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The Computation of Derivatives of Trigonometric Functions via the Fundamental Theorem of Calculus

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Abstract

We consider the real valued functions sine and cosine as defined in elementary trigonometry. We compute the derivatives of these functions by making use of several aids from analysis, but without using any nontrivial relationship between sine and cosine. More precisely, we compute the derivative of the inverse function arcsin of sine by using the fundamental theorem of calculus, before we compute the derivatives of sine and cosine. As an application, we prove an identity concerning sine and cosine by analysing some linear differential equation.

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1 Introduction

Trigonometric functions play an important role in many branches of mathematics and its applications, in particular in measurement and physics. The first systematic study of these functions was given by L. Euler in Chapter 8 of [Eu]; for more historical background the reader is referred to [Br], [Tr, Chapter B], and [Ze].

Several occurrences in nature may be – at least approximately – described in terms of periodic oscillations. To describe these motions quantitatively, one usually derives a differential equation from physical informations, and then mathematics shows that this differential equation is solved by some trigonometric function, cf. [Bu, § 17]. For that and related reasons, the computation of derivatives of trigonometric functions is very important.

In history of mathematics, there have been several approaches to compute the derivatives of trigonometric functions. The first compilation was given by R. Cotes (1682-1716) in his "Harmonia mensurarum" (published by R. Smith in 1722, cf. the extensive discussion in Part 2, Chapter 3, § 2, of [Br]). Cotes presented the respective formulae in verbal form. E.g., for $\sin'(\Theta) = \cos\Theta$ he wrote: "Variatio minima cujusvis arcus circularis est ad Variationem minimam Sinus eiusdem arcus ut Radius ad Sinum complementi".

If the trigonometric functions are defined in terms of elementary trigonometry, it is very difficult to compute their derivatives without any additional aids. In trigonometry, one usually derives several identities concerning these functions. Afterwards it becomes simpler to compute their derivatives; however, to prove the indicated identities in an elementary geometric manner, one has in general to consider a lot of cases.

A different approach to trigonometric functions is studied in analysis in terms of power series. In this way, it is much simpler to compute their derivatives. However, the relation

of these power series to geometry is not obvious.
In the present paper, we start from the definition of trigonometric functions in terms of elementary trigonometry and compute their derivatives by using the fundamental theorem of calculus. The decisive idea is as follows: In general, it is much simpler to compute some derivative than to compute some area. However, we shall study two functions f, F with $F' = f$ such that it is simpler to compute the area determined by the graph of f than to compute the derivative of F directly.
In our approach, we make only use of obvious facts from elementary trigonometry; nevertheless, we shall need several aids from analysis. Besides the fundamental theorem of calculus, we shall need standard rules for differentiation, for instance the chain rule, the rule to compute the derivative of the inverse function of a differentiable function $\varphi : I \longrightarrow \mathbb{R}$ defined on some interval I satisfying $\varphi'(x) > 0$ for all $x \in I$, and the mean value theorem.

In a closing section, we derive an identity concerning trigonometric functions by making use of their derivatives and the following fact: The set of solutions of an ordinary linear differential equation of second order is some 2-dimensional real vector space. This is of course well known; but for our special case we have included a rather short and self-contained elementary proof. Most of the other standard identities concerning trigonometric functions may be either proved in a similar manner or derived from identities already proved in this way – without going through a couple of cases as in elementary trigonometry.

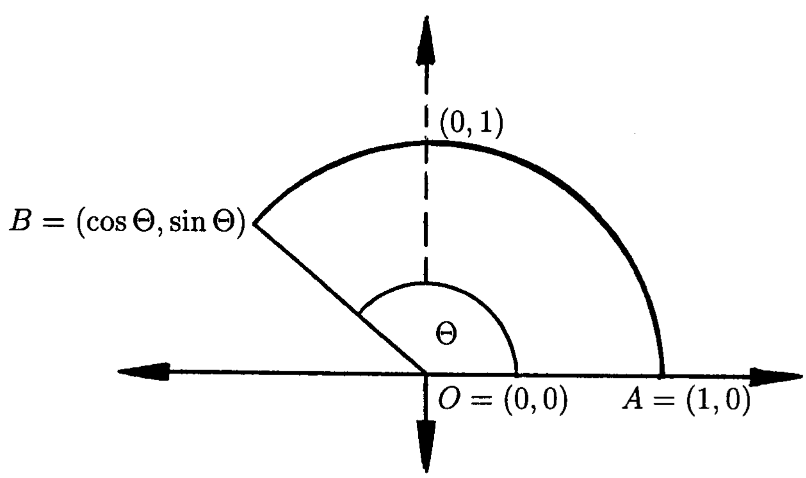
2 Computation of the Derivatives of Sine and Cosine

