A Note on Simultaneous Approximation

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The following result is well-known: If $f \in X$ (X: some normed function space) can be approximated of order $\|f-f_n\|_X \le c \inf_{g_n \in X_n} \|f-g_n\|_X = O(n^{-s-r})$ ($r, s > 0$ fixed) by elements $f_1, f_2, \dots$ of certain subspaces $X_1 \subseteq X_2 \subseteq \dots$ for which the Bernstein inequalities $\|g_n\|_Y \le c\, n^r \|g_n\|_X$, $g_n \in X_n$, hold true with some Banach space $Y \hookrightarrow X$ of smooth functions, then $\|f - f_n\|_Y = O(n^{-s})$. (Usually, $\|f\|_Y$ contains the norm of f and some norm of $f^{(r)}$, so that $\|f - f_n\|_Y = O(n^{-s})$ means simultaneous approximation of f and $f^{(r)}$ by f_n and $f_n^{(r)}$, respectively.)

We show that this result remains true if the order $O(a_n^{-1} n^{-r})$ is considered instead of $O(n^{-s-r})$, where a_n is strictly increasing and converges to infinity faster than n^ε (in a certain sense). We also present similar results in case $\sum (n^r \|f - f_n\|_X)^q (a_{n+1}^q - a_n^q) < \infty$ and in case of non-classical Bernstein inequalities, where $\{n^r\}$ is replaced by some other increasing sequence.

1. Introduction

Let X be a normed space and let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$ be linear subspaces of X. (For the sake of simplicity we do not consider nonlinear X_n and/or spaces X which are only quasi-normed. But this is also possible; see, e.g., [1] or [8].) Let us further suppose that, with some Banach space $Y \hookrightarrow X$, $Y \supseteq \bigcup X_n$ ("$\hookrightarrow$" means continuous embedding), the following Bernstein type inequality holds true, where $1 = A_1 < A_2 < A_3 < \dots$ are given positive numbers:

$$\|f_n\|_Y \le c\, A_n \|f_n\|_X \quad \text{for } f_n \in X_n \text{ and } n \in \mathbb{N} \tag{1.1}$$

where $c \ne c(n, f_n)$. (In the sequel we shall denote by c positive constants that may have different values at different places. By $c \ne c(n, f, \dots)$ we will indicate that c is independent of $n, f, \dots$.)

For fixed $1 \le q \le \infty$, we consider the following question: Under which conditions on a given sequence $1 = a_1 < a_2 < a_3 < \dots$, $\lim_{n\to\infty} a_n = \infty$, does the approximation order $\|\{a_n(q) A_n E_n(f)\}_{n=1}^\infty\|_q < \infty$ ($\|\,.\,\|_q$: l^q-norm), where

$$E_n(f) := \inf_{f_n \in X_n} \|f - f_n\|_X\,, \quad a_n(q) := \begin{cases} (a_{n+1}^q - a_n^q)^{1/q}, & 1 \le q < \infty, \\ a_n\,, & q = \infty, \end{cases}$$

imply that, for any sequence $\{f_n\}$, $f_n \in X_n$, with $\|f - f_n\|_X \le c\,E_n(f)$,

$$\big\|\big\{a_n(q)\,\|f - f_n\|_Y\big\}_{n=1}^{\infty}\big\|_q < \infty \quad \text{(especially, } f \in Y)\,?$$

In the weighted l^q-norms we use $a_n(q)$ instead of a_n, since we want to consider weights with $\|\{a_n(q)\}\|_q = \infty$ which do not need to satisfy any monotonicity condition (if $q < \infty$). One can use the following equivalence to find out how $a_n(q)$ behaves: If $a_{n+1} \le c\,a_n$ and $a_n = a(n)$ with $a \in C([1,\infty)) \cap \bigcap_{k\in\mathbb{N}} C^1[k,k+1]$ satisfying $a' > 0$ and $a'(\xi) \sim a'(n+0)$ for $\xi \in (n, n+1)$ and $n \in \mathbb{N}$, then $a_n(q) \sim a_n\big[(\ln a)'(n+0)\big]^{1/q}$ for all $n \in \mathbb{N}$. (We write $A \sim B$ for $n, \xi, \dots$ if $c_1 B \le A \le c_2 B$ with $0 < c_i \ne c_i(n, \xi, \dots)$.) This follows from the mean value theorem, applied to the difference $a^q(n+1) - a^q(n)$.

