Stochastic Price Processes with ε-correlated Returns

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Abstract

Empirical autocorrelation functions of returns of stochastic price processes show phenomena of correlation on small intervals of time, which decay to zero after a short time.

The paper deals with the concept of weakly correlated random processes to describe a mathematical model which takes into account this behaviour of statistical data. Weakly correlated functions have been applied to model numerous problems of physics and engineering. The main idea is, that the values of the functions at two points are uncorrelated if the distance between the points exceeds a certain quantity $\varepsilon > 0$. In contrast to the white noise model, for distances smaller than ε a correlation between the values is permitted.

A property of the introduced price model is, that the corresponding stochastic processes possess absolutely continuous sample paths. Therefore the used concept implies, that the so called condition of "No Free Lunch with Vanishing Risk" is not fulfilled.

1 Introduction

From statistical studies of the returns of price processes in liquid markets it is well-known, that the autocorrelation functions of these returns rapidly decay to zero (cf. [1], [7], [13]). Nevertheless, empirical autocorrelation functions of logarithmic returns show phenomena of correlation on small intervals of time.

The aim of the paper is to describe a mathematical model which also takes into account the non-zero behaviour of the autocorrelation function in those time intervals. For that purpose the concept of weakly correlated random processes is used. This class of random functions has been applied to model numerous problems of physics and engineering, cf. [10]. The concept of weakly correlated functions is based on the idea that the values of the functions at two points are uncorrelated if the distance between the points exceeds a certain quantity $\varepsilon > 0$. In contrast to the white noise model, for distances smaller than ε a correlation between the values is permitted.

The above idea is closely related to the theory of differential equations with random parameters, where a probabilistic analysis of the pathwise solutions is considered. It is well-known, that there are relations but also differences of these methods to the utilization of stochastic integrals. In the paper, both concepts are compared.

Consider a frictionless market with continuous trading and price processes possessing continuous sample paths. In order to take into account the mentioned correlation phenomena we model the price processes similar to the classical Black-Scholes model but replace the white noise process in this model by a weakly correlated process with continuous paths. Then the sample paths of the price process itself are absolutely continuous and therefore of bounded variation. From [3] and [9] it is known that such a model allows arbitrage (which is a riskless plan to make profit without any investment). Contrary to the derivations in the cited papers we show by using the Fundamental Theorem of Asset Pricing that the condition of "No Free Lunch with Vanishing Risk" is not fulfilled.

The paper is organized as follows. In Section 2 the basic definitions and applications concerning the theory of weakly correlated processes are given. In Section 3 the ideas of arbitrage and "Free Lunch" are considered and illustrated by the well-known example of a geometric Brownian motion. Section 4 contains a stochastic price model with ε-correlated returns. Following the considerations of [7], from the non-existence of an equivalent martingale measure it is deduced, that the property of "No Free Lunch with Vanishing Risk" is not fulfilled. For keeping the paper self-explanatory, some basic probabilistic concepts concerning stochastic processes and the theory of stochastic integration are given in the Appendix.

2 Weakly correlated processes

Throughout this chapter, ε is a (small) positive real number.

Definition 2.1
Let $\mathcal{P} = (x_i \in \mathbb{R}\,;\, i \in \mathcal{I})$ be a family of points of $\mathbb{R}$ (i. e. a mapping of the index set $\mathcal{I}$ into

$\mathbb{R}$). *For* $\tilde{\mathcal{I}} \subset \mathcal{I}$ *the set*

$$\mathcal{J}(\tilde{\mathcal{I}}) := \bigcup_{i \in \tilde{\mathcal{I}}} \{j \in \mathcal{I} : |x_i - x_j| \leq \varepsilon\}$$

consists of the indices of those points, which are in the ε*-neighbourhood of the points belonging to* $\tilde{\mathcal{I}}$.

$\mathcal{P}$ *is called* **ε-neighbouring**, *if for every partition* $(\mathcal{I}_1, \mathcal{I}_2)$ *of* $\mathcal{I} = \{1, 2, \ldots, k\}$ *(i. e.* $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}\,; \mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset\,;\, \mathcal{I}_1, \mathcal{I}_2 \neq \emptyset$*) it holds*

$$\mathcal{J}(\mathcal{I}_1) \cap \mathcal{J}(\mathcal{I}_2) \neq \emptyset\,.$$

A family of a single point, $\mathcal{P} = (x_1)$, *is also called* ε*-neighbouring.*

Definition 2.2
Let $\mathcal{P} = (x_i \in \mathbb{R}\,;\, i \in \mathcal{I})$ *be a family of points of* $\mathbb{R}$. *A subfamily* $\mathcal{P}^*$ *of* $\mathcal{P}$, $\mathcal{P}^* = (x_i \in \mathbb{R}\,;\, i \in \mathcal{I}^* \subset \mathcal{I})$ *is called* **maximally ε-neighbouring with respect to $\mathcal{P}$**, *if* $\mathcal{P}^*$ *is* ε*-neighbouring, but* $\mathcal{P}^{**} = (x_i \in \mathbb{R}\,;\, i \in \mathcal{I}^* \cup \{r\})$ *is not* ε*-neighbouring for all* $r \in \mathcal{I} \setminus \mathcal{I}^*$. *If a family* $\mathcal{P}$ *is* ε*-neighbouring, it is called maximally* ε*-neighbouring with respect to itself.*

In [10] it is proved, that every family $\mathcal{P}$ of real numbers has an (up to changing of indices) unique decomposition into maximally ε-neighbouring subfamilies.