For two points P, Q in the Euclidean plane $\mathbb{R}^2$, the line segment $\overline{PQ}$ is defined by

$$\overline{PQ} := \{\lambda \cdot P + (1 - \lambda) \cdot Q : \lambda \in [0, 1]\}.$$

We consider the unit disc

$$K := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}. \tag{2.1}$$



– Fig. 1 –

Moreover, we put

$$0 := (0,0),\ A := (1,0)$$

and let $B = (a, b)$ denote any point on the boundary of K; that is, $a^2 + b^2 = 1$. The angle Θ, encircled by the line segments $\overline{OA}$ and $\overline{OB}$ counterclockwise, can be defined as the unique real number $\Theta \in [0, 2\pi)$ such that the sector obtained by rotating $\overline{OA}$ counterclockwise until $\overline{OB}$ is covered exhibits the area $\frac{1}{2} \cdot \Theta$, cf. Figure 1.

Moreover, we define the real valued functions sine and cosine, shortly denoted by sin, cos: $\mathbb{R} \longrightarrow \mathbb{R}$, in terms of elementary trigonometry; that is, $B = (\cos\Theta, \sin\Theta)$ for B and $\Theta \in [0, 2\pi)$ as above. These functions are now determined completely by periodicity; more precisely, for all $\Theta \in [0, 2\pi)$ and all $n \in \mathbb{Z}$ one has

$$(2.2) \qquad \sin(\Theta + 2n \cdot \pi) = \sin\Theta\,, \quad \cos(\Theta + 2n \cdot \pi) = \cos\Theta\,.$$

The theorem of Pythagoras yields for all $\Theta \in \mathbb{R}$:

$$(2.3) \qquad \cos^2\Theta + \sin^2\Theta = 1\,.$$

Furthermore, by elementary trigonometry we get at once the following relations for all $\Theta \in \mathbb{R}$:

$$(2.4\text{ a}) \qquad \sin\left(\frac{\pi}{2} + \Theta\right) = \cos\Theta\,,$$

$$(2.4\text{ b}) \qquad \cos\left(\frac{\pi}{2} + \Theta\right) = -\sin\Theta\,,$$

$$(2.4\text{ c}) \qquad \sin(\pi - \Theta) = \sin\Theta\,,$$

$$(2.4\text{ d}) \qquad \cos(\pi - \Theta) = -\cos\Theta\,,$$

$$(2.4\text{ e}) \qquad \sin(-\Theta) = -\sin\Theta\,,$$

$$(2.4\text{ f}) \qquad \cos(-\Theta) = \cos\Theta\,.$$

Finally, we need the following fact which is an immediate consequence of the definition of sine and the fact that angles are invariant under congruence transformations: If the *hypotenuse* of a rectangular triangle exhibits the length 1, then the length of any edge s equals $\sin\Theta_s$, where Θ_s is the angle encircled by the other edges of the triangle.

We shall now prove the following

Theorem 2.1: *The functions sine and cosine are differentiable on $\mathbb{R}$; more precisely, for all $\Theta \in \mathbb{R}$ one has*

$$(2.5\text{ a}) \qquad \sin'(\Theta) = \cos\Theta\,,$$

(2.5 b) $\cos'(\Theta) = -\sin\Theta$.

Proof: Let us first consider sine on the interval $[0, \frac{\pi}{2}]$. More precisely, define $s : [0, \frac{\pi}{2}] \longrightarrow [0, 1]$ by

$$s(\Theta) := \sin\Theta .$$

It is immediate from the trigonometric definition of sine that s is strictly increasing on $[0, \frac{\pi}{2}]$ and bijective; as usual, we denote the corresponding inverse mapping $s^{-1} : [0, 1] \longrightarrow [0, \frac{\pi}{2}]$ by arcsin. (arcsin may be continued to $[-1, 1]$; however, for our purposes it suffices to consider arcsin on $[0, 1]$.) Bijectivity and strict monotony of s and s^{-1} on $[0, \frac{\pi}{2}]$ and $[0, 1]$, respectively, imply even that these functions are continuous. Thus, (2.2) and (2.4 a) - (2.4 f) imply that sine and cosine are continuous on $\mathbb{R}$.

