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# On the inverse problem of option pricing in the time-dependent case

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**Summary**

The paper formulates and analyses specific forward and inverse option pricing problems for the purely time-dependent case, which seems to be rather academic, but provides some interesting insight concerning the role of smoothness and no arbitrage of option data for the identification of local volatility functions. Forward and inverse problems under consideration here can be written by using Nemytskii operators based on Black-Scholes functions. In this context, the inverse option pricing problem consists in solving a nonlinear operator equation in Banach spaces of functions defined on a finite time interval. The solution process is decomposed into solving an outer nonlinear operator equation by inverting the associated Nemytskii operator and solving an inner linear operator equation by differentiation. In contrast to the ill-posed inner linear problem of differentiation the outer nonlinear problem is proved to be well-posed in spaces of continuous and integrable functions. This also implies well-posedness of the forward problem in $C$-spaces. Ideas for a discrete approach and some numerical case studies complete the paper.

MSC 2000 Subject Classifications: 35R30, 91B24, 47H30, 65J20

## 1 Introduction

The past twenty years can be characterized as a very active period in developing and trading of financial derivative securities in financial markets. This was the reason for an extremely growing interest in derivative pricing theory as a modern part of Financial Mathematics. Stochastic calculus could be applied successfully for the fair price calculation of options and other financial derivatives in arbitrage-free markets. There occur direct and inverse problems in the mathematical treatment of derivative valuations. On the one hand, the calculation of option prices provided that the required parameters of the underlying stochastic asset price process are known is a forward (direct) problem. On the other hand, problems of determining process parameters like volatility from observed option prices are of inverse nature. We will consider here such an inverse problem aimed at finding a time-dependent volatility function in the context of the *valuation of European vanilla call options* on a stock or a stock index. Call options, which we always assume to be issued at time $t = t^* := 0$, are contracts that gives the owner the right to buy an

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amount of the underlying asset (stock or index certificate) for a fixed price $K > 0$ (strike or exercise price) at the time to expiry (maturity) $t = T > 0$ of the option. For our mathematical considerations we assume for all $T \in (0, \overline{T}]$ the existence of options with such maturities. This is an idealization, since in real markets for a fixed asset and a fixed strike price $K$ options are available only for a discrete number of maturities. However, this idealization will provide some new insight into the structure of option prices in our time-dependent case.

Neglecting the role of dividends the option price $u$ at time $t = t_0$ $(0 \leq t_0 \leq T)$ is a function of the current time $t_0$ and the corresponding price $X = X(t_0) > 0$ of the underlying asset, moreover of the *strike* $K$, the *expiriy* $T$, the *risk-free interest rate* $r \geq 0$ and of the nonnegative *volatility* function $\sigma(t)$ $(t_0 \leq t \leq T)$ measuring the asset risk. The volatility $\sigma(t)$ expresses the instantaneous standard deviation of the expected asset returns at time $t$. Under the arguments of $u$ the function $\sigma$ is the only argument, which is not directly observable on the market. In this paper we will not consider the more general model of option pricing, where $\sigma$ also gets a function of the current asset price $X = X(t)$ (see, e.g., Reference [5]). If the value of $\sigma(t_0)$ is estimated from a sample of asset prices $X(t)$ $(t < t_0)$, then it is called *historical volatility*. On the other hand, we call an estimation of $\sigma(t_0)$ *implied volatility*, if it reflects the current market expectation concerning the asset returns. Both variants have deterministic character. In the literature (see, e.g., Reference [13]) there also occur approaches of *stochastic volatilities*, where $\sigma(t_0)$ itself is assumed to be random. In the sequel we only consider deterministic time-dependent functions $\sigma(t)$ $(0 \leq t \leq \overline{T})$ and call them *local volatility functions*. These functions are to be predicted from option price data $u$ at the initial time $t^* = 0$ by solving an inverse problem (see also [2], [6], [7] and [8]). If the local volatility functions are known, then they allow us to compute a wide class of option prices $u(X, t)$ for different asset prices $X$, strikes $K$ and maturities $T$ by solving forward problems. In this context, the value $u(X, t)$ $((X, t) \in (0, \infty) \times [0, T])$ expresses the option price at time $t$, which corresponds to the price $X = X(t)$ of the underlying asset at the same time. For simplicity, we omit here the arguments $K, T, r$ and $\sigma$ in the list of $u$ considering these parameters to be constant scalars and functions.

The paper is organized as follows: In the remaining part of the introduction we formulate for time-dependent local volatilities the option price formula using Black-Scholes functions and give definitions for the corresponding inverse and forward problems of option pricing. Both classes of problems can be written based on specific Nemytskii operators the properties of which are summarized in Section 2. In this context, the inverse problem consists in solving a nonlinear operator equation in Banach spaces of functions defined on a finite time interval. The solution process is decomposed into solving an outer nonlinear operator equation by inverting the associated Nemytskii operator and solving an inner linear operator equation by differentiation. Section 3 is dealt with the solution of the inverse problem for arbitrage-free smooth option data in spaces of continuous functions, whereas Section 4 considers the inverse problem solution with noisy data in $L^p$-spaces. In contrast to the ill-posed inner linear problem of differentiation the outer nonlinear problem is proved to be well-posed in spaces of continuous and integrable functions. This also implies well-posedness of the forward problem in $C$-spaces. Ideas for a discrete approach will be discussed in Section 5 including some numerical case studies aimed at identifying the volatility term-structure from computer-generated option price data.

Widely accepted option pricing principles as the basis of modern Financial Engineering go back to the seminal papers [4] and [19] of Fisher Black, Myron Scholes and Robert

Merton published in 1973. The created *Black-Scholes model* (for details see [14] and [21]) is based on a *geometric Brownian motion* as the stochastic process for the asset price $X(t) > 0$. With constant drift $\mu$, time-dependent volatilities $\sigma(t)$ and a standard Wiener process $B(t)$ the stochastic differential equation

$$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma(t)\, dB(t) \qquad (t \geq 0), \qquad X(0) := X^* > 0 \tag{1}$$

holds. For a European call option with strike $K$ and maturity $T$ it follows from stochastic considerations that fair option prices $u(X,t)$ on *arbitrage-free* markets satisfy the *Black-Scholes differential equation*

$$\frac{\partial u(X,t)}{\partial t} + \frac{1}{2} X^2 \sigma^2(t) \frac{\partial^2 u(X,t)}{\partial X^2} + r\, X \frac{\partial u(X,t)}{\partial X} - r\, u(X,t) = 0 \quad ((X,t) \in (0,\infty) \times (0,T)). \tag{2}$$