Let us explain why the answer to the above question is of interest and why it is necessary to generalize the known classical result which asserts that the implication is true if $A_n = n^r$ and $a_n = n^s$ ($r, s > 0$ fixed), i.e., that

$$\begin{aligned}&\big\|\big\{n^{r+s-(1/q)}E_n(f)\big\}\big\|_q < \infty \ \text{ implies } \ \big\|\big\{n^{s-(1/q)}\|f - f_n\|_Y\big\}\big\|_q < \infty,\\ &\text{supposed that } \|f - f_n\|_X \sim E_n(f) \text{ and (1.1) holds with } A_n = n^r.\end{aligned} \tag{1.2}$$

In approximation theory one usually considers an L^p-space X, a Sobolev space $Y = W^{p,r}$, and X_n consisting of all polynomials of degree less than n. For example, $X = L^p_{2\pi}$ (2π-periodic L^p-functions), $Y = W^{p,r}_{2\pi} = \{f : f^{(r)} \in L^p_{2\pi}\}$ ($\|f\|_Y = \|f\|_p + \|f^{(r)}\|_p$), $X_n = \mathcal{T}_n = \{\text{trig. polynomials of degree} < n\}$. Then it is known that (1.1) holds with $A_n = n^r$ and one wants to know whether $E^T_n(f)_p := \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_p = O(n^{-r}a_n^{-1})$ implies $\|f^{(r)} - T_n^{(r)}\|_p = O(a_n^{-1})$ for $T_n \in \mathcal{T}_n$ with $\|f - T_n\|_p \sim E^T_n(f)_p$. This corresponds to the case $q = \infty$ of the above question. Also the case $q < \infty$ is of interest. For example, the well-known Nikolskii inequality $\|T_n\|_{p_2} \le c\,n^{(1/p_1)-(1/p_2)}\|T_n\|_{p_1}$ ($p_1 < p_2$) can be interpreted as Bernstein type inequality with $X = L^{p_1}_{2\pi}$, $Y = L^{p_2}_{2\pi}$, and it follows from (1.2) that, for $s > (1/p_1) - (1/p_2)$, the Besov space $B^s_q(L^{p_1}_{2\pi}) = \big\{f \in L^{p_1}_{2\pi} : \|\{n^{s-(1/q)}E^T_n(f)_{p_1}\}_{n=1}^{\infty}\|_q < \infty\big\}$ is embedded into the Besov space $B_q^{s-(1/p_1)+(1/p_2)}(L^{p_2}_{2\pi})$. To obtain embedding theorems for generalized Besov spaces, one has to consider non-classical a_n. Generalized Besov spaces are needed, for example, in the theory of Cauchy singular and related integral equations (see [3, 5, 6, 7, 9, 10, 11, 12]). Numbers A_n different from n^r appear if one wants to study the mapping properties of an unbounded operator $A \in \mathcal{L}\big(\bigcup X_n, X\big)$: If $\|A\|_{X_n \to X} \le c\,A_n$, then (1.1) is satisfied with the completion Y of $\bigcup X_n$ w.r.t. $\|f\| := \|f\|_X + \|Af\|_X$. Thus, if the above implication is true, then A can be extended to an operator which maps the so-called approximation space $A(X, l^q(\{A_n a_n(q)\}); \{X_n\}) := \big\{f \in X : \{E_n(f)\}_{n=1}^{\infty} \in l^q(\{A_n a_n(q)\})\big\}$ (endowed with $\|f\|_X + \|\{A_n a_n(q) E_n(f)\}\|_q$; $l^q(\{b_n\}) = \{\{E_n\} : \{b_n E_n\} \in l^q\}\big)$ into $A(X, l^q(\{a_n(q)\}); \{A(X_n)\})$. This result is useful in the numerical analysis of operator equations in which unbounded operators occur (see [10]).

The key to the answer of the above question is the following result, which is a special case of [8, Theorem 2.2 and estimate (2.7)]. (Apply [8, Theorem 2.2] with $c_n = a_n(q)$, $d_n = 1$, and remark that only [8, (8.1)] is needed in the proof.)

