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Abstract

We derive various geometrical and analytical representations for diameter and thickness of a convex body in Minkowski space (i.e., in a real finite-dimensional Banach space).

1 Introduction

One-dimensional cross-section measures are one-dimensional section and projection measures of convex bodies in $\mathbb{E}^d$, $d \geq 2$. More precisely, this notion is used to describe, in a unified way, the *width function* and the *maximal chord-length function* of a convex body $K \subset \mathbb{E}^d$ (see the monograph [BF74, §30 and §33], where also the notions of outer 1-quermass and inner 1-quermass are used). References regarding these cross-section measures are collected in [Mar94] and [Gar95, Chapter 3]. Both these concepts are closely related to further important tools and notions from convexity, such as Minkowski addition, difference bodies, central symmetrals, affine diameters, antipodality, circum- and inradius, cf. the references above as well as [Grü67, §19.3], [Sch93, Chapters 3 and 7] and the survey [Mar96]. In particular, the minima and maxima of the width function and the maximal chord-length function coincide and are usually called *thickness* (minimal width) and *diameter*, yielding also a connection to the classical notions of *bodies of constant width* and *reduced bodies*.

We will consider the analogues of both these functions for convex bodies in Minkowski space, and therefore provide some analytical and geometrical descriptions of their corresponding extensions (a basic reference to the geometry of Minkowski spaces is [Tho96], see also [MSW01]). Collecting various approaches to these functions from the literature, we give a unified way to represent them in different terms, and prove some theorems about Minkowskian diameter, thickness, in- and circumradius.

2 Basic notations and background material

As usual, $\mathbb{E}^d$ denotes the d-dimensional Euclidean space with origin o, while $\langle\, . \,, . \,\rangle$ and $|\,.\,|$ denote the scalar product and the norm in $\mathbb{E}^d$, respectively. The unit ball

$\{x \in \mathbb{E}^d \mid |x| \leq 1\}$ in $\mathbb{E}^d$ is denoted by S_E. We shall always use the letter K for an arbitrary *convex body* in $\mathbb{E}^d$, i.e., a compact, convex set with nonempty interior. The notations $\operatorname{ext} K$ and $\exp K$ denote the set of all *extreme* and *exposed points* of a convex body K, respectively. Furthermore we write $\operatorname{bd} K$ for the *boundary* of K. If X is a set from $\mathbb{E}^d$, then $\operatorname{cl} X$ is its *closure*. Given points x, y in $\mathbb{E}^d$, $[x, y]$ denotes the line segment with endpoints x and y.

A real finite-dimensional Banach space of dimension d, $d \geq 2$, is said to be a d-dimensional *Minkowski space*. Since all linear spaces of a given dimension are isomorphic we always identify any Minkowski space of dimension d with $\mathbb{E}^d$. It is a well known fact that every Minkowski space M with norm $\|\,.\,\|$ is uniquely defined by its unit ball, i.e., by the set $\{x \in M \mid \|x\| \leq 1\}$. Moreover, each convex body B symmetric with respect to the origin defines a Minkowski space with this body as its unit ball. In view of this it is possible to introduce the notation $\mathcal{M}^d(B)$ for the d-dimensional Minkowski space with unit ball B. Then the norm of this space is denoted by $\|\,.\,\|_B$.

For a convex body K containing the origin in its interior, the body

$$K^* := \{y \in \mathbb{E}^d \mid \langle y, x\rangle \leq 1 \text{ for every } x \in K\}$$

is called the *polar dual* of K. Notice that the space $\mathcal{M}^d(B^*)$ is dual to $\mathcal{M}^d(B)$, as it is usually written also in functional analysis.

The *difference body* DK of K is introduced by

$$DK := \{x - y \mid x, y \in K\}.$$

The body DK is convex since it is the Minkowski sum of the convex bodies K and $-K := \{-x \mid x \in K\}$, see [Sch93, §3.1].

The *support function* of K (notation: $h_K(u)$) is defined by

$$h_K(u) := \max\{\langle x, u\rangle \mid x \in K\}, \tag{2.1}$$

where u ranges over $\mathbb{E}^d$. The *width function* of K is defined by means of the support function as follows:

$$w_K(u) := h_K(u) + h_K(-u), \tag{2.2}$$

Let $H_K(u)$ be the supporting hyperplane of the body K with outward normal $u \in \mathbb{E}^d \setminus \{o\}$. Trivially, $H_K(u) := \{x \in K \mid \langle x, u\rangle = h_K(u)\}$. Then the width function $w_K(u)$ is the distance between parallel supporting hyperplanes $H_K(u)$ and $H_K(-u)$.