Moreover, the payoff at expiry yields the terminal condition

$$u(X,T) = \max(X - K, 0). \tag{3}$$

If $\sigma(t) \equiv const. > 0$ $(0 \leq t \leq T)$, then well-known Black-Scholes formula expresses the solution of the backward parabolic problem (2) – (3). For an almost everywhere nonvanishing Lipschitz continuous function $\sigma(t) \geq 0$ $(0 \leq t \leq T)$ the uniquely determined classical solution of problem (2) – (3) obtaines the form

$$u(X,t) = X\, \Phi(\tilde{d}_1) - K\, e^{-r(T-t)} \Phi(\tilde{d}_2) \qquad ((X,t) \in (0,\infty) \times [0,T)) \tag{4}$$

with

$$\tilde{d}_1 := \frac{\ln\left(\frac{X}{K}\right) + r(T-t) + \frac{1}{2}\int_t^T \sigma^2(\tau)\, d\tau}{\sqrt{\int_t^T \sigma^2(\tau)\, d\tau}}, \quad \tilde{d}_2 := \tilde{d}_1 - \sqrt{\int_t^T \sigma^2(\tau)\, d\tau} \tag{5}$$

and the cumulative density function of the standard normal distribution

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx. \tag{6}$$

A complete proof of the *modified Black-Scholes formula* (4) – (6) is given, for example, by Kwok [16, p.71/72]. This formula remains true expressing a weak solution of the problem (2) – (3), if the a.e. nonvanishing local volatility function $\sigma(t) \geq 0$ $(0 \leq t \leq T)$ is at least *square-integrable.*

For parameters $X > 0$, $K > 0$, $r \geq 0$, $\tau \geq 0$ and $s \geq 0$ it is useful to introduce the *Black-Scholes function*

$$U_{BS}(X,K,r,\tau,s) := \begin{cases} X\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) & (s > 0) \\ \max(X - Ke^{-r\tau}, 0) & (s = 0) \end{cases} \tag{7}$$

with

$$d_1 := \frac{\ln\left(\frac{X}{K}\right) + r\tau + \frac{s}{2}}{\sqrt{s}}, \quad d_2 := d_1 - \sqrt{s} \tag{8}$$

and $\Phi(\cdot)$ from formula (6). In terms of the *continuous* auxiliary function

$$S(T) := \int_0^T \sigma^2(t)\,dt \qquad (0 \le T \le \overline{T}) \tag{9}$$

the solution of (2) – (3) can be written concisely as

$$u(X,t) = U_{BS}(X,K,r,T-t,S(T)-S(t)) \qquad ((X,t) \in (0,\infty)\times[0,T]). \tag{10}$$

Now we are going to formulate a pair of specific inverse and direct problems in the time-dependent case. In both problems we consider prices $u$ of option families, for which the maturity $T$ is continuously varying between zero and the upper bound $\overline{T}$.

At the initial time $t^* = 0$ we observe option prices for a fixed strike $K^*$ and try to identify the non-observable local volatility function $\sigma(t)$ $(0 \le t \le \overline{T})$. That means, we predict the volatility term-structure by solving the following inverse problem:

**Definition 1.1 (Inverse Problem - IP)** *At time $t^* = 0$ let be given call option prices*

$$u^*(T) := U_{BS}\left(X^*, K^*, r^*, T, \int_0^T \sigma^2(t)\,dt\right) = U_{BS}\left(X^*, K^*, r^*, T, S(T)\right) \qquad (0 \le T \le \overline{T})$$

*for a fixed current asset price $X^* = X(0) > 0$, a fixed interest rate $r^* \ge 0$, a fixed strike price $K^* > 0$ and varying maturities $T$. Find the associated square-integrable local volatility function $\sigma(t)$ $(0 \le t \le \overline{T})$ of the underlying asset.*

If the local volatility function $\sigma(t)$ $(0 \le t \le \overline{T})$ has been determined, then we can predict option prices for times $\hat{t} \ge t^*$ and arbitrary strike prices $\widehat{K} > 0$ by solving the following forward problem:

**Definition 1.2 (Forward Problem - FP)** *Given the square-integrable local volatility function $\sigma(t)$ $(0 \le t \le \overline{T})$, find at time $\hat{t}$ with $0 \le \hat{t} < \overline{T}$ the call option prices*

$$\hat{u}(T) := U_{BS}\left(\widehat{X}, \widehat{K}, \hat{r}, T-\hat{t}, \int_{\hat{t}}^T \sigma^2(t)\,dt\right) = U_{BS}\left(\widehat{X}, \widehat{K}, \hat{r}, T-\hat{t}, S(T)-S(\hat{t})\right) \qquad (\hat{t} \le T \le \overline{T})$$

*for a current asset price $\widehat{X} = X(\hat{t})$, a fixed interest rate $\hat{r} \ge 0$, a fixed strike price $\widehat{K} > 0$ and varying maturities $T$.*

# 2 Black-Scholes function and Nemytzkii operators

We first summarize the main properties of the *Black-Scholes function* $U_{BS}$ according to the formulae (7) – (8). The results of the following lemma can be proven straightforward by elementary calculations.

**Lemma 2.1** *Let the parameters $X > 0$, $K > 0$ and $r \geq 0$ be fixed. Then the function $U_{BS}(X, K, r, \tau, s)$ is continuous for $(\tau, s) \in [0, \infty) \times [0, \infty)$. Moreover, for $(\tau, s) \in [0, \infty) \times (0, \infty)$, this function is continuously differentiable with respect to $\tau$, where we have*

$$\frac{\partial U_{BS}(X, K, r, \tau, s)}{\partial \tau} = r\, K\, e^{-r\tau}\, \Phi(d_2) \geq 0, \tag{11}$$

*and continuously differentiable with respect to s, where we have*

$$\frac{\partial U_{BS}(X, K, r, \tau, s)}{\partial s} = \Phi'(d_1)\, X\, \frac{1}{2\sqrt{s}} > 0. \tag{12}$$

*Furthermore, we find the limit conditions*

$$\lim_{s \to 0} \frac{\partial U_{BS}(X, K, r, \tau, s)}{\partial s} = \begin{cases} \infty & (X = K\, e^{-r\tau}) \\ 0 & (X \neq K\, e^{-r\tau}) \end{cases} \quad \text{and} \quad \lim_{s \to \infty} U_{BS}(X, K, r, \tau, s) = X. \tag{13}$$

*On the other hand, the partial derivative*

$$\frac{\partial U_{BS}(X, K, r, \tau, s)}{\partial K} = -e^{-r\tau}\, \Phi(d_2) < 0 \tag{14}$$

*exists and is continuous for $(\tau, s) \in [0, \infty) \times (0, \infty)$.*

The Black-Scholes function $U_{BS}$ helps us to express *Nemytskii operators*, which occur both in the inverse problem (IP) and in the forward problem (FP) of option pricing. For fixed $X > 0$, $K > 0$ and $r \geq 0$ we define a Nemytskii operator

$$N = N_{\overline{T}}^{X,K,r} : D_{\overline{T}}^{+} \longrightarrow D_{\overline{T}}^{+}$$

mapping the domain $D_{\overline{T}}^{+}$ of *nonnegative real functions on the interval* $[0, \overline{T}]$ into itself:

$$[N(v)](\tau) = \left[N_{\overline{T}}^{X,K,r}(v)\right](\tau) = k(\tau, v(\tau)) := U_{BS}(X, K, r, \tau, v(\tau)) \quad (0 \leq \tau \leq \overline{T}). \tag{15}$$

From formula (12) of Lemma 2.1 we obtain $\frac{\partial k(\tau,s)}{\partial s} > 0$ for all $(\tau, s) \in [0, \overline{T}] \times (0, \infty)$ and hence the following lemma.