Theorem 1. *Let* $1 = a_1 < a_2 < a_3 < \dots$ *such that* $\lim_{n\to\infty} a_n = \infty$ *and*

$$a_{n+1} \le K a_n \quad (n \in \mathbb{N}) \quad \text{with some constant } K > 1\,. \tag{1.3}$$

Then, the numbers $n(j) := \max\{n \in \mathbb{N} : a_n \le K^j\}$ $(j = 0, 1, \dots)$ *are well-defined,* $a_{n(j)}$ *behaves like* K^j*, more precisely,*

$$K^{j-1} < a_{n(j)} \le K^j \tag{1.4}$$

(especially, $1 = n(0) < n(1) < n(2) < \dots$*), and the equivalence*

$$\left\| \{a_n(q) E_n\}_{n=1}^{\infty} \right\|_q \sim \left\| \left\{K^j E_{n(j)}\right\}_{j=0}^{\infty} \right\|_q \tag{1.5}$$

holds uniformly in all sequences $\{E_n\}_{n=1}^{\infty} \subseteq [0, \infty)$ *which satisfy*

$$C^{-1} E_{n(j+1)} \le E_n \le C E_{n(j)} \quad \text{for all } n \in [n(j), n(j+1)] \ (j \in \mathbb{N}_0)\,, \tag{1.6}$$

where $C \ge 1$ *is some constant which has to be given in advance. (This means that the constants in the equivalence (1.5) depend on* C*, but not on* $\{E_n\}$*.)*

Condition (1.3) means, in some sense, that $\{a_n\}$ cannot increase faster than exponential: Exponential increase $a_n = K^{n-1}$ is still allowed, but not more, since (1.3) implies $a_n \le K a_{n-1} \le K^2 a_{n-2} \le \dots \le K^{n-1}$.

In [1] and [8] it is shown that Theorem 1 is a powerful tool in the theory of approximation spaces. The following result underlines this fact.

2. The Main Result

Let $1 = a_1 < a_2 < a_3 < \dots$ be fixed numbers, take $X, Y, \{X_n\}_{n=1}^{\infty}, \{a_n(q)\}_{n=1}^{\infty}$ from Section 1, and suppose that (1.1) holds true with

$$1 = A_1 < A_2 < A_3 < \dots \quad \text{satisfying} \quad \lim_{n\to\infty} A_n = \infty\,.$$

A sequence $\{b_n\}_{n=1}^{\infty} \subseteq [0, \infty)$ is called almost increasing if it is equivalent to some increasing sequence. In other words: $\{b_n\}$ is almost increasing if and only if $b_m \le c\, b_n$ for all $m \le n$, where $c \ne c(n, m)$. (Indeed, if the last inequality holds for all $m \le n$, then $\max_{m \le n} b_m$ is equivalent to b_n).

Theorem 2. *Let* a_n *increase faster than some positive power of* A_n*, but not faster than exponential, in the following sense:*

$$a_{n+1} \le c\, a_n \quad \text{and} \quad \{A_n^{-\varepsilon} a_n\}_{n=1}^{\infty} \text{ is almost increasing for some } \varepsilon > 0. \tag{2.1}$$

Then, for all $f \in X$ *with* $\left\| \{a_n(q) A_n E_n(f)\}_{n=1}^{\infty} \right\|_q < \infty$ *and all* $f_n \in X_n$ $(n \in \mathbb{N})$ *with* $\|f - f_n\|_X \le M E_n(f)$ $(M \ne M(n)$, $M = M(f, \{f_n\}))$*, we have* $f \in Y$ *and*

$$\left\| \{a_n(q)\, \|f - f_n\|_Y\}_{n=1}^{\infty} \right\|_q \le c\, M \left\| \{a_n(q) A_n E_n(f)\}_{n=1}^{\infty} \right\|_q\,, \tag{2.2}$$

where $c \ne c(f, \{f_n\})$.