Definition 2.3
For $D \subseteq \mathbb{R}$ *let* $X^\varepsilon = (X_t^\varepsilon)_{t \in D}$ *be a stochastic process with*

$$\mathrm{E} X_t^\varepsilon = 0\,, \quad \forall\, t \in D\,.$$

$X^\varepsilon = (X_t^\varepsilon)_{t \in D}$ *is called* **weakly correlated with correlation length $\varepsilon > 0$**, *if for every finite family* $\mathcal{P}$ *of real numbers in* D, $\mathcal{P} = (t_i\,;\, i \in \mathcal{I}) = (t_1, \ldots, t_k)$, $k \in \mathbb{N}$, $\mathcal{I} = \{1, 2, \ldots, k\}$, *and the corresponding decomposition into maximally* ε*-neighbouring subfamilies* $(t_i\,;\, i \in \mathcal{I}_j)\,;\, j = 1, 2, \ldots, p\,;\, \mathcal{I} = \cup_{j=1}^p \mathcal{I}_j\,;\, \mathcal{I}_{j_1} \cap \mathcal{I}_{j_2} = \emptyset$ *for* $j_1 \neq j_2$, *it holds*

$$\mathrm{E}(X_{t_1}^\varepsilon \cdot X_{t_2}^\varepsilon \cdot \ldots \cdot X_{t_k}^\varepsilon) = \prod_{j=1}^p \mathrm{E}\left(\prod_{i \in \mathcal{I}_j} X_{t_i}^\varepsilon\right),$$

that means, every moment of order k, $k \in \mathbb{N}$, *can be decomposed into a product of the moments corresponding to the decomposition of* $(t_1, \ldots, t_k)$ *into maximally* ε*-neighbouring subfamilies.*

Definition 2.4
Let $X^\varepsilon = (X_t^\varepsilon)_{t \in D}$, $D \subseteq \mathbb{R}$ *be a stochastic process.* $X^\varepsilon = (X_t^\varepsilon)_{t \in D}$ *is called* **ε-correlated with correlation length $\varepsilon > 0$**, *if for all* $t_1, t_2 \in D$ *with* $|t_1 - t_2| > \varepsilon$ *it holds*

$$\mathrm{cov}\left(X_{t_1}^\varepsilon, X_{t_2}^\varepsilon\right) = \mathrm{E}\left(\left(X_{t_1}^\varepsilon - \mathrm{E} X_{t_1}^\varepsilon\right)\left(X_{t_2}^\varepsilon - \mathrm{E} X_{t_2}^\varepsilon\right)\right) = 0\,.$$

Remark 2.5 A weakly correlated process with correlation length ε is ε-correlated. An ε-correlated Gaussian process X^ε with $\mathrm{E} X_t^\varepsilon = 0\ \forall\, t \in D$ is weakly correlated with correlation length ε, see [12].

Definition 2.6
Let $(X^\varepsilon) = \big((X_t^\varepsilon)_{t\in D}\ ,\ \varepsilon > 0\big)$, $D \subseteq \mathbb{R}$, *be a family of weakly correlated processes. Let* $\tilde{D} \subseteq D$ *denote the set of inner points of* D *and* $t_0 \in \tilde{D}$. *The* **intensity** $a(t_0)$ *is defined by*

$$a(t_0) := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathrm{E}\left(X_{t_0}^\varepsilon X_{t_0+s}^\varepsilon\right)\, ds\,,$$

if the integral and the limit exist.

Remark 2.7 Weakly correlated wide-sense stationary processes $(X_t^\varepsilon)_{t\in\mathbb{R}}$ possess constant intensities, $a(t) \equiv a$, $t \in \mathbb{R}$.

In applications of weakly correlated processes limit theorems for integral functionals of such functions play an important role, cf. [10], [11] and [12]. The fundamental result consists in a convergence of these functionals to Gaussian processes. We present a limit theorem given in [11].

Theorem 2.8
Let D_i, $i = 1, 2, \ldots, n$, *be intervals in* $\mathbb{R}$ *and let* $(X^\varepsilon) = \big((X_t^\varepsilon)_{t\in\mathbb{R}}\ ,\ \varepsilon > 0\big)$ *be a family of weakly correlated processes, where the intensity* $a(t)$ *of* (X^ε) *exists on* $D_i \cap D_j$, $i, j = 1, 2, \ldots, n$. *Let* $\mathrm{E}|X_t^\varepsilon|^p \le c_p < \infty$ *for all* $t \in \mathbb{R}$, *for all* ε *and for all* $p \ge 1$, *where the sample functions of* X_t^ε *are assumed to be continuous. We consider integral functionals*

$$r_i^\varepsilon(\omega) = \int_{D_i} G_i(t) X_t^\varepsilon(\omega)\, dt\,, \qquad i = 1, 2, \ldots, n\,,$$

where $G_i(t)$ *are deterministic functions on* D_i *fulfilling the conditions*

$$\int_{D_i} |G_i(t)|\, dt < \infty \qquad \text{and} \qquad \int_{D_i} (G_i(t))^2\, dt < \infty\,, \qquad i = 1, 2, \ldots, n\,.$$