**Lemma 2.2** *The Nemytskii operator $N = N_{\overline{T}}^{X,K,r}$ defined by formula (15) is injective on its domain $D_{\overline{T}}^{+}$.*

From Lemma 2.1 it follows that the function $k(\tau, v) := U_{BS}(X, K, r, \tau, v)$ generating the Nemytskii operator $N$ is continuous and uniformly bounded with $|k(\tau, v)| < X$ due to the formulae (12) and (13) for all $(\tau, v) \in [0, \overline{T}] \times [0, \infty)$. Then Krasnoselskii's *Carathéodory condition* and *growth condition* (see [17, Chap. 17] or [22, Th. 25.2, p.92]) hold and we have continuity of $N$ between spaces of power-integrable functions on the interval $[0, \overline{T}]$ as the following lemma asserts.

**Lemma 2.3** *The Nemytskii operator $N_{\overline{T}}^{X,K,r}$ with domain $D_{\overline{T}}^{+} \cap L^p(0, \overline{T})$ maps continuously from $L^p(0, \overline{T})$ into $L^q(0, \overline{T})$ for all $1 \leq p, q < \infty$.*

As obvious throughout this paper we denote by $L^p(a,b)$ $(1 \le p < \infty)$ the Banach space of integrable real functions $x(t)$ $(a \le t \le b)$ with a norm $\|x\|_{L^p(a,b)} := \left(\int\limits_a^b |x(t)|^p\,dt\right)^{1/p} < \infty$ and by $C[a,b]$ the Banach space of continuous functions $x$ defined on $[a,b]$ with the norm $\|x\|_{C[a,b]} := \max\limits_{t\in[a,b]} |x(t)|$.

If we restrict the domain of $N$ to the set

$$D_{\overline{T}}^{0} := \{v \in C[0,\overline{T}] : v(0)=0,\ v(\tau) \ge 0\ (0 < \tau \le \overline{T})\},$$

then because of Lemma 2.1 we have

$$N = N_{\overline{T}}^{X,K,r} :\ D_{\overline{T}}^{0} \subset C[0,\overline{T}] \longrightarrow\ D_{\overline{T}}^{+} \cap C[0,\overline{T}],$$

i.e., $N$ transforms nonnegative continuous functions vanishing at $\tau = 0$ to nonnegative continuous functions. Using the substitutions $w := \sqrt{\frac{v}{\tau}}$ as well as $\bar{k}(\tau,w) := k(\tau,v)$ we obtain for all $\tau > 0$ and $w > 0$

$$0 < \frac{\partial \bar{k}(\tau,w)}{\partial w} = X\,\sqrt{\tau}\,\Phi'(\bar{d}_1) \le \frac{X\,\sqrt{\tau}}{\sqrt{2\pi}}$$

with

$$\bar{d}_1 := \frac{\ln\left(\frac{X}{K}\right) + \tau\left(r + \frac{w^2}{2}\right)}{\sqrt{\tau}\,w}.$$

Consequently, for $v, \tilde{v} \in D_{\overline{T}}^{0}$ with $v(\tau) = \tau w^2(\tau)$ and $\tilde{v}(\tau) = \tau \tilde{w}^2(\tau)$, there are pointwise estimations

$$|[N(v)](\tau) - [N(\tilde{v})](\tau)| \le \left|\frac{\partial \bar{k}(\tau,w_\tau)}{\partial w}\right| \frac{1}{\sqrt{\tau}} \left|\sqrt{v(\tau)} - \sqrt{\tilde{v}(\tau)}\right| \qquad (0 < \tau \le \overline{T})$$

with an intermediate value $w_\tau$ between $w(\tau)$ and $\tilde{w}(\tau)$ and

$$|[N(v)](\tau) - [N(\tilde{v})](\tau)| \le \frac{X}{\sqrt{2\pi}} \left|\sqrt{v(\tau)} - \sqrt{\tilde{v}(\tau)}\right| \qquad (0 \le \tau \le \overline{T}) \tag{16}$$

From (16) we directly obtain:

**Lemma 2.4** *The Nemytskii operator $N_{\overline{T}}^{X,K,r}$ with domain $D_{\overline{T}}^{0}$ maps continuously from $C[0,\overline{T}]$ into $C[0,\overline{T}]$.*

Now we can represent the problems (IP) and (FP) via Nemytskii operators based on the Black-Scholes function $U_{BS}$. We denote by $B_1, B_2$ and $B_3$ appropriate Banach spaces of functions defined on the interval $[0,\overline{T}]$. Then the inverse problem (IP) consists in solving a *nonlinear operator equation*

$$F(\sigma^2) := N\left(I(\sigma^2)\right) = u^* \qquad (\sigma^2 \in D_{\overline{T}}^{+} \subset B_1,\ u^* \in D_{\overline{T}}^{+} \subset B_2), \tag{17}$$

where the nonlinear operator $F : D_{\overline{T}}^{+} \subset B_1 \longrightarrow B_2$ is *decomposed* into an *inner* linear Volterra integral operator $I : B_1 \longrightarrow B_3$ with

$$[I(w)](T) := \int\limits_0^T w(\tau)\,d\tau \qquad (0 \le T \le \overline{T}) \tag{18}$$

and an *outer* nonlinear Nemytskii operator $N = N_{\overline{T}}^{X^*,K^*,r^*} : D_{\overline{T}}^+ \subset B_3 \longrightarrow B_2$. Note that our decomposition $F(\cdot) = N(I(\cdot))$ is reverse to the situation in [18, Chap. 7.5], where composite operators $\widetilde{F}(\cdot) = A(N(\cdot))$ with an inner Nemytskii and an outer bounded linear operator $A$ are analysed.