Remark 1. *In the proof we will see that this theorem remains true if $\{E_n(f)\}$ is not the sequence of best approximation errors of f, but any other decreasing sequence of nonnegative numbers which may depend on f. We have $c \neq c(f, \{f_n\}, \{E_n(f)\})$ in this case. If $q = \infty$, then this can be used to delete the restriction $b_n \leq c\, b_{n+1}$ in the assertion $\|f - f_n\|_Y = O(b_n)$:*

If $A_{n+1} \leq cA_n$ and if $\{b_n\}_{n=1}^{\infty} \subseteq (0, \infty)$ is a fixed sequence for which, with some $\varepsilon > 0$, $\{b_n A_n^{\varepsilon}\}$ is almost decreasing, then $\|f - f_n\|_X \leq M(f) A_n^{-1} b_n$ ($f \in X$, $f_n \in X_n$, $n \in \mathbb{N}$) implies $f \in Y$ and $\|f - f_n\|_Y \leq cM(f)\, b_n$, where $c \neq c(n, f, \{f_n\})$. Indeed, for fixed m, we may apply the above assertion with $\widetilde{f} = f - f_m$, $\widetilde{f}_n = 0$ for $n < m$, $\widetilde{f}_n = f_n - f_m$ for $n \geq m$, $a_n = A_n^{\varepsilon}$: $\|\widetilde{f} - \widetilde{f}_n\|_X \leq M(f) A_{\max\{n,m\}}^{-1} a_{\max\{n,m\}}^{-1} b_{\max\{n,m\}} A_{\max\{n,m\}}^{\varepsilon} \leq cM(f) A_n^{-1} a_n^{-1} b_m A_m^{\varepsilon}$ and we may set $M = cM(f)\, b_m A_m^{\varepsilon}$ and $E_n = A_n^{-1} a_n^{-1}$. It follows $\|\widetilde{f} - \widetilde{f}_n\|_Y \leq cM(f)\, b_m A_m^{\varepsilon} a_n^{-1}$, $n \in \mathbb{N}$, where c does not depend on $\widetilde{f}$ and $\{\widetilde{f}_n\}$, i.e., $c \neq c(n, m, f, \{f_n\})$. For $n = m$ we get $\|f - f_m\|_Y \leq cM(f)\, b_m$.

Before we come to the proof of Theorem 2, let us compare it with known results: In the classical setting $X = L_{2\pi}^p$, $X_n = \mathcal{T}_n$, $Y = W_{2\pi}^{p,r}$, $A_n = n^r$ ($1 \leq p \leq \infty$, $r \in \mathbb{N}$), it follows from (2.2) and Jackson's inequality $E_n^T(f)_p \leq c\, n^{-r} E_n^T(f^{(r)})_p$ ([4, (7.2.17)]) that $E_n^T(f^{(r)})_p \in l^q(\{a_n(q)\})$ and $\|f - T_n\|_p \sim E_n^T(f)_p$ imply $\|f^{(r)} - T_n^{(r)}\|_p \in l^q(\{a_n(q)\})$. In this form, the assertion is not new, since $\|f - T_n\|_p \leq M E_n^T(f)_p$ ($f \in W_{2\pi}^{p,r}$) even implies $\|f^{(r)} - T_n^{(r)}\|_p \leq cM E_n^T(f^{(r)})_p$ ([4, Theorem 7.2.8 and its proof]). But Theorem 2 asserts more than only the equivalence of $E_n^T(f^{(r)})_p \in l^q(\{a_n(q)\})$ and $\|f^{(r)} - T_n^{(r)}\|_p \in l^q(\{a_n(q)\})$ (if $\|f - T_n\|_p \sim E_n^T(f)_p$): If $\{a_n\}$ satisfies (2.1) with $A_n = n^r$, then $E_n^T(f)_p \in l^q(\{n^r a_n(q)\})$ implies $E_n^T(f^{(r)})_p \in l^q(\{a_n(q)\})$. This is a new result (as far as we know). For this reason, we explicitly state the following