Then it follows the convergence in distribution

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\varepsilon}} (r_1^\varepsilon, r_2^\varepsilon, \ldots, r_n^\varepsilon) = (\xi_1, \xi_2, \ldots, \xi_n)\ ,$$

where $(\xi_1, \xi_2, \ldots, \xi_n)$ *is a Gaussian vector with mean*

$$\mathrm{E}\xi_i = 0\,, \qquad i = 1, 2, \ldots, n$$

and covariance relations

$$\mathrm{E}\,(\xi_i \xi_j) = \int_{D_i \cap D_j} G_i(t) G_j(t) a(t)\, dt\,, \qquad i, j = 1, 2, \ldots, n\,.$$

3 Stochastic financial models, arbitrage and "Free Lunch"

Let $S = (S_t)_{t\geq 0}$ be a stochastic process that reflects economically meaningful ideas, for instance with respect to the description of a price of a financial asset. For details of the used concepts it is referred to [4], [6], [8] and [13].

Usually, financial markets are modelled by (S^0, S), where S^0 denotes a bank account (or a bond) and $S = \left(S^1, \ldots, S^d\right)$ is a vector of several financial assets. For simplification, in the following considerations we restrict to the case $d = 1$ and normalize the bond to $S^0 = \left(S^0_t\right)_{t\geq 0} \equiv 1$.

Example 3.1 In order to describe the dynamics of financial processes, the model of a geometric Brownian motion is well-known, it is a basic assumption of the Black-Merton-Scholes model and the Black-Scholes formula for the rational price of a standard European call option.

The price process $S = (S_t)_{t\geq 0}$ is here modelled by the formula

$$S_t = S_0 \, \exp\left(H_t\right) \, ,$$

where H_t is given by

$$H_t = \left(\mu - \frac{\sigma^2}{2}\right) \, t + \sigma \, B_t \, .$$

$(B_t)_{t\geq 0}$ is a standard Brownian motion, μ and $\sigma > 0$ are real constants. With the help of Itô's formula it follows

$$S_t = S_0 + \mu \int_0^t S_s \, ds + \sigma \int_0^t S_s \, dB_s \, ,$$

which can be written as

$$dS_t = S_t \left(\mu \, dt + \sigma \, dB_t\right) \, .$$

Definition 3.2
Let $S = (S_t)_{t\geq 0}$ be a stochastic (price-) process with $S_t > 0 \ \forall t$. Let $\Delta t > 0$. The **logarithmic return** *$Z_{\Delta t}(t)$ is defined by*

$$Z_{\Delta t}(t) = \ln\left(\frac{S_{t+\Delta t}}{S_t}\right) \, .$$

We remark, that Δt stands for a representative time interval depending on the availability of the financial data.

Remark 3.3 The model of Example 3.1,

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma \, B_t\right) \, ,$$

leads to

$$Z_{\Delta t}(t) = \left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma \left(B_{t+\Delta t} - B_t\right) \, ,$$

especially $Z_{\Delta t}(s)$ and $Z_{\Delta t}(t)$ are uncorrelated if and only if it holds $|t - s| \geq \Delta t$.

As mentioned above, in comparison to Remark 3.3 the aim is to present a model which allows correlation between $Z_{\Delta t}(s)$ and $Z_{\Delta t}(t)$ for (small) time intervals $[s,t]$ with $\Delta t < |t-s| < \varepsilon$, where ε is a fixed number. For that purpose we want to use the concept of weakly correlated random functions introduced in Section 2. Since the concept of arbitrage and conditions of its absence play an important role in the theory of financial markets, the consequences of the model with respect to arbitrage possibilities are studied. Let S be a semimartingale fulfilling several standard assumptions (cf. Assumption A.15) and let H be a process, such that the stochastic integral

$$(HS)_t := \int_0^t H(s,\omega)\, dS_s$$

is well defined. Besides the restriction to the choice of integrands H to make sure, that $(HS)_t$ exists, further restrictions are desired to avoid problems coming from the well-known "doubling strategies". The main idea is, that there should be a lower bound on the players loss.

Definition 3.4
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. Fix a semimartingale* $S = (S_t)_{t\geq 0}$. *A predictable process* $H = (H_t)_{t\geq 0}$ *is called* **admissible integrand** *for* S, *if*

- H *is* S*-integrable, i. e. the stochastic integral* $(HS)_t$ *is well-defined,*
- $H(0) = 0$ *and*
- *there exists a positive constant* a *such that* $(HS)_t \geq -a$ (P-a. s.) *for all* $t \geq 0$.

Definition 3.5
Let H *be an admissible integrand for* S. *The stochastic integral* $(HS)_t$ *is called* **payoff at time** t. *In the case of the existence,* $(HS)_\infty = \lim_{t\to\infty} (HS)_t$ *is called* **payoff**.

Definition 3.6
We say, that the process S *fulfills the* **Arbitrage** *property, if there exists an admissible integrand* H, *such that the payoff*

$$(HS)_\infty = \lim_{t\to\infty} \int_0^t H(s,\omega)\, dS_s$$

exists and it holds

$$\mathrm{P}((HS)_\infty < 0) = 0 \quad \textit{and} \quad \mathrm{P}((HS)_\infty > 0) > 0\,.$$

The process S *fulfills the* **No Arbitrage (NA)** *property, if the Arbitrage property is not fulfilled.*

In the discussion of the absence of arbitrage, in [2] the following property is used.