The problem of solving the operator equation (17) can be decomposed into solving, successively, the nonlinear outer operator equation

$$N(S) = u^* \qquad (S \in D_{\overline{T}}^+ \subset B_3,\, u^* \in D_{\overline{T}}^+ \subset B_2) \tag{19}$$

and the linear inner operator equation

$$I(\sigma^2) = S \qquad (\sigma^2 \in D_{\overline{T}}^+ \subset B_1,\, S \in D_{\overline{T}}^+ \subset B_3). \tag{20}$$

In order to get a square-integrable local volatility function $\sigma$ from equation (20), $S$ has to be an absolutely continuous function belonging to the set

$$D_{\overline{T}}^{\angle} := \{S \in C[0,\overline{T}] : S(0) = 0,\ S(t_1) \le S(t_2)\ (0 \le t_1 < t_2 \le \overline{T})\}.$$

Solving the *forward problem* (FP) corresponds with the application of a Nemytskii operator to the solution $S \in D_{\overline{T}}^{\angle}$ of the outer problem (19) in the form

$$\widehat{u} := N_{\overline{T}-\widehat{t}}^{\widehat{X},\widehat{K},\widehat{r}}\, v, \quad \text{where} \quad v(\tau) := S(\widehat{t}+\tau) - S(\widehat{t})\ \ (0 \le \tau \le \overline{T} - \widehat{t}). \tag{21}$$

For solving the problem (FP), it is sufficient to know the function $S$ defined by (9), whereas the local volatility function $\sigma(t)$ $(0 \le t \le \overline{T})$ is not used explicitly. Hence, providing data for the direct problem requires only to solve the outer equation (19). This is an advantage, since the inner equation (20) aimed at finding the derivative $\sigma^2(t) = S'(t)$ $(0 \le t \le \overline{T})$ of the function $S$ is *ill-posed* in usual Banach spaces $B_1$ and $B_3$ of integrable or continuous functions on the interval $[0,\overline{T}]$ and leads to *ill-conditioned* problems after discretization (see, e.g., [10]). In the Hilbert space setting $B_1 := B_3 := L^2(0,\overline{T})$ the differentiation problem is weakly ill-posed and has an ill-posedness degree of one (see, e.g., [15, p.235] and [12, p.33ff]).

On the other hand, in view of the continuity of the Nemytskii operator $N$ (see Lemma 2.4) the variant (21) of the forward problem that consists in finding the output function $\widehat{u}$ from a given input function $S$ is *well-posed* in a $C$-space setting. That means, $\widehat{u}$ is well-defined for given $S \in D_{\overline{T}}^{\angle}$ and small errors in $S$ imply only small errors in $\widehat{u}$ if we measure perturbations of input and output functions both in the maximum norm.

# 3 Solving the inverse problem in C-spaces for smooth arbitrage-free option data

In this section we are going to solve the *inverse problem* (IP) for a given smooth function $u(T)$ $(0 \le T \le \overline{T})$ of observed option price data that approximate the fair price function $u^*(T) = [F(\sigma^{*2})](T) = [N(S^*)](T)$ $(0 \le T \le \overline{T})$ corresponding with the square-integrable local volatility function $\sigma^*(t) \ge 0$ $(0 \le t \le \overline{T})$ and $S^* := I(\sigma^{*2})$.

For the data $u$ we first pose the following assumption, which is a consequence of an arbitrage-free market (see, e.g., [19]):

**Assumption 3.1** *The data function* $u(T)$ $(0 \le T \le \overline{T})$ *is assumed to be continuous and strictly increasing with*

$$u(0) = \max(X^* - K^*, 0), \quad \max(X^* - K^* e^{-r^* T}, 0) < u(T) < X^* \quad (0 < T \le \overline{T}). \tag{22}$$

Now the outer equation (19) attains here the form

$$[N(S)](T) = k^*(T, S(T)) := U_{BS}(X^*, K^*, r^*, T, S(T)) = u(T) \qquad (0 \le T \le \overline{T}). \tag{23}$$

If there *exists* a solution $S \in D^+_{\overline{T}}$ of equation (23) for given data $u$, then from the *injectivity* of the Nemytskii operators $N$ (see Lemma 2.2) it follows that this solution is *unique*. Moreover, the following theorem shows that we can even find a uniquely determined function $S \in D^0_{\overline{T}} \subset C[0, \overline{T}]$ satisfying (23).

**Theorem 3.2** *Under the Assumption 3.1 there exists a uniquely determined continuous function* $S(T)$ $(0 \le T \le \overline{T})$ *with* $S(0) = 0$ *and* $0 < S(T) \le \overline{S}$ $(0 < T \le \overline{T})$ *solving the equation (23), where* $\overline{S}$ *satisfies the equation* $k^*(0, \overline{S}) = u(\overline{T}) = \|u\|_{C[0,\overline{T}]}$.

**Proof:** As a consequence of Lemma 2.1 the function $k^*(T, s) := U_{BS}(X^*, K^*, r^*, T, s)$ with

$$\frac{\partial k^*(T,s)}{\partial T} \ge 0 \qquad \text{and} \qquad \frac{\partial k^*(T,s)}{\partial s} > 0$$

is continuous in both variables $T$ and $s$, increasing with respect to $T$ and stricly increasing with respect to $s$ for $(T, s) \in [0, \overline{T}] \times (0, \infty)$. Moreover, we have for all $T \in [0, \overline{T}]$

$$\lim_{s \to 0} k^*(T, s) = k^*(T, 0) = \max(X^* - K^* e^{-r^* T}, 0) \; < \; \lim_{s \to \infty} k^*(T, s) = X^*$$

Since the data $u$ with $u(T) \le u(\overline{T})$ $(0 \le T \le \overline{T})$ satisfy the condition (22), from the family of equations

$$k^*(T, s) \; = \; u(T) \tag{24}$$

in $s$, where the parameter $T$ varies in the interval $[0, \overline{T}]$, we find in a unique manner values $s = S(T) > 0$ for all $T \in (0, \overline{T}]$ and $s = S(0) = 0$ for $T = 0$ because of $k^*(0, 0) = u(0)$. The value $\overline{S}$ satisfying $k^*(0, \overline{S}) = u(\overline{T})$ is also uniquely determined. From the estimation $k^*(0, S(T)) \le k^*(T, S(T)) = u(T) \le u(\overline{T}) = k^*(0, \overline{S})$ we get $S(T) \le \overline{S}$. Finally, the continuity of the function $S(T)$ $(0 \le T \le \overline{T})$ follows from the *implicit function theorem* (see, e.g., [9, p.421]) considering that $k^*(T, s)$ is continuous in both variables and strictly monotone with respect to $s$. This proves the theorem ∎

Note that for any maturity $T > 0$ the corresponding value $S(T)$ only depends on the option price $u(T)$ and can be found easily by a line search solving the equation (24). However, Theorem 3.2 based on Assumption 3.1 cannot ensure $S \in D^{\nearrow}_{\overline{T}}$, although the monotonicity of $S$ is required in order to find a nonnegative volatility function $\sigma$ from equation (20). This gap can be closed by posing a further assumption:

**Assumption 3.3** *In addition to Assumption 3.1 the data function* $u(T)$ *is assumed to be continuously differentiable for* $0 < T \le \overline{T}$ *with*

$$u'(T) - K^* r^* e^{-r^* T} \, \Phi(d_2^*) \ge 0 \;\; (0 < T \le \overline{T}), \quad d_2^* := \frac{\ln\left(\frac{X^*}{K^*}\right) + r^* T - \frac{S(T)}{2}}{\sqrt{S(T)}}, \tag{25}$$

*where* $u$ *implies the function* $S \in D^0_{\overline{T}}$ *via equation (23) in a unique manner.*

The condition (25) is also a consequence of an arbitrage-free market. Namely, by comparing approriate porfolios it can be shown that option prices $\mathbf{u}$ at fixed time $t^*$ considered as differentiable functions of strike price $K$ and maturity $T$ satisfy inequalities of the form (see [1, p.11])

$$\frac{\partial \mathbf{u}}{\partial T} + K\,r\,\frac{\partial \mathbf{u}}{\partial K} \geq 0. \tag{26}$$

For the inverse problem (IP) we have $t^* = 0$ and $\mathbf{u} = U_{BS}(X^*, K^*, r^*, T, S(T))$, where

$$\frac{\partial \mathbf{u}}{\partial T} = u'(T) \quad \text{and with (14)} \quad \frac{\partial \mathbf{u}}{\partial K} = \frac{\partial U_{BS}(X^*, K^*, r^*, T, S(T))}{\partial K} = -e^{-r^*T}\,\Phi(d_2^*).$$

Consequently, the inequality (26) attains here the form (25).