Given a convex body K containing the origin, its *radius function* is introduced by

$$r_K(u) := \max\{\alpha \mid \alpha u \in K\} \quad (u \in \mathbb{E}^d \setminus \{o\}). \tag{2.3}$$

The *maximal chord-length function* of K is defined by

$$l_K(u) := \max\{\alpha \mid x, y \in K,\ x - y = \alpha u \text{ for some } \alpha > 0\}, \tag{2.4}$$

where $u \in \mathbb{E}^d \setminus \{o\}$.The functions $h_K(u)$ and $w_K(u)$ are homogeneous of order one, while $r_K(u)$ and $l_K(u)$ are homogeneous of order -1. Furthermore, it turns out that these functions are continuous (see, for instance, [Gar95, pp.16,18]). Hence, whenever we consider the supremum or the infimum over a compact set of algebraic expressions constructed by the functions $h_K(u), r_K(u), w_K(u)$ or $l_K(u)$, we can always replace them by maximum and minimum, respectively.

There are the following familiar relations for the functions defined above:

$$\begin{aligned} w_K(u) &= h_{DK}(u), \\ l_K(u) &= r_{DK}(u). \end{aligned}$$

We notice that for a convex body B symmetric with respect to the origin we obtain $h_B(u) = \|u\|_{B^*}$ just applying the definition of the support function and the fact that $\|\,.\,\|_{B^*}$ is the norm of the space dual to $\mathcal{M}^d(B)$.

In the sequel we need the following characterizations of inclusion, which might be considered as foklore.

Lemma 1. *Let B_1 and B_2 be convex bodies in $\mathbb{E}^d$ symmetric with respect to the origin. Then the following conditions are equivalent.*

(i) The body B_1 is a subset of B_2.

(ii) For every direction u we have $r_{B_1}(u) \le r_{B_1}(u)$.

(iii) For every direction u we have $h_{B_1}(u) \le h_{B_1}(u)$.

(iv) The body B_2^* is a subset of B_1^*.

□

3 Cross-section measures in Minkowski spaces

Further on, we introduce the notion of one-dimensional *cross-section measures* in Minkowski spaces, starting with the respective generalization of the width function. Although there are several ways to define the width function in Minkowski spaces, the following seems to be the most natural one. We consider two parallel hyperplanes of a body K having the normal $u \in \mathbb{E}^d \setminus \{o\}$ and the Minkowskian distance between them (this means here the distance between their closest points in the Minkowskian metric). This quantity is said to be the Minkowskian width of K at direction u and denoted by $w_{K,B}(u)$. By defintion we have:

$$w_{K,B}(u) := \min\left\{\|x_1 - x_2\|_B \,|\, x_1 \in H_K(u), x_2 \in H_K(-u)\right\}. \tag{3.1}$$

Let us consider an arbitrary point x in $H_K(-u)$ and a ball $\alpha B + x$ with $\alpha > 0$ chosen so that $H_K(u)$ supports $\alpha B + x$. Obviously, any point y from $(\alpha B + x) \cap H_K(u)$ is at Minkowskian distance α from x, and for every other point y from $H_K(u)$ we have $\|y - x\|_B > \alpha$. Thus, α is the Minkowskian distance from x to the hyperplane $H_K(u)$. Consider also the unit ball B and its supporting hyperplane $H_B(u)$. If we apply the transformation of multiplying, by the homothety constant α, and then translating by the vector x to both B and $H_B(u)$, then B transforms to $\alpha B + x$ and $H_B(u)$ to $H_K(u)$. But the Euclidean distance from $H_K(u)$ to x is $w_K(u)$, while the Euclidean distance from o to $H_B(u)$ is $h_B(u)$. Therefore we have $\alpha h_B(u) = w_K(u)$. Consequently, the distance from x to $H_K(u)$ does not depend on the choice of x in $H_K(-u)$ and is the distance between the hyperplanes $H_K(u)$ and $H_K(-u)$. Thus,

$$w_{K,B}(u) = \frac{w_K(u)}{h_B(u)}. \tag{3.2}$$

Notice that in [BS78] the width is defined as in this paper, i.e., actually by (3.1). In [CG83] the width function is introduced by (3.2), and in [Tho96] it is given by

$$w_{K,B}(u) = w_K(u), \tag{3.3}$$

where $u \in \operatorname{bd} B^*$, which is also in accordance with our derivations since, if $u \in \operatorname{bd} B^*$, then $h_B(u) = \|u\|_{B^*} = 1$, and therefore (3.3) can be reduced to (3.2).