Definition 3.7
We say, that the process S fulfills the property of **Free Lunch with Vanishing Risk (FLVR)**, *if there exists a sequence of admissible integrands $\{H_n\}_{n\geq 1}$ with the properties*

- $f_n := (H_n S)_\infty$ *exists for all* $n \geq 1$,
- *the sequence* $\{f_n^-\}_{n\geq 1}$ *with* $f_n^- := \begin{cases} 0 & \text{if } f_n \geq 0\,, \\ f_n & \text{if } f_n < 0 \end{cases}$ *tends to zero uniformly and*
- (P-a. s.)- $\lim_{n\to\infty} f_n = f_0$ *with* $f_0 \in [0,\infty)$ *and* $\mathrm{P}(f_0 > 0) > 0$.

The process S fulfills the **No Free Lunch with Vanishing Risk (NFLVR)** *property, if the FLVR-property is not fulfilled.*

Obviously, from the Arbitrage property follows the FLVR-property and consequently the NFLVR-property is a (slightly) stronger version of the No Arbitrage-property. In [13] the NFLVR-property is called $\overline{NA}_+$-property.

In [2] it is shown, that for a locally bounded semimartingale S the condition of "No Free Lunch with Vanishing Risk" is equivalent to the existence of an equivalent local martingale measure for the process S.

Definition 3.8
The measures P *and* Q *on $\mathcal{F}$ are called* **equivalent**, *if a set in $\mathcal{F}$ has* P*-measure zero if and only if it has* Q*-measure zero.*

Theorem 3.9
Let $S = (S_t)_{t\geq 0}$ be a locally bounded semimartingale. The following assertions are equivalent.

(i) *There exists a probability measure* Q *which is equivalent to* P*, such that S is a local martingale under* Q.

(ii) *S satisfies the condition of "No Free Lunch with Vanishing Risk".*

Theorem 3.9 is known as the **Fundamental Theorem of Asset Pricing**.

Remark 3.10 To illustrate the construction of an equivalent local martingal measure in the case of the model introduced in Example 3.1 we set

$$Y_t = \exp\left(-\frac{\mu}{\sigma}\, B_t - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t\right).$$

For an arbitrary $T < \infty$ then it can be shown, that it holds $\mathrm{E}Y_T = 1$ and that $Y = (Y_t, \mathcal{F}_t)_{0\leq t\leq T}$ is an uniformly integrable martingale. Furthermore, Y_T is positive (P-a. s.) and we can define a probability measure Q_T on $(\Omega, \mathcal{F}_T)$ by

$$\mathrm{Q}_T(A) = \int_A Y_T(\omega) P(d\omega) = \mathrm{E}\left[I_A\, Y_T\right], \quad A \in \mathcal{F}_T\,.$$

If P_T denotes the restriction of the measure P to $\mathcal{F}_T$, then Q_T is equivalent to P_T. Under the measure Q_T the process $\tilde{B} = \left(\tilde{B}_t, \mathcal{F}_t\right)_{0\le t\le T}$ defined by

$$\tilde{B}_t = B_t + \frac{\mu}{\sigma}\, t$$

is a standard Brownian motion (Girsanov's theorem).

It follows, that for each $T > 0$ the process $S = (S_t, \mathcal{F}_t)_{0\le t\le T}$ introduced in Example 3.1 becomes a martingale with respect to Q_T, and it holds

$$S_t = S_0 + \sigma \int_0^t S_s\, d\tilde{B}_s\,, \quad \text{i. e.} \quad dS_t = \sigma\, S_t\, d\tilde{B}_t\,.$$

The measure Q_T is unique in the sense that, if $\hat{\mathrm{Q}}_T$ is another measure such that $\hat{\mathrm{Q}}_T$ is equivalent to P_T and $S = (S_t, \mathcal{F}_t)_{0\le t\le T}$ is a local martingale with respect to $\hat{\mathrm{Q}}_T$, then $\hat{\mathrm{Q}}_T = \mathrm{Q}_T$.

Proposition 3.11
Assume, that the stock price $S = (S_t)_{t\ge 0}$ is modelled as described in Example 3.1. Then S fulfills the NFLVR-property.

4 A stochastic price model with ε-correlated returns

Consider a family of wide-sense stationary, weakly correlated processes, $(X^\varepsilon) = ((X^\varepsilon_t)_{t\ge 0})$ fulfilling the assumptions of Theorem 2.8 and possessing intensity $a = 1$. The existence of such processes is shown in [10]. Furthermore, we define for $0 \le a_i \le b_i < \infty$ $(i = 1, 2, \ldots, n)$ the integral functionals

$$r_i^\varepsilon(\omega) = \int_{a_i}^{b_i} X_t^\varepsilon(\omega)\, dt\,, \qquad i = 1, 2, \ldots, n\,.$$

Then from Theorem 2.8 it follows the convergence in distribution of

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\varepsilon}}\,(r_1^\varepsilon, r_2^\varepsilon, \ldots, r_n^\varepsilon) = (\xi_1, \xi_2, \ldots, \xi_n)\ ,$$

where $(\xi_1, \xi_2, \ldots, \xi_n)$ is a Gaussian vector with mean

$$\mathrm{E}\xi_i = 0\,, \qquad i = 1, 2, \ldots, n$$

and covariance relations

$$\mathrm{E}\,(\xi_i\xi_j) = \lambda([a_i, b_i] \cap [a_j, b_j])\,, \qquad i, j = 1, 2, \ldots, n\,.$$

Thereby, λ is the Lebesgue measure on $\mathbb{R}$.