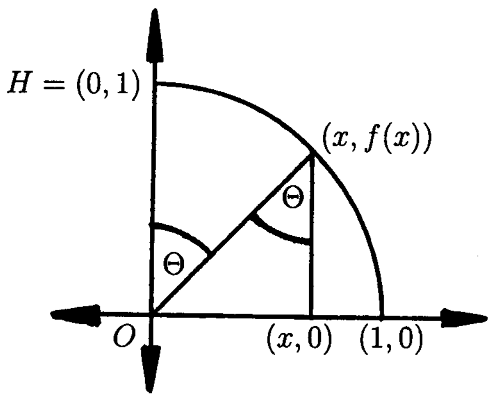
To compute the derivative of sine, we first compute the derivative of arcsin by using the fundamental theorem of calculus. To this end, define $f : [0, 1] \longrightarrow \mathbb{R}$ by

(2.6) $$f(x) := \sqrt{1 - x^2} .$$

Clearly, $\text{graph}(f) := \{(x, f(x)) : x \in [0, 1]\}$ is just a quarter of the boundary of the unit disc K. Consider the point $H := (0, 1)$ and, for fixed $x \in [0, 1]$, put

$$P := (x, 0) , \quad Q := (x, f(x)) .$$

Moreover, let $F(x)$ denote the area of the domain $G(x)$ bounded by the three line segments $\overline{OH}, \overline{OP}, \overline{PQ}$ as well as by graph(f), cf. Figure 2.



– Fig. 2 –

Then, on the one hand, the fundamental theorem of calculus yields that F is a differentiable function on $[0, 1]$ and that for all $x \in [0, 1]$ we get

(2.7) $$F'(x) = f(x) = \sqrt{1 - x^2} .$$

On the other hand, we can partition the domain $G(x)$ into a sector which encircles the angle $\Theta = \arcsin x$ and a rectangular triangle exhibiting x and $f(x)$ as the lengths of its mutually perpendicular edges. Thus we get

$$F(x) = \frac{1}{2} \cdot \arcsin x + \frac{1}{2} \cdot x \cdot \sqrt{1-x^2}\,;$$

that is

$$\text{(2.8)} \qquad \arcsin x = 2 \cdot F(x) - x \cdot \sqrt{1-x^2} \quad \text{for all } \; x \in [0,1]\,.$$

Now we combine (2.7) and (2.8) and conclude that the function arcsin is differentiable on $[0,1)$. Namely, for $0 \le x < 1$ we get

$$\begin{aligned} \arcsin'(x) &= 2 \cdot F'(x) - \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} \\ &= \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

This means that s is differentiable on $[0, \frac{\pi}{2})$, too, and for $0 \le \Theta < \frac{\pi}{2}$ we get, together with (2.3):

$$\sin'(\Theta) = \frac{1}{\arcsin'(\sin\Theta)} = \sqrt{1-\sin^2\Theta} = \cos\Theta\,.$$

Note that (2.4 e) implies that these equations also hold for $\Theta = 0$.
For $\frac{\pi}{2} < \Theta \le \pi$ we obtain by (2.4 c), (2.4 d), and the chain rule:

$$\sin'(\Theta) = -\sin'(\pi - \Theta) = -\cos(\pi - \Theta) = \cos\Theta\,.$$

For $\Theta_0 = \frac{\pi}{2}$ and $\Theta \in [0,\pi] \setminus \{\Theta_0\}$, the mean value theorem implies

$$\frac{\sin\Theta - \sin\Theta_0}{\Theta - \Theta_0} = \sin'(\tilde{\Theta}) = \cos\tilde{\Theta} \quad \text{for some } \; \tilde{\Theta} \text{ with } \Theta_0 < \tilde{\Theta} < \Theta \text{ or } \Theta < \tilde{\Theta} < \Theta_0\,.$$

Thus, it follows also that $\sin'(\Theta_0) = \cos\Theta_0 = 0$, because we have already seen that cosine is continuous. Furthermore, for $-\pi \le \Theta \le 0$ we get by (2.4 e), (2.4 f), and the chain rule:

$$\sin'(\Theta) = (-1)^2 \cdot \sin'(-\Theta) = \cos(-\Theta) = \cos\Theta\,.$$

Thus we have proved (2.5 a) for all $\Theta \in [-\pi, \pi]$. By (2.2), it follows that (2.5 a) holds for all $\Theta \in \mathbb{R}$.

