all infinite sequences from $\mathcal{V}_A$ – including those with c tending to infinity – possess a $\mathcal{P}$-convergent subsequence.

If conversely, A is rich, then every sequence from $\mathcal{V}_A$ with c tending to infinity has a $\mathcal{P}$-convergent subsequence. All other sequences $h \subset \mathcal{V}_A$ have an infinite subsequence g such that all c involved in g are in some bounded set $U \subset \mathbb{Z}^n$. Since U has only finitely many elements, there is even an infinite constant subsequence f of g. Summarizing this, every sequence h in $\mathcal{V}_A$ (with c tending to infinity or not) has a $\mathcal{P}$-convergent subsequence, i.e. $\mathcal{V}_A$ is relatively $\mathcal{P}$-sequentially compact. ■

As trivial Proposition 4.7 seems, it opens the door to seeing rich operators and their operator spectra in a somewhat different light. For instance, we just have to recall Proposition 4.3 to find and prove the following interesting fact:

Proposition 4.8 *For every rich operator $A \in \mathrm{BDO}^p_{\$}$, every local as well as the global operator spectrum of A is $\mathcal{P}$-sequentially compact.*

Proof. If A is rich, then, by Proposition 4.7, $\mathcal{V}_A$ is relatively $\mathcal{P}$-sequentially compact. Consequently, $\mathcal{P}\text{-clos}\mathcal{V}_A$ is $\mathcal{P}$-sequentially compact. Clearly, $\sigma^{\mathsf{op}}(A)$ (as well as every local operator spectrum of A) is a subset of the latter set. And as a $\mathcal{P}$-closed subset (cf. Proposition 4.3) of a $\mathcal{P}$-sequentially compact set, it is $\mathcal{P}$-sequentially compact itself. ■

4.3 Employing the $\mathcal{P}$-Compactness of $\sigma^{\mathsf{op}}(A)$

Proposition 4.8 is a first indication that the operator spectrum of a rich operator has enough topological structure that condition ❷ can be dropped.

However, for ordinary operators the answer is "No", as the following example shows: Let $(a_m)_{m=1}^{\infty} \subset [0, 1)$ be a sequence of pairwise distinct numbers. For every $m \in \mathbb{N}$, put $U_m := a_m + [0, 1]$. So all these intervals U_m differ from each other. Now put

$$A_m := \frac{1}{m} P_{U_m} + Q_{U_m}, \qquad m = 1, 2, ...$$

and let A be the so called *inflation operator* of the sequence (A_m) (arising from a construction suggested by Roch), having all operators $A_1, A_2, ...$ in its operator spectrum.

It turns out that the operator spectrum of A, besides $A_1, A_2, ...$ (and shifts of those), only contains the identity operator I (which is due to the choice of the intervals U_m). This set is elementwise invertible – but not uniformly since $\|A_m^{-1}\| = m$.

If this operator A were rich, then by Proposition 4.8, the sequence (A_m) would have some $\mathcal{P}$-convergent subsequence which clearly is not possible due to the incompatibility of the intervals U_m. So in this example, A was just ordinary.

4.3.1 1^{st} try

Suppose we have some rich operator A whose operator spectrum is elementwise – but not uniformly – invertible, i.e. ❶ holds, but not so ❷.

Choose a sequence $A_1, A_2, ...$ from $\sigma^{\mathsf{op}}(A)$ such that $\|A_m^{-1}\| \to \infty$ as $m \to \infty$. From Proposition 4.8 we know that (A_m) has some $\mathcal{P}$-convergent subsequence with $\mathcal{P}$-limit B in $\sigma^{\mathsf{op}}(A)$ again.

The question springing to mind now is whether this operator B can be invertible or not. (If not, this were some contradiction to $B \in \sigma^{\mathsf{op}}(A)$!) Answer:

Although the norm-limit of a sequence whose inverses tend to infinity cannot be invertible, the $\mathcal{P}$-limit B can. For instance, the sequence

$$A_m := P_{mC} + \frac{1}{m} Q_{mC}, \qquad m = 1, 2, ... \tag{1}$$

$\mathcal{P}$-converges to the identity I as $m \to \infty$, although its inverses are growing like m.