The generalization of the maximal chord-length function goes more or less automatically. We consider a Minkowski space $\mathcal{M}^d(B)$, a measured body K and a direction u. Then the length of the longest (in Minkowskian sense) chord of K of direction u is said to be the value of the Minkowskian maximal chord-length function at direction u (notation: $l_{K,B}(u)$). This verbal definition can be expressed in an analytical way as follows:

$$l_{K,B}(u) = \max\{\|x-y\|_B \mid x, y \in K,\ x-y = \alpha u, \text{ for some } \alpha > 0\}. \tag{3.4}$$

The difference $(x-y)$ is a point of DK and its norm is equal to $\alpha\|u\|_B$. Thus

$$l_{K,B}(u) = \|u\|_B \max\{\alpha > 0 \mid \alpha u \in DK\} = \|u\|_B l_K(u). \tag{3.5}$$

We notice that $r_B(u) = \max\{\alpha > 0 \mid \|\alpha u\|_B \leq 1\} = 1/\|u\|_B$. Therefore, modifying (3.5), we arrive at

$$l_{K,B}(u) = \frac{l_K(u)}{r_B(u)}. \tag{3.6}$$

A chord $[x, y]$ of a body K is said to be *diametrical* (see [CG83, p.53]) if there exist different parallel supporting hyperplanes of K, say H_1 and H_2, such that x belongs to H_1 and y to H_2.

From the next theorem we can see the relation between the notions of diametrical chord and maximal chord-length function.

Theorem 2. *Let K be a convex body in Minkowski space $\mathcal{M}^d(B)$ and let $[x, y]$, $x, y \in \operatorname{bd} K$, be a chord of K. Then the following conditions are equivalent:*

(i) The chord $[x, y]$ of body K is diametrical;

(ii) The point $(x-y)$ lies in the boundary of DK;

(iii) The chord $[x, y]$ is the longest one (in the Minkowskian sense) among all chords of K having the same direction.

Proof. The equivalence of *(i)* and *(ii)* is noticed in [BM40, p.30] for the two-dimensional case. The authors do not give the proof and refer to an article of Rademacher [Rad26, p.65] who did not give, however, the explicit formulation of the statement. It turns out that the proof of Rademacher may be applied for arbitrary dimension. By (3.4) the equivalence of *(ii)* and *(iii)* is easily derived. □

4 In- and circumballs

A convex body K_1 in $\mathbb{E}^d$ is said to be *inscribed* in a convex body K_2 if K_1 is contained in K_2 and any larger homothetical copy $\alpha K_1 + p$ of K_1, with $\alpha > 1$ and p in $\mathbb{E}^d$, is not contained in K_2. In this case we shall also say that K_2 is *circumscribed* to K_1. If K_1 is symmetric with respect to some point, i.e., a ball αB, $\alpha > 0$, of some Minkowski space $\mathcal{M}^d(B)$, we say that K_1 is an *inball* of K_2 in this space, and the radius α of K_1 is called the *inradius* of K_2 (notation: $\underline{r}_B(K_2)$). The notions *circumball* and *circumradius* (notation: $\overline{r}_B(K_2)$) are introduced analoguosly.

The following theorem gives several equivalent definitions for the inscribed bodies in case when both K_1 and K_2 are centrally symmetric with respect to the same point (for simplicity this point is supposed to be the origin).

Theorem 3. *Let B_1 and B_2 be convex bodies symmetric with respect to the origin and let B_1 be contained in B_2. Then the following conditions are equivalent.*

(i) The body B_1 is inscribed in B_2.

(ii) For every $\alpha > 1$, αB_1 is not contained in B_2.

(iii) For some direction $u_0 \in \mathbb{E}^d \setminus \{o\}$ we have $r_{B_1}(u_0) = r_{B_2}(u_0)$.

(iv) For some direction $u_0 \in \mathbb{E}^d \setminus \{o\}$ we have $h_{B_1}(u_0) = h_{B_2}(u_0)$.

(v) The body B_2^* is inscribed in B_1^*.

(vi) There exists an $x \in B_1$ with $\|x\|_{B_2} = 1$.

(vii) There exists an $x \in \operatorname{cl}\exp B_1$ with $x \in \operatorname{bd} B_2$.