Corollary 1. *Let $1 \leq p \leq \infty$, $r \in \mathbb{N}$, and let (2.1) be satisfied with $A_n = n$. If $f \in L_{2\pi}^p \to f_n \in \mathcal{T}_n$ is an optimal approximation method, i.e., $\|f - f_n\|_p \leq c\, E_n^T(f)_p$ with $c \neq c(n, f)$, then, for all $f \in W_{2\pi}^{p,r}$ and all $k = 0, \ldots, r$,*

$$\left\| \{ n^{r-k} a_n(q)\, \|f^{(k)} - f_n^{(k)}\|_p \}_{n=1}^{\infty} \right\|_q \sim \left\| \{ n^{r-k} a_n(q) E_n^T(f^{(k)})_p \}_{n=1}^{\infty} \right\|_q \sim \left\| \{ n^r a_n(q) E_n^T(f)_p \}_{n=1}^{\infty} \right\|_q .$$

Moreover, every $f \in L_{2\pi}^p$ with $\left\| \{n^r a_n(q) E_n^T(f)_p\} \right\|_q < \infty$ belongs to $W_{2\pi}^{p,r}$.

Proof. First we prove that Theorem 2 can be applied with a_n replaced by

$$\widetilde{a}_n := \begin{cases} n^{r-k} a_n & , \quad q = \infty, \\ \left\| \{ m^{r-k} a_{m-1}(q) \}_{m=1}^{n} \right\|_q \ (a_0(q) := 1) & , \quad q < \infty, \end{cases}$$

i.e., that (2.1) (with $A_n = n$) is satisfied for $\widetilde{a}_n$. For $q = \infty$ this is clear and for $q < \infty$ we show that $\widetilde{a}_n \sim n^{r-k} a_n$: Obviously, $\widetilde{a}_n \leq n^{r-k} \left\| \{a_{m-1}(q)\}_{m=1}^{n} \right\|_q =$

$n^{r-k}a_n$. On the other hand, it follows from $(m^{-\varepsilon}a_m)^q \le C(n^{-\varepsilon}a_n)^q$, $m \le n$, that, for $M := (C+1)^{1/(\varepsilon q)}$ and $n > M$,

$$\begin{aligned}
\widetilde{a}_n^q &\ge \left(\frac{n}{M}\right)^{q(r-k)} \left\| \left\{a_{m-1}(q)\right\}_{m=[n/M]+1}^{n} \right\|_q^q = \left(\frac{n}{M}\right)^{q(r-k)} \left(a_n^q - a_{[n/M]}^q\right) \\
&\ge \left(\frac{n}{M}\right)^{q(r-k)} \left[\frac{n}{M}\right]^{\varepsilon q} \left(M^{\varepsilon q} n^{-\varepsilon q} a_n^q - \left[\frac{n}{M}\right]^{-\varepsilon q} a_{[n/M]}^q\right) \\
&= \left(\frac{n}{M}\right)^{q(r-k)} \left[\frac{n}{M}\right]^{\varepsilon q} \left(n^{-\varepsilon q} a_n^q + C n^{-\varepsilon q} a_n^q - \left[\frac{n}{M}\right]^{-\varepsilon q} a_{[n/M]}^q\right) \ge c\, n^{q(r-k)} a_n^q .
\end{aligned}$$

Now we apply Theorem 2 with $\widetilde{a}_n$ instead of a_n and with $Y = W_{2\pi}^{p,k}$, $A_n = n^k$. Taking into account that $\widetilde{a}_n(q) \sim n^{r-k}a_n(q)$, it follows

$$\left\| \left\{ n^{r-k} a_n(q) \, \| f^{(k)} - f_n^{(k)} \|_p \right\}_{n=1}^{\infty} \right\|_q \le c \left\| \left\{ n^r a_n(q) E_n^T(f)_p \right\}_{n=1}^{\infty} \right\|_q , \tag{2.3}$$

supposed that the right hand side is finite (which implies $f \in W_{2\pi}^{p,k}$). Jackson's inequality $E_n^T(f)_p \le c\, n^{-k} E_n^T(f^{(k)})_p$ shows that the right hand side of (2.3) can be estimated by a multiple of $\left\| \left\{ n^{r-k} a_n(q) E_n^T(f^{(k)})_p \right\}_{n=1}^{\infty} \right\|_q$. Of course, the last expression is less than or equal to the left hand side of (2.3), i.e., both terms in (2.3) are equivalent to this expression. ∎