We define the process $\tilde{B}^\varepsilon = (\tilde{B}^\varepsilon_t)_{t\ge 0}$ by

$$\tilde{B}^\varepsilon_t(\omega) := \frac{1}{\sqrt{\varepsilon}} \int_0^t X_s^\varepsilon(\omega)\, ds$$

and consider the behaviour of $\tilde{B}^\varepsilon$ for $\varepsilon \downarrow 0$.

It holds $\mathrm{P}(\tilde{B}_0^\varepsilon = 0) = 1$ uniformly with respect to ε. By Theorem 2.8, it follows, that for $0 = t_0 < t_1 < \ldots < t_n < \infty$ the vector of increments $(\tilde{B}_{t_1}^\varepsilon - \tilde{B}_{t_0}^\varepsilon, \tilde{B}_{t_2}^\varepsilon - \tilde{B}_{t_1}^\varepsilon, \ldots, \tilde{B}_{t_n}^\varepsilon - \tilde{B}_{t_{n-1}}^\varepsilon)$ converges to a Gaussian random vector $(\xi_1, \xi_2, \ldots, \xi_n)$, whose components are stochastically independent. Furthermore, for $0 \leq s < t$, the difference $\tilde{B}_t^\varepsilon - \tilde{B}_s^\varepsilon$ converges in distribution to a Gaussian random variable ξ with $\xi \sim N(0, t-s)$. A comparison of these facts with Definition A.12 shows the close relation of $\tilde{B}^\varepsilon$ to the Brownian motion.

Moreover, for fixed $\varepsilon > 0$ the process $\tilde{B}^\varepsilon$ has the following properties,

- $\mathrm{P}(\tilde{B}_0^\varepsilon = 0) = 1$,
- $\mathrm{E}\tilde{B}_t^\varepsilon = 0$ for all $t \geq 0$,
- the trajectories of $\tilde{B}^\varepsilon$ are continuous,
- the variance of the increments $\tilde{B}_t^\varepsilon - \tilde{B}_s^\varepsilon$ depends for $0 \leq s < t$ only on the difference $t - s$.

The latter statement follows from

$$\begin{aligned}
\mathrm{E}\left(\tilde{B}_t^\varepsilon - \tilde{B}_s^\varepsilon\right)^2 &= \frac{1}{\varepsilon}\,\mathrm{E}\left(\int_s^t \int_s^t X_u^\varepsilon X_v^\varepsilon \; du\, dv\right) \\
&= \frac{1}{\varepsilon}\int_s^t \int_s^t \mathrm{E}\left(X_u^\varepsilon X_v^\varepsilon\right) \; du\, dv \\
&= \frac{1}{\varepsilon}\int_0^{t-s} \int_0^{t-s} \mathrm{E}\left(X_{u+s}^\varepsilon X_{v+s}^\varepsilon\right) \; du\, dv \\
&= \frac{1}{\varepsilon}\int_0^{t-s} \int_0^{t-s} \mathrm{E}\left(X_u^\varepsilon X_v^\varepsilon\right) \; du\, dv\,,
\end{aligned}$$

where the property of X^ε to be wide-sense stationary has been used.

According to Example 3.1, in [7] the following model for the price $\tilde{S}^\varepsilon = (\tilde{S}_t^\varepsilon)_{t \geq 0}$ of a financial asset is considered,

$$\tilde{S}_t^\varepsilon = \tilde{S}_0^\varepsilon \; \exp\left(\tilde{H}_t^\varepsilon\right)\,,$$

where

$$\tilde{H}_t^\varepsilon = \left(\mu - \frac{\sigma^2}{2}\right)\, t + \sigma\, \tilde{B}_t^\varepsilon = \left(\mu - \frac{\sigma^2}{2}\right)\, t + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t X_s^\varepsilon \, ds\,.$$

We consider the properties of $\tilde{S}^\varepsilon$. Let $\varepsilon > 0$ be fixed. Then it holds

$$\frac{d\tilde{S}_t^\varepsilon}{dt} = \tilde{S}_0^\varepsilon \left(\left(\mu - \frac{\sigma^2}{2}\right) + \frac{\sigma}{\sqrt{\varepsilon}}\, X_t^\varepsilon\right) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\, t + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t X_s^\varepsilon \, ds\right)\,,$$

Because of the continuity of the paths of X^ε, on each compact interval of $[0, \infty)$ the paths of $\frac{d\tilde{S}_t^\varepsilon}{dt}$ are bounded and consequently the paths of $\tilde{S}^\varepsilon$ are of bounded variation on each compact interval of $[0, \infty)$, i. e. $\tilde{S}^\varepsilon$ is a process of bounded variation.

Consequently (cf. Proposition A.19) there exists no probability measure Q which is equivalent to P such that $\tilde{S}^\varepsilon$ is a local martingale under Q. From Theorem 3.9 it follows, that $\tilde{S}^\varepsilon$ satisfies the condition of "Free Lunch with Vanishing Risk".