4.3.2 Learning from this setback

The reason why the sequence (1) however could have an invertible $\mathcal{P}$-limit, was that those "parts" of A_m which were responsible for the badly growing inverses, were "running away" towards infinity when $m \to \infty$, and so they had no contribution to the $\mathcal{P}$-convergence and hence, to the $\mathcal{P}$-limit B.

Luckily, we here have to do with operator spectra and those have one more nice feature, stated in Proposition 4.5: We stay in $\sigma^{\mathsf{op}}(A)$ if we study appropriate shifts $V_{-c_m} A_m V_{c_m}$ instead of A_m, where every $c_m \in \mathbb{Z}^n$ shall be chosen such that the "bad parts" of $V_{-c_m} A_m V_{c_m}$ cannot run away to infinity any longer and consequently must have some "bad impact" on the $\mathcal{P}$-limit B as well! The outcome of this idea is as follows:

Theorem 4.9 *The (global as well as every local) operator spectrum of an operator $A \in \mathrm{BDO}^\infty_\$$ is automatically uniformly invertible, provided it is elementwise invertible.*

Proof. Let $A \in \mathrm{BDO}^\infty_\$$, and suppose $\sigma^{\mathsf{op}}(A)$ is elementwise invertible with some sequence $(A_m) \subset \sigma^{\mathsf{op}}(A)$ such that $\|A_m^{-1}\| > m$ for $m = 1, 2, ...$

Then for every m, there is an $x_m \in L^\infty$ with

$$\|x_m\| = 1 \qquad \text{and} \qquad \|A_m x_m\| \leq \frac{1}{m}.$$

Now fix some bounded and measurable set $U \subset \mathbb{R}^n$ with $U \supset C$. Clearly, there are translation vectors $c_m \in \mathbb{Z}^n$ such that

$$\|P_{c_m+U} x_m\|_\infty > 1/2 \, \|x_m\|_\infty \, (= 1/2)$$

holds for every[1] m. Now put

$$y_m := V_{-c_m} x_m \qquad \text{and} \qquad B_m := V_{-c_m} A_m V_{c_m}, \tag{2}$$

which leaves us with

$$\|y_m\| = \|V_{-c_m} x_m\| = \|x_m\| = 1, \tag{3}$$

$$\|P_U y_m\| = \|P_U V_{-c_m} x_m\| = \|P_{c_m+U} x_m\| > \frac{1}{2}\|y_m\| \tag{4}$$

and

$$\|B_m y_m\| = \|V_{-c_m} A_m V_{c_m} V_{-c_m} x_m\| = \|A_m x_m\| \le \frac{1}{m}. \tag{5}$$

Moreover, by Proposition 4.5, the sequence (B_m) is in $\sigma^{\mathsf{op}}(A)$ again. So by Proposition 4.8, we can pass to a subsequence of (B_m) with $\mathcal{P}$-limit $B \in \sigma^{\mathsf{op}}(A)$. For simplicity, suppose (B_m) itself already be this sequence.

Now some of the "bad parts" of B_m are located inside U, and hence, they cannot run away when $m \to \infty$. In fact, now we can prove that the $\mathcal{P}$-limit B has inherited some "very bad parts" inside U – bad enough to make B non-invertible:

As an element of $\sigma^{\mathsf{op}}(A)$, the operator B is invertible by our premise. Consequently, there is some constant $a > 0$ (for instance, put $a := 1/\|B^{-1}\|$) such that

$$\|Bx\| \ge a\|x\| \qquad \forall x \in L^\infty. \tag{6}$$

Now fix some continuous function $\varphi : \mathbb{R}^n \to [0,1]$ which is identically 1 on C and vanishes outside of $2C$. By φ_r denote the function $\varphi_r(t) := \varphi(t/r)$ for every $r > 0$. So φ_r is supported somewhere in $2rC$, while on rC it is equal to 1.