Proof. The implication from *(i)* to *(ii)* is obvious. Let us prove the converse implication by contradiction. We suppose that there exists a body $\beta B_1 + p$ with $\beta > 1$ and some $p \in \mathbb{E}^d \setminus \{o\}$ which is contained in B_2. Then, by symmetry of B_1 and B_2, also $\beta B_1 - p$ is contained in B_2. Therefore $\beta B_1 \subset \operatorname{conv}\{(\beta B_1 + p) \cup (\beta B_1 - p)\} \subset B_2$, a contradiction.

The equivalence of *(ii)* and *(iii)-(vi)* is, in each case, more or less trivial. Thus, it is left to prove the equivalence of *(vii)* and one of the remaining conditions, for instance *(iii)*. The implication from *(vii)* to *(iii)* is trivial. To prove the converse implication we suppose that *(vii)* does not hold and show that then *(iii)* does not hold, too. Indeed, by the assumption we have $\operatorname{cl}\exp B_1 \subset \operatorname{int} B_2$ and therefore $B_1 = \operatorname{conv}\operatorname{cl}\exp B_1 \subset \operatorname{int} B_2$. Thus, for every direction u we have $r_{B_1}(u) < r_{B_2}(u)$, which means that *(iii)* is not fulfilled. □

We notice that Condition *(iii)* of Theorem 3 implies the uniqueness of both the inball and the circumball which have the same center of symmetry as the body.

5 Representations of diameter and thickness in Minkowski spaces

As in Euclidean spaces, the *diameter* of a convex body K in Minkowski space $\mathcal{M}^d(B)$ (notation: $\mathrm{diam}_B(K)$) is defined to be the largest distance occurring between points of K, i.e.: $\mathrm{diam}_B(K) := \max\{\|x-y\|_B \mid x, y \in K\}$. We denote the maximum and the minimum of the function $w_{K,B}(u)$ by $\overline{w}_B(K)$ and $\underline{w}_B(K)$, respectively, and call these values the *maximal* and the *minimal width* of K in $\mathcal{M}^d(B)$, respectively. For the maximum and the minimum of the function $l_{K,B}(u)$ we introduce the notations $\bar{l}_B(K)$ and $\underline{l}_B(K)$, respectively.

The following two theorems provide various geometrical and analytical representations of the diameter and the thickness of a convex body.

Theorem 4. *For an arbitrary Minkowski space $\mathcal{M}^d(B)$ and a convex body K the diameter of K in $\mathcal{M}^d(B)$ is equal to*

(i) the maximal width of K in $\mathcal{M}^d(B)$,

(ii) the maximum of the maximal chord length function of K in $\mathcal{M}^d(B)$,

(iii) the Minkowskian circumradius of DK,

(iv) the Minkowskian length of the longest diametrical chord of K,

(v) the analytical expression $\max\{\langle x, u\rangle \mid x \in DK,\ u \in B^*\}$.

Proof. In view of Theorem 3 we obtain that DK is inscribed in the bodies $\overline{w}_B(K)\cdot B$, $\bar{l}_B(K)\cdot B$, and $\overline{r}_B(DK)\cdot B$. Consequently $\overline{w}_B(K) = \bar{l}_B(K) = \overline{r}_B(DK)$. Further on, $\mathrm{diam}_B(K) = \max\{\|x\|_B \mid x \in DK\} = \overline{r}_B(DK)$. It is left to prove *(iv)* and *(v)*. By Theorem 2 we have that the length of the longest chord of K is equal to $\max\{\|x\|_B \mid x \in \mathrm{bd}\, DK\} = \{\|x\|_B \mid x \in DK\} = \mathrm{diam}_B(K)$, which proves *(iv)*. To prove *(v)* we replace $\|x\|_B$ by $h_{B^*}(x)$ and arrive at $\mathrm{diam}_B(K) = \max\{h_{B^*}(x) \mid x \in DK\}$. Then, simply applying the definition of the support function, we obtain $\mathrm{diam}_B(K) = \max\{\langle x, u\rangle \mid x \in DK,\ u \in B^*\}$. □

Theorem 5. *For an arbitrary Minkowski space $\mathcal{M}^d(B)$ and a convex body K in $\mathcal{M}^d(B)$ the following values are equal:*

(i) the minimal width of K in $\mathcal{M}^d(B)$,

(ii) the minimum of the maximal chord length function,

(iii) the Minkowskian inradius of DK,

(iv) the Minkowskian length of the shortest diametrical chord of K,

(v) the analytical expression $(\max\{\langle x, u\rangle \mid u \in (DK)^*,\ x \in B\})^{-1}$.