Finally, we consider the correlation behaviour of the returns $Z_{\Delta t}(s)$ and $Z_{\Delta t}(t)$ (cf. Remark 3.3) under the model of $\tilde{S}^\varepsilon$. It holds

$$Z_{\Delta t}(t) = \ln\left(\frac{\tilde{S}^\varepsilon_{t+\Delta t}}{\tilde{S}^\varepsilon_t}\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \frac{\sigma}{\sqrt{\varepsilon}}\left(\int_t^{t+\Delta t} X^\varepsilon_s \, ds\right),$$

and because of the assumptions to X^ε_t the returns $Z_{\Delta t}(s)$ and $Z_{\Delta t}(t)$ are uncorrelated in case of $|t-s| \geq \varepsilon + \Delta t$, i. e. $(Z_{\Delta t}(t))_{t\geq 0}$ is (for fixed $\Delta t > 0$) an $\tilde{\varepsilon}$-correlated process with $\tilde{\varepsilon} = \varepsilon + \Delta t$. In case of $|t-s| < \tilde{\varepsilon}$ the correlation between $Z_{\Delta t}(s)$ and $Z_{\Delta t}(t)$ is described by the correlation function of X^ε. The problem, whether it is possible to adapt the behaviour of this correlation function to statistical data is in the interest of further research.

It is clear, that the smoothness of the trajectories of the modelled process is a very critical assumption. This possibility of describing correlation effects leads to the effect of ruling out the process as equilibrium price model. For the construction of corresponding arbitrage strategies it is referred to [3] and [9].

Nevertheless, the described correlation phenomena leading to arbitrage may (and in fact do) arise in real markets. The close relation of the model using weakly correlated processes to the classical price model seems to make it desirable to study financial markets where the price processes are modelled by using weakly correlated functions.

Appendix A Probabilistic concepts

We recall some well-known facts concerning the theory of stochastic integration, for details reference should be made to [4], [5], [8] and [13].

Definition A.1
A **stochastic basis** *is defined as a filtered probability space*

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right),$$

where the σ-algebra $\mathcal{F}$ is P*-complete, $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of σ-algebras with $\mathcal{F}_t \subseteq \mathcal{F}$, $t \geq 0$, $\mathcal{F}_0$ contains all sets $N \in \mathcal{F}$ with $\mathrm{P}(N) = 0$ and $(\mathcal{F}_t)_{t\geq 0}$ is assumed to be right continuous, i. e.*

$$\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t \,.$$

We consider stochastic processes $X = \{X_t : 0 \leq t < \infty\}$ (denoted by $X = (X_t)_{t\geq 0}$), which are families of random variables on the sample space $(\Omega, \mathcal{F})$ and take values in the space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition A.2
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. The stochastic process* $X = (X_t)_{t\geq 0}$ *is called* **measurable**, *if the mapping*

$$(t,\omega) \mapsto X_t(\omega) \,:\, ([0,\infty)\times\Omega, \mathcal{B}([0,\infty))\otimes\mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable.

Definition A.3
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. The stochastic process* $X = (X_t)_{t\geq 0}$ *is called* **adapted** *to* $(\mathcal{F}_t)_{t\geq 0}$ *if, for each* $t \geq 0$, *the random variable* X_t *is* $\mathcal{F}_t$*-measurable.*

Definition A.4
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. The stochastic process* $X = (X_t)_{t\geq 0}$ *is called* **progressively measurable**, *if the mapping*

$$(t,\omega) \mapsto X_t(\omega) \,:\, ([0,s]\times\Omega, \mathcal{B}([0,s])\otimes\mathcal{F}_s) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable, for each $s \geq 0$.

Remark A.5 Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ be a stochastic basis. Let $X = (X_t)_{t\geq 0}$ be adapted and let every sample path of X be right-continuous (or left-continuous). Then X is progressively measurable.

Definition A.6
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. The random variable* $\tau : \Omega \to [0,\infty]$ *with*

$$\{\omega\in\Omega \,:\, \tau(\omega)\leq t\} \in \mathcal{F}_t \qquad \forall t\in[0,\infty)$$

is called **stopping time**.

Remark A.7 Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ be a stochastic basis. The stochastic process $X = (X_t)_{t\geq 0}$ is assumed to be progressively measurable. Furthermore let τ be a stopping time. It is possible to show, that $(X_{t\wedge\tau})_{t\geq 0}$ defined by

$$X_{t\wedge\tau}(\omega) := \begin{cases} X_t(\omega) & \text{for } t \leq \tau(\omega) \\ X_{\tau(\omega)}(\omega) & \text{for } t > \tau(\omega) \end{cases}$$

is a progressively measurable process. $(X_{t\wedge\tau})_{t\geq 0}$ is called **stopped process**.

Definition A.8
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. The stochastic process* $X = (X_t)_{t\geq 0}$ *is called a* **martingale** *with respect to the flow of* σ*-algebras* $(\mathcal{F}_t)_{t\geq 0}$ *and the probability measure* P, *if* X_t *for* $t \geq 0$ *fulfills the conditions*

X_t *is* $\mathcal{F}_t$*-measurable,*
$\mathrm{E}\,|X_t| < \infty$,
$\mathrm{E}\,(X_t \,|\, \mathcal{F}_s) = X_s \quad$ (P-a. s.) $\quad$ *for* $\quad s \leq t$,

we write $X = (X_t, \mathcal{F}_t)_{t\geq 0}$.