**Theorem 3.4** *Under the Assumptions 3.1 and 3.3 the uniquely determined solution $S(T) \geq 0$ $(0 \leq T \leq \overline{T})$ of equation (23) with $S(0) = 0$ is an increasing and absolutely continuous function with a continuous and integrable derivative $S'(T) \geq 0$ $(0 < T \leq \overline{T})$, where $S(T) = \int_0^T S'(t)\,dt$ $(0 < T \leq \overline{T})$ and*

$$S'(T) = \frac{2\sqrt{S(T)}\,\left[u'(T) - K^* r^* e^{-r^*T}\,\Phi(d_2^*)\right]}{\Phi'(d_1^*)\,X^*} \geq 0 \qquad (0 < T \leq \overline{T}) \tag{27}$$

*with*

$$d_1^* := \frac{\ln\left(\frac{X^*}{K^*}\right) + r^*T + \frac{S(T)}{2}}{\sqrt{S(T)}}, \qquad d_2^* := d_1^* - \sqrt{S(T)}.$$

**Proof:** Considering the formulae (11), (12) and (25) for $0 < T \leq \overline{T}$ from the implicit function theorem (see, e.g., [9, p.423ff.]) we obtain continuous differentiability of $S$ with $S'(T) \geq 0$ and formula (27). Hence $S(T)$ $(0 \leq T \leq \overline{T})$ is increasing and based on [20, Thms. 4 and 5, p.236f] we have an integrable derivative $S' \in L^1(0, \overline{T})$ with $\int_0^T S'(t)\,dt \leq S(T) - S(0) = S(T)$ $(0 \leq T \leq \overline{T})$. Choosing $\varepsilon$ from the interval $0 < \varepsilon < T$ we get

$$\int_0^T S'(t)\,dt = \int_0^\varepsilon S'(t)\,dt + S(T) - S(\varepsilon) = S(T)$$

and absolute continuity of $S$, since $\int_0^\varepsilon S'(t)\,dt - S(\varepsilon)$ is a constant and tends to zero as $\varepsilon \to 0$. This proves the theorem ∎

As a consequence of Theorem 3.4 we obtain for arbitrage-free and sufficiently smooth option data $u(T)$ $(0 \leq T \leq \overline{T})$ by solving equation (23) a function $S \in D_{\overline{T}}^{\nearrow}$, which is continuously differentiable for positive $T$ and provides a square-integrable local volatility function

$$\sigma(t) := \sqrt{S'(t)} \geq 0 \qquad (0 < t \leq \overline{T}). \tag{28}$$

The function $\sigma(t)$ is continuous for $t > 0$, but may tend to infinity as $t$ tends to zero.

The next theorem will show that solving the equation (23) for *smooth arbitrage-free data* $u$ as a variant of the operator equation (19) with Banach spaces $B_2 := B_3 := C[0, \overline{T}]$

is a *well-posed problem.* On the other hand, the problem of determining $\sigma^2$ from $S$ as a variant of solving the operator equation (20) is *ill-posed* for spaces of continuous or integrable functions on the interval $[0,\overline{T}]$.

**Theorem 3.5** *Let $\{u_n = N(S_n)\}_{n=1}^{\infty}$ with $N$ according to (23) be a sequence of time-dependent arbitrage-free noisy option price functions satisfying the Assumptions 3.1 and 3.3 that converges in the Banach space $B_2 := C[0,\overline{T}]$ to the fair option price function $u^* = N(S^*)$. Then the associated sequence of functions $\{S_n\}_{n=1}^{\infty}$ also converges to $S^*$ in the Banach space $B_3 := C[0,\overline{T}]$.*

**Proof:** In view of the positivity and continuity of the partial derivative $\dfrac{\partial U_{BS}(X^*,K^*,r^*,T,s)}{\partial s}$ on the domain $(T,s) \in [0,\overline{T}] \times (0,\infty)$ (see Lemma 2.1) we have, for fixed $T \in (0,\overline{T}]$,

$$|S_n(T) - S^*(T)| \le \left(\frac{\partial U_{BS}(X^*,K^*,r^*,T,S_T)}{\partial s}\right)^{-1} |u_n(T) - u^*(T)| \qquad (29)$$

with an intermediate value $S_T$ between the positive values $S_n(T)$ and $S^*(T)$. Now, for given sufficiently small $\varepsilon > 0$ we choose $T_\varepsilon \in (0,\overline{T}]$ such that $S^*(T_\varepsilon) = \frac{\varepsilon}{4}$. Since all the data $u_n$ and $u^*$ have to satisfy the condition (22), we find $0 < S_{min} < S_{max} < \infty$ and a positive integer $n_1$ depending on $\varepsilon$ such that

$$S_{min} \le S_n(T) \le S_{max}, \; S_{min} \le S^*(T) \le S_{max}(T) \quad (T_\varepsilon \le T \le \overline{T}, \; n \ge n_1).$$

Then we obtain

$$\|S_n - S^*\|_{C[T_\varepsilon,\overline{T}]} \le C\,\|u_n - u^*\|_{C[T_\varepsilon,\overline{T}]} \qquad (n \ge n_1(\varepsilon))$$

with the constant

$$C := \max_{(T,s)\in[T_\varepsilon,\overline{T}]\times[S_{min},S_{max}]} \left(\frac{\partial U_{BS}(X^*,K^*,r^*,T,s)}{\partial s}\right)^{-1}.$$

Moreover, there exists an integer $n_2$ depending on $\varepsilon$ with

$$|u_n(T) - u^*(T)| \le \frac{\varepsilon}{2C} \quad (0 \le T \le \overline{T}, \; n \ge n_2).$$

This provides

$$\|S_n - S^*\|_{C[T_\varepsilon,\overline{T}]} \le \frac{\varepsilon}{2} \qquad (n \ge \max(n_1,n_2)).$$

Using the growth of the functions $S_n$ and $S^*$ (see Theorem 3.4) and the triangle inequality we get for $n \ge \max(n_1,n_2)$ the estimations

$$\|S_n - S^*\|_{C[0,T_\varepsilon]} \le S_n(T_\varepsilon) + S^*(T_\varepsilon) \le |S_n(T_\varepsilon) - S^*(T_\varepsilon)| + 2S^*(T_\varepsilon) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

and

$$\|S_n - S^*\|_{C[0,\overline{T}]} \le \varepsilon,$$

which prove the theorem ∎

# 4 Solving the inverse problem in $L^p$-spaces for noisy option data

In this section we measure deviations of the time-dependent functions $u, S$ and $\sigma$ on the interval $[0, \overline{T}]$ by means of $L^p$-norms. For the Banach spaces of the inverse problem (IP) written as an operator equation (17), which is decomposed into the operator equations (19) and (20), there are used $B_1 := L^1(0, \overline{T}), B_2 := L^q(0, \overline{T})$ and $B_3 := L^p(0, \overline{T})$ with $1 \le p, q < \infty$. The positive function $u^\delta(T)$ $(0 \le T \le \overline{T})$ of observed maturity-dependent option prices is not necessarily smooth and arbitrage-free in the sense of Assumptions 3.1 and 3.3, but it satisfies the Assumption 4.1.