From $A \in \mathrm{BDO}^p_{\mathfrak{S}} \subset \mathrm{BDO}^p$, Proposition 3.3 and [6, Theorem 2.15] we know that the norm of the commutator $[\, B\,,\, M_{\varphi_r}\,]$ tends to 0 as $r \to \infty$. Choose r large enough that $U \subset rC$ and

$$\|\,[\, B\,,\, M_{\varphi_r}\,]\,\| < \frac{a}{6}. \tag{7}$$

Since $B_m \xrightarrow{\mathcal{P}} B$, there is some m_0 such that

$$\|P_{2rC}(B - B_m)\| < \frac{a}{6} \qquad \forall m > m_0. \tag{8}$$

From $M_{\varphi_r} = M_{\varphi_r} P_{2rC}$ and inequalities (7) and (8) we conclude that

$$\begin{aligned}
\|BM_{\varphi_r} - M_{\varphi_r} B_m\| &\le \|BM_{\varphi_r} - M_{\varphi_r} B\| + \|M_{\varphi_r}(B - B_m)\| \\
&\le \|BM_{\varphi_r} - M_{\varphi_r} B\| + \|M_{\varphi_r}\| \cdot \|P_{2rC}(B - B_m)\| \\
&< \frac{a}{6} + 1 \cdot \frac{a}{6} = \frac{a}{3}.
\end{aligned}$$

[1] A typical L^∞-argument. L^p is more sophisticated here since U has to be chosen sufficiently large, where it is not clear if one U works for all x_m.

The latter shows that for every $x \in L^\infty$,

$$\begin{aligned}\|BM_{\varphi_r}x\| &\leq \|M_{\varphi_r}B_m x\| + \|BM_{\varphi_r}x - M_{\varphi_r}B_m x\| \\ &\leq \|M_{\varphi_r}B_m x\| + \|BM_{\varphi_r} - M_{\varphi_r}B_m\| \cdot \|x\| \\ &\leq \|M_{\varphi_r}B_m x\| + \frac{a}{3}\|x\| \qquad (9)\end{aligned}$$

holds if $m > m_0$. Taking everything together, keeping in mind that $P_U = P_U M_{\varphi_r}$, we get:

$$\begin{aligned}\frac{a}{2} &\overset{(3)}{=} \frac{a}{2}\|y_m\| \overset{(4)}{<} a\|P_U y_m\| \leq a\|P_U\| \cdot \|M_{\varphi_r}y_m\| = a\|M_{\varphi_r}y_m\| \overset{(6)}{\leq} \|BM_{\varphi_r}y_m\| \\ &\overset{(9)}{\leq} \|M_{\varphi_r}B_m y_m\| + \frac{a}{3}\|y_m\| \overset{(3)}{\leq} \|M_{\varphi_r}\| \cdot \|B_m y_m\| + \frac{a}{3} \cdot 1 \overset{(5)}{\leq} 1 \cdot \frac{1}{m} + \frac{a}{3}\end{aligned}$$

If we still subtract $a/3$ at both ends of the chain, we arrive at

$$\frac{a}{6} \leq \frac{1}{m} \qquad \forall m > m_0$$

which is perfectly contradicting $a > 0$. ■

"Well, we've knocked the bastard off!"

EDMUND HILLARY ON MAY 29, 1953

So in L^∞ the answer is: "Yes, we may drop ❷!"
Clearly, the same proof works for ℓ^∞. Moreover, if A is a rich operator on L^1 (or ℓ^1), then we know that A^* is a rich operator on L^∞ (or ℓ^∞), and the operator spectra $\sigma^{\mathsf{op}}(A)$ and $\sigma^{\mathsf{op}}(A^*)$ can be identified elementwise by taking (pre-)adjoints. Since this identification clearly preserves elementwise and uniform invertibility, we are done with L^1 and ℓ^1 as well.

4.4 Personal remark

Having presented the news to some specialists of the subject, one of them recognized a characteristic argumentation in its proof and indeed found it in some part of the proof of a theorem on the so called Wiener algebra in [11]. This step shows that Theorem 4.9 holds in ℓ^∞ for operators in the Wiener algebra. Looking a bit deeper, it turns out that membership in the Wiener algebra actually nowhere is needed in this particular step of the proof!

So afterwards, it turned out that a (quite – but not completely – different) almost complete proof of Theorem 4.9 in the case of ℓ^∞ had been within reach for several years. One reason why this was not realized, clearly is accidentally associating this result with the Wiener algebra only – where it was stated in [11]. From another point of view, it is quite understandable why Theorem 4.9 was not explicitly stated for ℓ^∞ that time: This is some quite breadless art if one does not have Theorem 4.2 for $p = \infty$.

However, the huge bastard, still to be knocked off, is L^p (or ℓ^p) with $1 < p < \infty$.

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