Definition A.9
Let $N^\star$ be a class of stochastic processes with a certain property. We say, that the process $X = (X_t)_{t\geq 0}$ fulfills this property **locally**, *if there exists a nondecreasing sequence $(\tau_n)_{n\geq 1}$ of finite stopping times with $\tau_n : \Omega \to [0,\infty)$ and $\mathrm{P}(\lim_{n\to\infty} \tau_n = \infty) = 1$ such that the stopped processes $(X_{t\wedge\tau_n})_{t\geq 0}$ are in $N^\star$ for each $n \geq 1$.*

Remark A.10 Of certain interest is the case $N^\star = \{X : X \text{ is martingale}\}$, which leads to the class of **local martingales**.

Definition A.11
Let $X = (X_t, \mathcal{F}_t)_{t\geq 0}$ be a martingale and let every sample path of X be right-continuous. X is called **square-integrable**, *if $\sup\limits_{t\geq 0} \mathrm{E}X_t^2 < \infty$.*

Definition A.12
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ be a stochastic basis. A **(standard, one-dimensional) Brownian motion** *is an adapted process $B = (B_t)_{t\geq 0}$ with continuous paths fulfilling the properties*

- $\mathrm{P}(B_0 = 0) = 1$,
- *$B_t - B_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t$ and*
- *$B_t - B_s \sim N(0, t-s)$ for $0 \leq s < t$.*

Remark A.13 A Brownian motion is a process with stationary independent increments, i. e. if we consider a sequence $0 = t_0 < t_1 < \ldots < t_n < \infty$ then the increments $\{B_{t_j} - B_{t_{j-1}}\}$ are independent and the distribution of $B_{t_j} - B_{t_{j-1}}$ depends only on the difference $t_j - t_{j-1}$, $j = 1, \ldots, n$.

The filtration $(\mathcal{F}_t)_{t\geq 0}$ is a part of the definition, but if it is known, that B has stationary independent increments and that it holds $(B_t - B_0) \sim N(0,t)$, then $B = (B_t)_{t\geq 0}$ is a Brownian motion with respect to the filtration $\left(\mathcal{F}_t^B\right)_{t\geq 0}$ which is defined as follows. Let $\mathcal{F}_t^0 = \sigma\{B_s : 0 \leq s \leq t\}$ be the σ-algebra generated by the variables B_s, $s \leq t$ and $\mathcal{N} = \{A \in \mathcal{F} : \mathrm{P}(A) = 0\}$ be the set of zero-probability events in $\mathcal{F}$. Then $\mathcal{F}_t^B$ for $t \geq 0$ is defined as $\mathcal{F}_t^B := \sigma\left(\mathcal{F}_t^0 \cup \mathcal{N}\right)$. Note, that in contrast to $\left(\mathcal{F}_t^B\right)_{t\geq 0}$ the filtration $\left(\mathcal{F}_t^0\right)_{t\geq 0}$ in general has not the property of right-continuity.

The main ideas of the theory of stochastic integration are based on the construction of integrals of the form

$$I_t(f) := \int_0^t f(s,\omega)\, dB_s\,,$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion.

Thereby, the construction starts with the definition of integrals for simple functions and the class of integrands is step by step extended to measurable and adapted functions

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

with

$$\mathrm{P}\left(\int_0^t (f(s,\omega))^2\,ds < \infty\right) = 1$$

for each $t \geq 0$.

Proposition A.14
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis and let* $a = (a(t,\omega))_{t\geq 0}$ *and* $b = (b(t,\omega))_{t\geq 0}$ *be two measurable adapted processes such that for* $t \geq 0$ *it holds*

$$\mathrm{P}\left(\int_0^t |\,a(s,\omega)|\,ds < \infty\right) = 1 \quad \text{and} \quad \mathrm{P}\left(\int_0^t (b(s,\omega))^2\,ds < \infty\right) = 1\,,$$

furthermore let $B = (B_t, \mathcal{F}_t)_{t\geq 0}$ *be a Brownian motion and* X_0 *be a* $\mathcal{F}_0$*-measurable random variable. We define*

$$X_t = X_0 + \int_0^t a(s,\omega)\,ds + \int_0^t b(s,\omega)\,dB_s\,.$$

$X = (X_t)_{t\geq 0}$ *is called* **Itô-process**.

Let $F(t,x)$ *be a function defined in* $\mathbb{R}^+ \times \mathbb{R}$ *which possesses continuous derivatives* $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ *and* $\frac{\partial^2 F}{\partial x^2}$. *Under these assumptions, we have the following formula, well-known as* **Itô's formula** *for* $F(t, X_t)$,

$$F(t,X_t) = F(0,X_0) + \int_0^t \left(\frac{\partial F}{\partial s} + a(s,\omega)\frac{\partial F}{\partial x} + \frac{1}{2}b^2(s,\omega)\frac{\partial^2 F}{\partial x^2}\right) ds + \int_0^t b(s,\omega)\frac{\partial F}{\partial x}\,dB_s\,.$$

Assumption A.15 We consider a stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ and an adapted process $X = (X_t)_{t\geq 0}$, where the trajectories $t \mapsto X_t(\omega)$, $\omega \in \Omega$ are right-continuous with limits from the left. In the French literature such a process is called **un processus càdlàg**.