**Assumption 4.1** *We assume to know an upper uniform bound $\overline{\sigma} > 0$ of the nonnegative square-integrable local volatility function $\sigma^*$ implying the auxiliary function $S^* := I({\sigma^*}^2)$ such that*

$$0 \le \sigma^*(t) \le \overline{\sigma}, \quad 0 \le S^*(t) \le \kappa := \overline{T}\overline{\sigma}^2 \qquad (0 \le t \le \overline{T}). \tag{30}$$

*Moreover, the positive data function $u^\delta \in L^q(0, \overline{T})$ $(1 \le q < \infty)$ approximates with the estimate*

$$\|u^\delta - u^*\|_{L^q(0,\overline{T})} \le \delta \tag{31}$$

*the fair option price function $u^* = F({\sigma^*}^2) = N(S^*)$ for a given noise level $\delta > 0$.*

So we can apply a variant of the *method of $\delta$-quasisolutions* exploiting the fact that

$$D_{\overline{T}}^{\kappa} := \{S \in D_{\overline{T}}^{+} : 0 \le S(t) \le \kappa \ (0 \le t \le \overline{T}),\ S(t_1) \le S(t_2) \ (0 \le t_1 < t_2 \le \overline{T})\}$$

is a *compactum* in the Banach space $L^p(0, \overline{T})$ $(1 \le p < \infty)$. As an approximate solution of the outer problem (19) we use a $\delta$-quasisolution associated with the data $u^\delta$, which is a minimizer $S^\delta \in D_{\overline{T}}^{\kappa}$ of the extremal problem

$$\|N(S) - u^\delta\|_{L^q(0,\overline{T})} \longrightarrow \min, \qquad \text{subject to} \qquad S \in D_{\overline{T}}^{\kappa}. \tag{32}$$

If we measure the distance of two functions $x, y \in D_{\overline{T}}^{\kappa}$ in a uniform metrics on a subinterval $[a, b] \subseteq [0, \overline{T}]$, the symbol $\|x - y\|_{C[a,b]} := \sup_{t \in [a,b]} |x(t) - y(t)|$ will be used.

Then we can prove the following convergence assertion:

**Theorem 4.2** *Let $S^{\delta_n}$ be a sequence of $\delta_n$-quasisolutions associated with a sequence of data $u^{\delta_n}$ satisfying the inequality (31), where $\delta_n \to 0$ as $n \to \infty$. Then the convergence properties*

$$\lim_{n \to \infty} \|S^{\delta_n} - S^*\|_{L^p(0,\overline{T})} = 0 \tag{33}$$

*and*

$$\lim_{n \to \infty} \|S^{\delta_n} - S^*\|_{C[0,\gamma]} = 0 \quad \text{for all} \quad 0 < \gamma < \overline{T} \tag{34}$$

*hold.*

**Proof:** Since the Nemytskii operator

$$N = N_{\overline{T}}^{X^*,K^*,r^*} : \quad D_{\overline{T}}^{\kappa} \subset L^p(0, \overline{T}) \longrightarrow L^q(0, \overline{T})$$

is injective and continuous (see Lemmas 2.2 and 2.3), we obtain the first limit condition (33) immediately from Tikhonov's theorem on the continuity of the inverse of an operator, which is injective, continuous and defined on a compactum (see, e.g., [3, Lemma 2.2]). Moreover, from [3, Theorem 2.8] based on the continuity of the function $S^*$ we can formulate a further limit condition

$$\lim_{n\to\infty} \|S^{\delta_n} - S^*\|_{C[\beta,\gamma]} = 0 \quad \text{for all} \quad 0 < \beta < \gamma < \overline{T}, \tag{35}$$

where the approximate solution $S^{\delta_n} \in D^{\kappa}_{\overline{T}}$ may have discontinuities. Using the triangle inequality and the growth of the functions $S^*$ and $S^{\delta_n}$ we find

$$\|S^{\delta_n} - S^*\|_{C[0,\beta]} \le S^{\delta_n}(\beta) + S^*(\beta) \le \|S^{\delta_n} - S^*\|_{C[\beta,\gamma]} + 2S^*(\beta)$$

for arbitrarily small values $\beta > 0$. For any given $\varepsilon > 0$ there is a value $\beta_0 > 0$ such that $S^*(\beta_0) < \frac{\varepsilon}{4}$, since $\lim\limits_{\beta\to 0} S^*(\beta) = 0$. For sufficiently large $n$ we moreover have with (35) $\|S^{\delta_n} - S^*\|_{C[\beta_0,\gamma]} < \frac{\varepsilon}{2}$ and hence $\|S^{\delta_n} - S^*\|_{C[0,\gamma]} < \varepsilon$. This implies the limit condition (34) and proves the theorem ∎

As is well-known, the quasisolution method does not provide convergence rates. That means, the convergence (33) may be arbitrarily slow even on any subinterval. In particular, the $L^q$-data $u^\delta$ do not allow pointwise error estimations as given in formula (29). Moreover note that $D^{\kappa}_{\overline{T}} \cap L^\infty(0,\overline{T})$ fails to be a compactum in the Banach space $L^\infty(0,\overline{T})$. Therefore the uniform convergence of approximate solutions $S^\delta$ to $S^*$ cannot be shown on the whole interval $[0,\overline{T}]$. In contrast to monotonicity requirements the nonnegativity alone is not able to stabilze the solution process of an ill-posed problem. Therefore, the reconstruction of $\sigma^2 \in L^1(0,\overline{T})$ from data $S^\delta \in L^p(0,\overline{T})$ by solving the inner equation (20) remains here again the unstable part in solving the inverse problem (IP).