Now, (2.4 a) and (2.4 b) yield that cosine is differentiable on $\mathbb{R}$, too, and that for all $\Theta \in \mathbb{R}$ we get

$$\cos'(\Theta) = \sin'\left(\frac{\pi}{2} + \Theta\right) = \cos\left(\frac{\pi}{2} + \Theta\right) = -\sin\Theta\,,$$

whence (2.5 b) is also proved. ■

Remark 2.2: Note that the well known formula

$$\text{(2.9)} \qquad \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}\,,$$

which we have proved for $0 \le x < 1$, has been a decisive step in the above proof; we have derived (2.5 a) from (2.9). If we consider arcsin defined on $[-1, 1]$, then (2.4 e) implies that arcsin is an odd function just as sine. This implies that (2.9) holds for all $x \in (-1, 1)$. □

3 Proof of some Identity concerning Sine and Cosine via some Differential Equation

In this closing section we want to prove the following result, which is based on Theorem 2.1.

Theorem 3.1: *For all $\alpha, \beta \in \mathbb{R}$ one has*

$$\text{(3.1)} \qquad \sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta\,.$$

First we prove

Lemma 3.2: *Assume $f : \mathbb{R} \longrightarrow \mathbb{R}$ exhibits at least two derivatives, and for all $\Theta \in \mathbb{R}$ one has $f''(\Theta) = -f(\Theta)$. Then there exist constants $a, b \in \mathbb{R}$ such that the following equation holds for all $\Theta \in \mathbb{R}$:*

$$\text{(3.2)} \qquad f(\Theta) = a \cdot \sin\Theta + b \cdot \cos\Theta\,.$$

Proof: Consider the functions $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\text{(3.3 a)} \qquad g(\Theta) := \sin\Theta \cdot f(\Theta) + \cos\Theta \cdot f'(\Theta)\,,$$

$$\text{(3.3 b)} \qquad h(\Theta) := \cos\Theta \cdot f(\Theta) - \sin\Theta \cdot f'(\Theta)\,.$$

Then for all $\Theta \in \mathbb{R}$ we get

$$g(\Theta) \cdot \sin\Theta + h(\Theta) \cdot \cos\Theta = \sin^2\Theta \cdot f(\Theta) + \cos^2\Theta \cdot f(\Theta) = f(\Theta)\,.$$

Thus it remains to show that g and h are constant functions. To this end, it suffices to prove that $g'(\Theta) = h'(\Theta) = 0$ holds for all $\Theta \in \mathbb{R}$. We get by Theorem 2.1:

$$\begin{aligned} g'(\Theta) &= \cos\Theta \cdot f(\Theta) + \sin\Theta \cdot f'(\Theta) - \sin\Theta \cdot f'(\Theta) + \cos\Theta \cdot f''(\Theta) = 0\,, \\ h'(\Theta) &= -\sin\Theta \cdot f(\Theta) + \cos\Theta \cdot f'(\Theta) - \cos\Theta \cdot f'(\Theta) - \sin\Theta \cdot f''(\Theta) = 0\,, \end{aligned}$$

since $f(\Theta) + f''(\Theta) = 0$, and the lemma follows. ■

Remark: Lemma 3.2 is of course well known within the theory of Ordinary Linear Differential Equations. However, to make the paper more self-contained, we decided to give this rather short elementary proof. □

Proof of Theorem 3.1: Assume $\alpha \in \mathbb{R}$ is fixed, and define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(\Theta) := \sin(\alpha + \Theta)\,. \tag{3.4}$$

Applying (2.5 a) and (2.5 b) in Theorem 2.1, we obtain

$$f''(\Theta) = -f(\Theta) \text{ for all } \Theta \in \mathbb{R}.$$

Thus Lemma 3.2 implies that there exist $a, b \in \mathbb{R}$ such that for all $\Theta \in \mathbb{R}$ one has:

$$\sin(\alpha + \Theta) = a \cdot \sin\Theta + b \cdot \cos\Theta\,. \tag{3.5}$$

We can take derivatives and obtain for all $\Theta \in \mathbb{R}$:

$$\cos(\alpha + \Theta) = a \cdot \cos\Theta - b \cdot \sin\Theta\,. \tag{3.6}$$

Now put $\Theta = 0$. Then (3.5) yields $b = \sin\alpha$, while (3.6) implies $a = \cos\alpha$. By (3.5) this means

$$\sin(\alpha + \Theta) = \cos\alpha \cdot \sin\Theta + \sin\alpha \cdot \cos\Theta$$

for all $\alpha, \Theta \in \mathbb{R}$, and the theorem follows. ■

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