Definition A.16
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. A process* $X = (X_t)_{t\geq 0}$ *fulfilling Assumption A.15 is said to be of* **bounded variation**, *if* P*-almost surely the paths of* X *are of bounded variation on each compact interval of* $[0,\infty)$.

Definition A.17
A process $(X_t)_{t\geq 0}$ *fulfilling Assumption A.15 is called* **semimartingale**, *if* X_t, $t \geq 0$ *is representable as a sum*

$$X_t = X_0 + A_t + M_t\,,$$

where $A = (A_t)_{t\geq 0}$ *is an adapted process of bounded variation with* $\mathrm{P}(A_0 = 0) = 1$ *and* $M = (M_t)_{t\geq 0}$ *is a local martingale with* $\mathrm{P}(M_0 = 0) = 1$.

Definition A.18
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *be a stochastic basis. Let* $\mathcal{A}$ *be the* σ*-algebra generated by all adapted processes on* $[0,\infty) \times \Omega$ *with left-continuous trajectories. A stochastic process* $X = (X_t)_{t\geq 0}$ *defined on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P}\right)$ *is called* **predictable** *if the map* $(t,\omega) \mapsto X(t,\omega)$ *is* $\mathcal{A}$*-measurable.*

For the construction of stochastic integrals with respect to arbitrary semimartingales certain constraints to f are needed. With the help of results of the theory of martingales the representation of each semimartingale X from Definition A.17 can be modified as follows,

$$X_t = X_0 + A'_t + M'_t\,,$$

where A' is of bounded variation and M' is a locally square-integrable martingale and it holds $\mathrm{P}(A'_0 = 0) = \mathrm{P}(M'_0 = 0) = 0$.

For locally bounded measurable functions f the Lebesgue-Stieltjes integrals

$$\int_0^t f(s,\omega)\, dA'_s$$

(for each $\omega \in \Omega$) are well defined. If, in addition, f is predictable, the stochastic integrals

$$\int_0^t f(s,\omega)\, dM'_s$$

can be defined similar to the case of integration with respect to a Brownian motion by means of an isometry. Although the above decomposition has not to be unique, it can be shown, that the integral

$$\int_0^t f(s,\omega)\, dX_s := \int_0^t f(s,\omega)\, dM'_s + \int_0^t f(s,\omega)\, dA'_s$$

does not depend (up to stochastic equivalence) on this decomposition, hence the stochastic integral

$$\int_0^t f(s,\omega)\, dX_s$$

is well-defined for each locally bounded predictable function f and each semimartingale X.

The consequences of the price model introduced in Section 4 with respect to the FLVR property are based on a result of the Doob-Meyer decomposition for semimartingales. We restrict the following considerations to the case of continuous processes.

Proposition A.19
Let $A = (A_t)_{t\geq 0}$ be a continuous process of bounded variation with $A_0 = 0$. If A is a local martingale, then it holds $A \equiv 0$.

Proof. Because of the importance of Proposition A.19 for the considerations in Section 4 the proof given in [4] is reproduced. Let a local martingale $A = (A_t)_{t\geq 0}$ be a process of bounded variation. Since the paths of A are assumed to be continuous, it is sufficient to show, that it holds $\mathrm{P}(A_t = 0) = 1$ for all $t \geq 0$. Furthermore, since a continuous local martingale possesses a localizing sequence of finite stopping times (which ensure the boundedness), using the technique of localization we can assume, that A is a bounded martingale and show that $\mathrm{E}\,(A_t)^2 = 0$ for all $t \geq 0$.

Let $0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = t\ \forall n$, be a sequence of partitions of $[0,t]$ with

$$\lim_{n\to\infty} \left(\max\left\{\left|t_{j+1}^n - t_j^n\right| \,:\, j = 0,1,\ldots,k_n - 1\right\}\right) = 0\,.$$

Then it holds

$$A_t^2 - 2\sum_{j=0}^{k_n-1} A_{t_j^n}\left(A_{t_{j+1}^n} - A_{t_j^n}\right) = \sum_{j=0}^{k_n-1}\left(A_{t_{j+1}^n} - A_{t_j^n}\right)^2$$
$$\leq \sup_i \left|A_{t_{i+1}^n} - A_{t_i^n}\right| \sum_{j=0}^{k_n-1}\left|A_{t_{j+1}^n} - A_{t_j^n}\right| \to 0\,.$$

This convergence follows due to the fact that the trajectory $s \mapsto A_s(\omega)$ is for all $\omega \in \Omega$ uniformly continuous on $[0,t]$ and therefore it holds

$$\sup_i \left|A_{t_{i+1}^n} - A_{t_i^n}\right| \to 0\,,$$

while the term

$$\sum_{j=0}^{k_n-1}\left|A_{t_{j+1}^n} - A_{t_j^n}\right|$$

is bounded by the variation of the trajectory $s \mapsto A_s(\omega)$ in $[0,t]$. It follows

$$2\sum_{j=0}^{k_n-1} \mathrm{E}\, A_{t_j^n}\left(A_{t_{j+1}^n} - A_{t_j^n}\right) \to \mathrm{E}\,(A_t)^2\ .$$

From the martingale property we conclude

$$2\sum_{j=0}^{k_n-1} \mathrm{E}\, A_{t_j^n}\left(A_{t_{j+1}^n} - A_{t_j^n}\right) = 0\,,$$

which completes the proof. □

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