# 5 The discrete approach and some case studies

Now we are going to address to the realistic situation of financial markets that we have option data $u_j := u^\delta(T_j)$ approximating fair prices $u_j^* := u^*(T_j)$ only for a discrete set of maturities $T_0 = 0 < T_1 < T_2 < \ldots < T_k = \overline{T}$. We assume according to formula (22)

$$u_0 = \max(X^* - K^*, 0), \quad \max(X^* - K^* e^{-rT_j}) < u_j < X^* \quad (j = 1, 2, \ldots, k). \tag{36}$$

Using the decomposition $F(\sigma^2) = N(I(x)) = u$ with $x := \sigma^2$ we will consider a discrete approach for solving the inverse problem. In the first step we determine a vector $\underline{S} = (S_1, \ldots, S_k)^T \in \mathbb{R}^k_+$ of nonnegative components by solving the nonlinear equations

$$U_{BS}(X^*, K^*, r^*, T_j, S_j) = u_j \qquad (j = 1, 2, \ldots k). \tag{37}$$

Each of these $k$ equations can be solved by a simple line search algorithm. Since $U_{BS}(X^*, K^*, r^*, T_j, s)$ is strictly increasing with respect to $s > 0$, due to (12), (36) and the second limit condition in (13) all values $S_j$ are uniquely determined from (37). The second step contains a numerical differentiation, which is regularised according to

$$\|\underline{I}\,\underline{x} - \underline{S}\|_2^2 + \alpha\,\|\underline{L}\,\underline{x}\|_2^2 \longrightarrow \min, \qquad \text{subject to} \qquad \underline{x} \in \mathbb{R}^k_+, \tag{38}$$

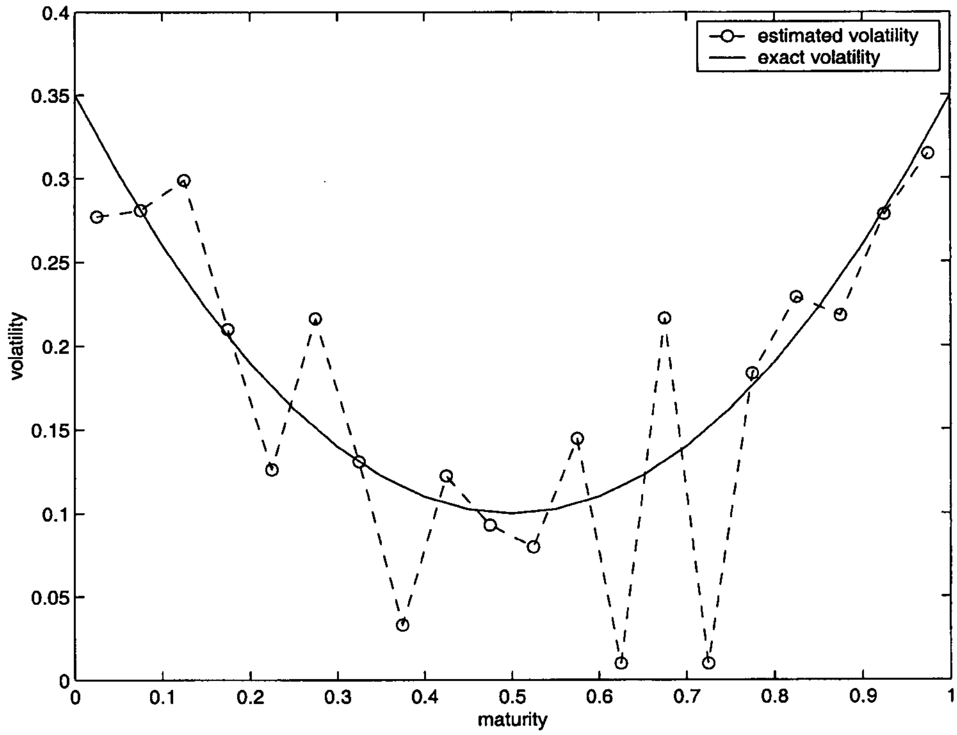


Figure 1: Unregularised solution ($\delta = 0.001$, $\alpha = 0$)

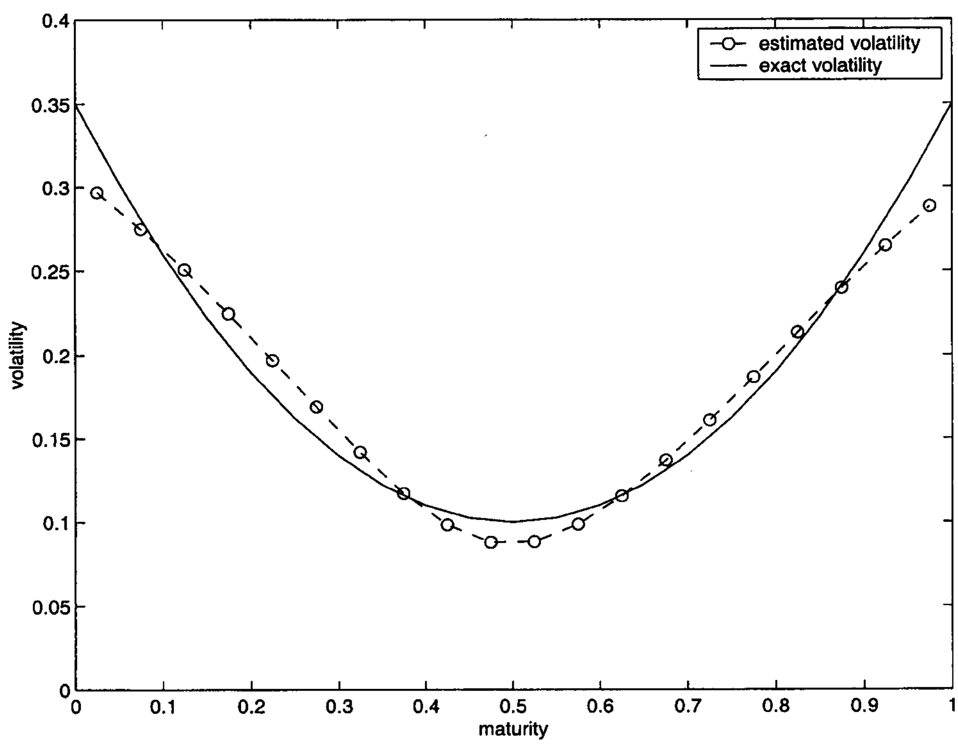


Figure 2: Regularised solution ($\delta = 0.001$, $\alpha = 7.1263 \cdot 10^{-7}$ from L-curve method)