Now we come back to the setting of Theorem 2. Also in this general framework there is a known result, but only for the case $q = \infty$: Under the assumptions of Remark 1, the implication "$\|f - f_n\|_X \sim E_n(f) = O(A_n^{-1}b_n) \Rightarrow \|f - f_n\|_Y = O(b_n)$" is a part of [2, Theorem 4.2], at least for certain sequences $\{A_n\}$ (e.g., for $A_n = n^r$, $r > 0$; [2, Part c) of Theorem 2.1 and Lemma 2.3]). Maybe it is possible to use the ideas from this paper to generalize the result to the case of arbitrary $q \in [1, \infty]$. But here we go another way. Namely, we use Theorem 1, since we already know from [8] and [1], that this theorem is a powerful tool if one wants to prove general results concerning the behaviour of best approximation errors.

3. Proof of the Main Result

Lemma 1 ([4], Lemma 2.3.4). *Let $K > 1$ be some constant. Then,*

$$\left\| \{ K^j \textstyle\sum_{i=j}^{\infty} b_i \}_{j=0}^{\infty} \right\|_q \le c \left\| \{ K^j b_j \}_{j=0}^{\infty} \right\|_q \quad \textit{for all } \{b_j\} \subseteq [0, \infty) \quad (c \ne c(\{b_j\})).$$

Lemma 2. *If (2.1) is fulfilled and if $n(j)$ are the numbers from Theorem 1, then $A_{n(j+1)} \le c\, A_{n(j)}$ for all j.*

Proof. We have $a_{n(j)} \sim a_{n(j+1)}$ ((1.4)). Hence, $A_{n(j)}^{-\varepsilon} = [A_{n(j)}^{-\varepsilon} a_{n(j)}]\, a_{n(j)}^{-1} \le c\, [A_{n(j+1)}^{-\varepsilon} a_{n(j+1)}]\, a_{n(j+1)}^{-1} = c\, A_{n(j+1)}^{-\varepsilon}$. ∎

Lemma 3. *Take the assumptions of Lemma 2 and let $f \in X$. If there are $f_i \in X_{n(i)}$ such that $\sum_{i=0}^{\infty} A_{n(i)} \|f - f_i\|_X < \infty$, then $f \in Y$ and*

$$\|f - f_j\|_Y \le c \sum_{i=j}^{\infty} A_{n(i)} \|f - f_i\|_X \quad \text{for all } j \in \mathbb{N}_0\,, \quad \text{where } c \ne c(f, j, \{f_j\}).$$

Proof. We have $f_k - f_j = \sum_{i=j}^{k-1} (f_{i+1} - f_i)$ for $k > j$. Thus,

$$\begin{aligned} \|f_k - f_j\|_Y &\le \sum_{i=j}^{k-1} \|f_{i+1} - f_i\|_Y \le c \sum_{i=j}^{k-1} A_{n(i+1)} \|f_{i+1} - f_i\|_X \\ &\le c \sum_{i=j}^{\infty} A_{n(i+1)} \|f_{i+1} - f\|_X + c \sum_{i=j}^{\infty} A_{n(i+1)} \|f - f_i\|_X \end{aligned}$$

Together with Lemma 2 we obtain $\|f_k - f_j\|_Y \le c \sum_{i=j}^{\infty} A_{n(i)} \|f - f_i\|_X$ for all $k > j \ge 0$. We have supposed that the right hand side converges to zero for $j \to \infty$. Thus, $\{f_j\}$ is a Cauchy sequence in Y. Of course, the Y-limit is equal to the X-limit, i.e., equal to f (since $\|f - f_j\|_X \to 0$ because of the assumption $\sum_{j=0}^{\infty} A_{n(j)} \|f - f_j\|_X < \infty$). Consequently, $f \in Y$ and the limit $k \to \infty$ in the above estimate for $\|f_k - f_j\|_Y$ yields the assertion. ∎

Proof of Theorem 2. In view of Lemma 3, $\|f - f_n\|_Y$ can be estimated by

$$\|f - f_{n(j)}\|_Y + \|f_{n(j)} - f_n\|_Y \le cM \sum_{i=j}^{\infty} A_{n(i)} E_{n(i)}(f) + \|f_{n(j)} - f_n\|_Y .$$