Proof. The equality of the values described in *(i)-(ii)* can be proved just in the same way as we did it for values *(i)-(ii)* in Theorem 4. In order to prove *(iv)* we use Theorem 3 which implies that the shortest diametrical chord has length $\gamma := \min\{\|x\|_B \mid x \in \operatorname{bd} DK\}$. But then γB is inscribed in DK, which yields that γ is equal to the inradius of DK. At last we prove the equality of the inradius of DK and the value in *(v)*. Since $\underline{r}_B(DK) \cdot B$ is inscribed in DK, we get that B is inscribed in $\frac{1}{\underline{r}_B(DK)} \cdot DK$. This implies that $\max\{\|x\|_{DK} \mid x \in B\} = \frac{1}{\underline{r}_B(DK)}$. Notice that in the latter equality we use the norm of the Minkowski space constructed by the body itself, namely $\|.\|_{DK}$. Let us replace this norm by the corresponding support function, yielding $\frac{1}{\underline{r}_B(DK)} = \max\{h_{DK^*}(x) \mid x \in B\}$. Taking the power -1 of the latter equality and applying the definition of the support function, we derive that $\underline{r}_B(DK)$ is equal to the value in *(v)*, and the proof is finished. □

Remarks

1. In view of the latter theorem we introduce the *thickness* of a body K in $\mathcal{M}^d(B)$ to be any of the equal values described by *(i)-(v)*, and we denote it by $\Delta_B(K)$.

2. The proof of the equalities $\operatorname{diam}_B(K) = \overline{w}_B(K) = \overline{l}_B(K)$ and $\Delta_B(K) = \underline{w}_B(K) = \underline{l}_B(K)$ for the Euclidean space can be found in [Web94, 7.6]. The descriptions by means of in- and circumballs are also given there, but implicitly only as a tool to prove the latter relations.

3. In [BS78, Theorem 4.2] there is a proof of the equality of diameter and maximal width in the general case, but it is done using another approach.

4. Whenever we consider a scalar product of a convex combination $\lambda x_1 + (1-\lambda)x_2$, where $x_1, x_2 \in \mathbb{E}^d$ and $\lambda \in [0,1]$, and a point y from $\mathbb{E}^d$, we can estimate this from above as follows: $\langle \lambda x_1 + (1-\lambda)x_2, y\rangle \leq \max\{\langle x_1, y\rangle, \langle x_2, y\rangle\}$. Thus, in the expressions given in Parts *(v)* of the last two theorems we can ignore non-extreme points of the sets over which we take the maxima not changing these maxima. Leaving only the points belonging to the closure of exposed points we obtain:

$$\operatorname{diam}_B(K) = \max\{\langle x, u\rangle \mid x \in \operatorname{cl}\exp DK, u \in \operatorname{cl}\exp B^*\}, \tag{5.1}$$

$$\Delta_B(K) = \Big(\max\{\langle x, u\rangle \mid u \in \operatorname{cl}\exp(DK)^*, x \in \operatorname{cl}\exp B\}\Big)^{-1}. \tag{5.2}$$

If both K and B are polytopes, the latter relations become discrete and can be investigated in the spirit of computational geometry.

The next theorem shows in which directions we can expect the optimal values for lengths of diametrical chords or the width of a convex body.

Theorem 6. *For an arbitrary Minkowski space $\mathcal{M}^d(B)$ and a convex body $K \subset \mathcal{M}^d(B)$ the following statements hold true.*

I. The width of K is maximal at one of the directions from $\operatorname{cl}\exp B^*$, *i.e, we have*

$$\operatorname{diam}_B(K) = \max_{u \in \operatorname{cl}\exp B^*} w_{K,B}(u). \tag{5.3}$$

II. There exists a diametrical chord of K having the maximal length and being parallel to one of the directions from $\mathrm{cl}\exp DK$, *i.e.,*

$$\mathrm{diam}_B(K) = \max_{x\in \mathrm{cl}\exp DK} l_{K,B}(x). \tag{5.4}$$

III. The width of a convex body is necessarily minimal at one of the directions from $\mathrm{cl}\exp(DK)^*$, *i.e.,*

$$\Delta_B(K) = \min_{u\in \mathrm{cl}\exp(DK)^*} w_{K,B}(u). \tag{5.5}$$

IV. There exists a diametrical chord of K having the minimal length and parallel to one of the directions from $\mathrm{cl}\exp B$, *i.e., we have*