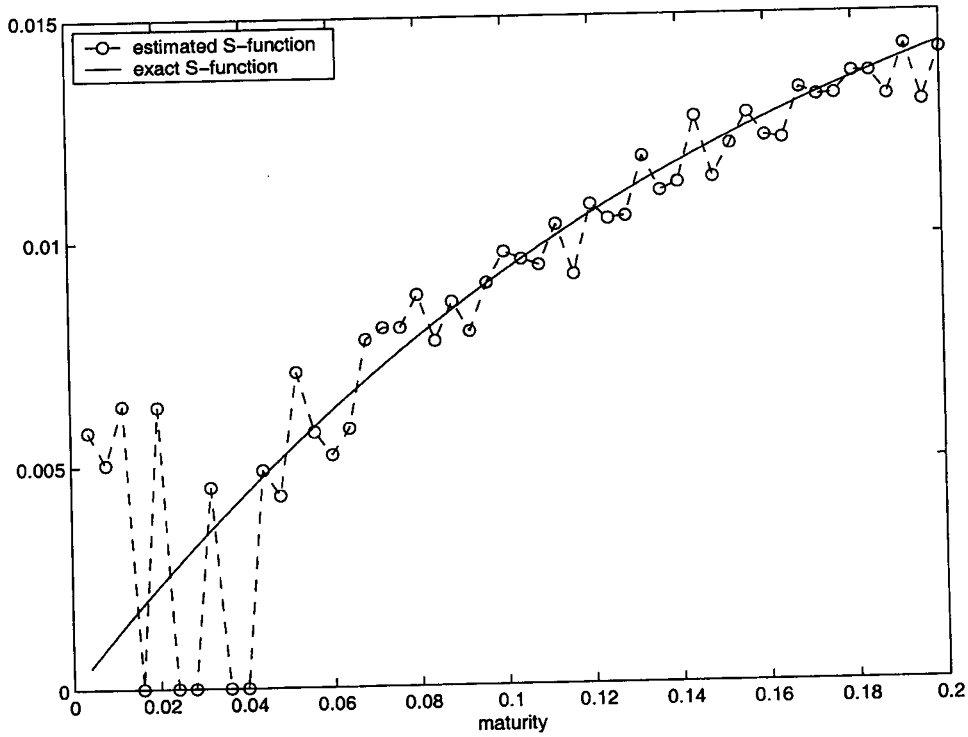


Figure 3: Pointwise reconstruction of $S(T)$ ($\delta = 0.001$, $k = 50$ grids on $[0, 0.2]$)

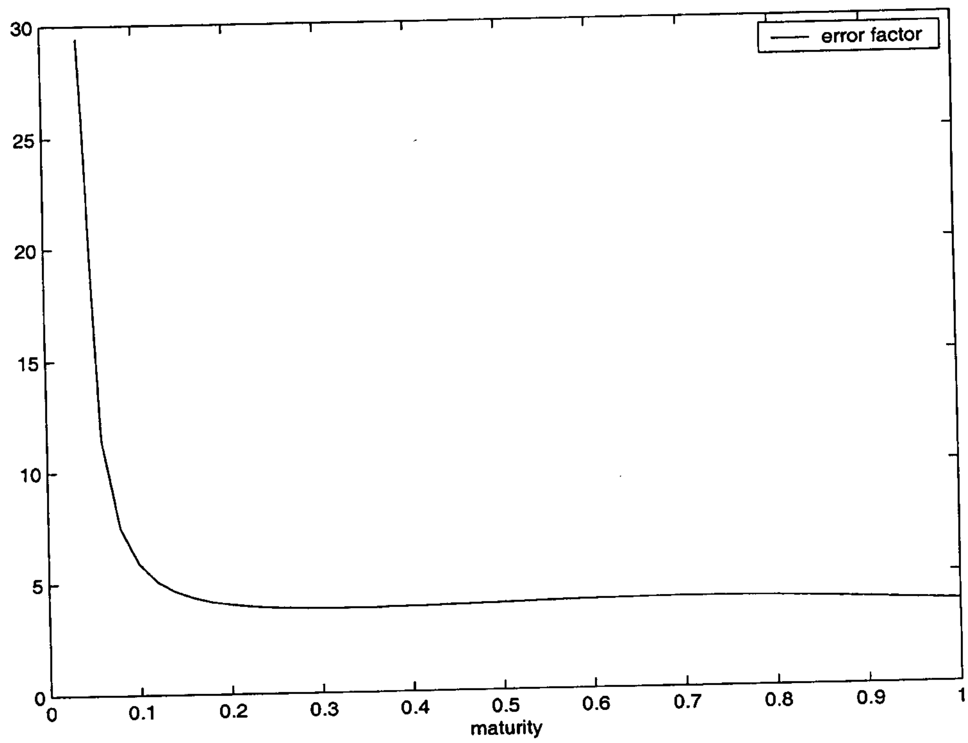


Figure 4: Behaviour of the error factor $\left(\frac{\partial U_{BS}(X^*,K^*,r^*,T,s)}{\partial s}\right)^{-1}\Big|_{s=S^*(T)}$ depending on $T$

with a minimizing vector $\underline{x}^\alpha = (x_1^\alpha, \ldots, x_k^\alpha)^T \in \mathbb{R}_+^k$, where $\alpha > 0$ is the regularization parameter, $\|\cdot\|_2$ denotes the Euclidean norm, $\underline{I}$ is a discretization of the linear Volterra integral operator $I$ and $\|\underline{L}\,\underline{x}\|_2^2$ expresses the obvious discretization of the $L^2$-norm square $\|x''\|^2_{L^2(0,\overline{T})}$ of the second derivative of the function $x = \sigma^2$. Finally we find with $\sigma_\alpha(T_j) := \sqrt{x_j^\alpha}$ $(j = 1, \ldots, k)$ the estimated local volatilities.

For a case study with computer-generated option price data we use the values $X^* := 0.6$, $K^* := 0.5$, $r^* := 0.05$, $T_j := \frac{j}{k}$ $(j = 1, \ldots, k := 20)$ and a convex volatility term-structure

$$\sigma^*(T) := (T - 0.5)^2 + 0.1, \qquad 0 \le T \le \overline{T} := 1$$

to be recovered. The exact data $\underline{u}^* = (u_1^*, \ldots, u_k^*)^T$ are computed by using the modified Black-Scholes formula (4) – (6). Perturbed with a random noise vector $\underline{e} = (e_1, \ldots, e_k)^T \in \mathbb{R}^k$ they yield the noisy data in the form

$$u_j := u_j^* + \delta \, \frac{\|\underline{u}^*\|_2}{\|\underline{e}\|_2} \, e_j \qquad (j = 1, \ldots, k)$$

for a given relative error $\delta > 0$.

The case study results are presented by figures showing on the one hand the exact solution as a solid line and on the other hand the linearly interpolated approximate solution as a dashed line. Figure 1 makes clear the oscillating character of the unregularized volatility reconstruction, even if the data error is rather small with 0.1%. For the same situation a quite good regularized solution is presented by Figure 2, where the regularisation parameter choice is based on Hansen's L-curve criterium (see [11]). As shown in Section 3, arbitrage-free option data $u$ yield in a unique and stable manner increasing functions $S$. When, however, the noisy discrete option data $u^\delta(T_j)$ are not necessarily arbitrage-free, then also for very small $\delta$ the monotonicity may be lost for values $S(T_j)$ obtained by a pointwise inversion of the Nemytskii operator $N$. In particular, if the remaining term $T$ of the option is small, the corresponding values $S(T_j)$ tend to oscillate (see Figure 3). This phenomenon is a consequence of the fact that $S(T)$ tends to zero as $T$ tends to zero. Namely, as shown in Figure 4, the error factor (cf. (29)) $\left(\frac{\partial U_{BS}(X^*,K^*,r^*,T,s)}{\partial s}\right)^{-1}\Big|_{s=S_T}$ grows to infinity as $T$ tends to zero.

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