(Later we will see that the sum is finite and, consequently, $f \in Y$.) This holds for all n, j and, hence, also for $j = j_n$ defined by $n \in [n(j), n(j+1))$, where n is arbitrary. For $j = j_n$ we have $\|f_{n(j)} - f_n\|_Y \le c\, A_{n(j+1)} \|f_{n(j)} - f_n\|_X \le cM A_{n(j+1)} E_{n(j)}(f)$. Together with $A_{n(j+1)} \sim A_{n(j)}$ (Lemma 2) it follows

$$\|f - f_n\|_Y \le cM \sum_{i=j_n}^{\infty} A_{n(i)} E_{n(i)}(f) =: E_n .$$

$\{E_n\}$ is decreasing, since $\{j_n\}$ is increasing. Thus, (1.5) can be applied:

$$\left\|\{a_n(q)\, \|f - f_n\|_Y\}\right\|_q \le \left\|\{a_n(q) E_n\}\right\|_q \sim M \left\| \left\{ K^j \sum_{i=j}^{\infty} A_{n(i)} E_{n(i)}(f) \right\}_{j=0}^{\infty} \right\|_q .$$

By Lemma 1, the last expression remains equivalent if the sum is replaced by its first addend. Hence,

$$\left\|\{a_n(q)\, \|f - f_n\|_Y\}_{n=1}^{\infty}\right\|_q \le cM \left\|\{K^j\, A_{n(j)} E_{n(j)}(f)\}_{j=0}^{\infty}\right\|_q .$$

We apply again (1.5), but now with $E_n := A_n E_n(f)$. This is possible, since E_n satisfies (1.6) with the constant C from $A_{n(j+1)} \le C A_{n(j)}$. Thus,

$$\left\|\{a_n(q)\, \|f - f_n\|_Y\}_{n=1}^{\infty}\right\|_q \le cM \left\|\{a_n(q) A_n E_n(f)\}_{n=1}^{\infty}\right\|_q$$

and the theorem is proved. ∎

References

[1] J. M. Almira, U. Luther, Generalized approximation spaces and applications, Accepted for publication in *Math. Nachr.*

[2] P. L. Butzer, S. Jansche, R. L. Stens, Functional analytic methods in the solution of the fundamental theorems on best-weighted algebraic approximation, In: G. A. Anastassiou (ed.), *Approximation Theory*, Lecture Notes in Pure and Applied Mathematics, Vol. **138**, Marcel Dekker, 1992, pp. 151-205.

[3] M. R. Capobianco, G. Criscuolo, P. Junghanns, U. Luther, Uniform convergence of the collocation method for Prandtl's integro-differential equation, *ANZIAM J.* **42** (2000), 151-168.

[4] R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer, 1993.

[5] P. Junghanns, U. Luther, Cauchy singular integral equations in spaces of continuous functions and methods for their numerical solution, *J. Comp. Appl. Math.* **77** (1997), 201-237.

[6] P. Junghanns, U. Luther, Uniform convergence of the quadrature method for Cauchy singular integral equations with weakly singular perturbation kernels, *Rendiconti del Circolo Matematico di Palermo*, Serie II, **52** (1998), 551-566.

[7] P. Junghanns, U. Luther, Uniform convergence of a fast algorithm for Cauchy singular integral equations, *Linear Algebra and its Applications* **275-276** (1998), 327-347.

[8] U. Luther, Representation, interpolation, and reiteration theorems for generalized approximation spaces, Accepted for publication in *Ann. Mat. Pura Appl.*

[9] U. Luther, Cauchy singular integral operators in weighted spaces of continuous functions, Submitted to *Integr. Equ. Oper. Theory*

[10] U. Luther, Approximation spaces in the numerical analysis of operator equations, Submitted to *Advances in Comp. Math.*

[11] U. Luther, M. G. Russo, Boundedness of the Hilbert transformation in some weighted Besov spaces, *Integr. Equ. Oper. Theory* **36** (2000), 220-240.

[12] G. Mastroianni, M. G. Russo, W. Themistoclakis, The boundedness of the Cauchy singular integral operator in weighted Besov type spaces with uniform norms, *Integr. Equ. Oper. Theory* **42** (2002), no. 1, 57-89.

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