$$\Delta_B(K) = \min_{x\in \mathrm{cl}\exp B} l_{K,B}(x). \tag{5.6}$$

Proof. I. By Theorem 4, DK is inscribed in $\mathrm{diam}_B(K)\cdot B$. Consequently B^* is inscribed in $\mathrm{diam}_B(K)\cdot(DK)^*$. Therefore, for some $x\in \mathrm{cl}\exp B^*$ we have $x\in \mathrm{diam}_B(K)\cdot \mathrm{bd}(DK)^*$. Then $r_{B^*}(x) = \mathrm{diam}_B(K)\cdot r_{(DK)^*}(x)$. Simple equivalent transformations of the latter equality yield $\mathrm{diam}_B(K) = w_K(x)/h_B(x) = w_{K,B}(x)$.

II. Using again that DK is inscribed in $\mathrm{diam}_B(K)\cdot B$, we find a point x from $\mathrm{cl}\exp DK$ with $x\in \mathrm{diam}_B(K)\cdot \mathrm{bd}\, B$. Consequently, $\mathrm{diam}_B(K)\cdot r_B(x) = r_{DK}(x)$ which is equivalent to $\mathrm{diam}_B(K) = l_{K,B}(u)$. Since x corresponds to some diametrical chord of K having the same direction as the vector x, the proof is complete.

The remaining Parts III and IV are proved in the same manner.

□

Remarks

1. When B is a polytope, the directions from $\mathrm{cl}\exp B^*$ are the facet normals of B. Thus Part I of the last theorem has then the following form: the width of K is necessarily maximal in one of the normal directions of some facet of B. As an example, we consider a Minkowski plane $\mathcal{M}^2(B)$ with Manhattan norm, i.e., $\|x\|_B = |x_1| + |x_2|$, where $x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}$. In this case the unit ball B is a square (see Fig. 1). Therefore, if we wish to measure the diameter of some body K in $\mathcal{M}^2(B)$ we only need to measure the width in two directions, namely $(1,1)$ and $(-1,1)$ (see Fig. 2).

2. From Part II of Theorem 6 we derive the generalization of the well-known Euclidean statement, namely that the diameter is attained at a chord parallel to some radius vector of a vertex of DK provided K is a polytope.

3. Part IV implies that if B is a polytope, then the minimal diametrical chord of K is parallel to the radius vector of some vertex of B. Again we illustrate this property by an example in the Minkowski plane $\mathcal{M}^2(B)$ with Manhattan norm. We obtain that, if we measure the thicknes of a body, we need to consider only diametrical chords parallel to any of the two diagonals of the square B, i.e., it sufficies to consider only two diametrical chords (see Fig. 3).

4. We wish to apply Part III of Theorem 6 for the case when K is a polygon in the Minkowski plane $\mathcal{M}^2(B)$. Then DK is a polygon each side of which is parallel to some side of K. Thus $\mathrm{cl}\exp DK^*$ consists of side normals of K. Hence, there is a normal of a side of K such that in the direction of this normal the width is minimal, i.e., we have just the same property as in the Euclidean plane.

5. We notice that the latter remark cannot be extended to higher dimensions. Although it is true that the width is minimal in the normal direction of some facet of DK (also by Part III), there are polytopes K (for instance, a simplex) such that DK has facets not parallel to any facet of K.

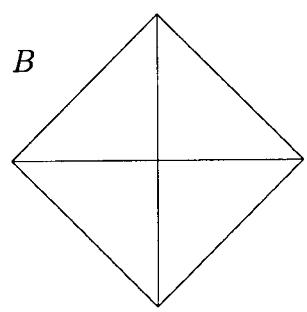

Figure 1:

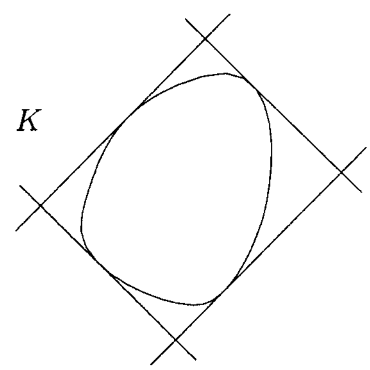

Figure 2:

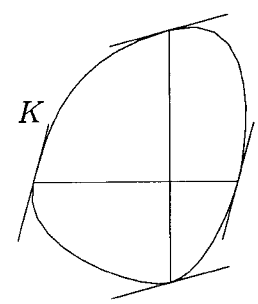

Figure 3:

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