Integral functionals of $\varepsilon-$correlated random fields

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Abstract

In this paper asymptotic expansions for second-order moments of integral functionals of a family of wide-sense homogeneous $\varepsilon-$correlated fields are considered. The asymptotic expansions are derived for $\varepsilon \to 0$ and extend the results for integrals of scalar and vector valued wide-sense stationary processes to the case of multidimensional integrals.

1 Introduction

In this paper asymptotic expansions for second-order moments of integral functionals of the type

$$ {}^{\varepsilon}r(\omega) = \int_{\mathcal{D}} Q(\boldsymbol{s})\, {}^{\varepsilon}f(\boldsymbol{s},\omega)\, d\boldsymbol{s}, \qquad \boldsymbol{\varepsilon} > \mathbf{0}, \tag{1} $$

are considered, where $Q(\cdot)$ is a deterministic complex matrix valued function, $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, denotes a family of wide-sense homogeneous and $\boldsymbol{\varepsilon}-$correlated random fields and the domain of integration $\mathcal{D}$ is a subset of $\mathbf{R}^m$, i.e. $\mathcal{D} \subset \mathbf{R}^m$, $m \in \mathbf{N}$. The given results extend the corresponding expansions for integral functionals of real valued (cf. [5]) and vector valued (cf. [6]) processes to the case of random fields. The region of non-zero correlation of the values of the underlying random fields is described by a m-dimensional vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$ with positive entries. This is a m-dimensional analogon to the correlation length in the case of integral functionals of random processes. The asymptotic expansions are derived as $\varepsilon_i \to 0, i = 1, \ldots, m$.

In the paper the following notations are used:

$\mathbf{0} = (0, \ldots, 0), \quad \mathbf{1} = (1, \ldots, 1), \quad \boldsymbol{\infty} = (\infty, \ldots, \infty),$

$\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\tau} \in \mathbf{R}^m, \quad \boldsymbol{k} = (k_1, \ldots, k_m) \in \mathbf{N}_0^m$

$|\boldsymbol{k}| = k_1 + \ldots + k_m, \quad \boldsymbol{k}! = k_1! \cdot \ldots \cdot k_m!$

$D^{\boldsymbol{k}} Q(\boldsymbol{s}) = \dfrac{\partial^{|\boldsymbol{k}|}}{\partial s_1^{k_1} \ldots \partial s_m^{k_m}} Q(s_1, \ldots, s_m),$

$\boldsymbol{u}^{\boldsymbol{k}} = u_1^{k_1} \cdot \ldots \cdot u_m^{k_m}, \quad \boldsymbol{\varepsilon}^{\boldsymbol{k}} = \varepsilon_1^{k_1} \cdot \ldots \cdot \varepsilon_m^{k_m}$, especially $\boldsymbol{\varepsilon}^{\mathbf{1}} = \varepsilon_1 \cdot \ldots \cdot \varepsilon_m,$

$\boldsymbol{\varepsilon u} = (\varepsilon_1 u_1, \ldots, \varepsilon_m u_m), \quad \dfrac{\boldsymbol{s}}{\boldsymbol{\varepsilon}} = \left(\dfrac{s_1}{\varepsilon_1}, \ldots, \dfrac{s_m}{\varepsilon_m}\right).$

Equalities, inequalities, sums, products, quotients of m-vectors or multiindices are in general defined componentwise. The vectors ${}^{\varepsilon}r$, ${}^{\varepsilon}f(\cdot)$ are assumed to be columns, * denotes the adjoint of a vector or matrix.
Analogously to [5] and [6] we will suppose for the family $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, of random fields

Assumption 1

1. $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, *are complex vector valued wide-sense homogeneous random fields with matrix correlation functions*

$$\mathbf{E}\{{}^{\varepsilon}f(\boldsymbol{s})\,{}^{\varepsilon}f^*(\boldsymbol{t})\} = {}^{\varepsilon}R(\boldsymbol{t}-\boldsymbol{s});$$

2. $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, *are centered random fields, i.e.* $\mathbf{E}\{{}^{\varepsilon}f(\boldsymbol{s})\} = 0$*;*

3. $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, *are* $\boldsymbol{\varepsilon}$*-correlated random fields, i.e.* ${}^{\varepsilon}R(\boldsymbol{s}) = 0$ *if* $|s_i| \geq \varepsilon_i$, $i = 1, \ldots, m$*;*

4. *the correlation functions* ${}^{\varepsilon}R(\cdot), \boldsymbol{\varepsilon} > \mathbf{0}$, *are generated by a correlation function* $R(\cdot)$ *of a* $\mathbf{1}$*-correlated wide-sense homogeneous random field, i.e.*

$${}^{\varepsilon}R(\boldsymbol{s}) = R\left(\frac{\boldsymbol{s}}{\boldsymbol{\varepsilon}}\right),\ \boldsymbol{s} \in \mathbf{R}^m,\ \boldsymbol{\varepsilon} > \mathbf{0};$$

5. *the correlation function* $R(\cdot)$ *is continuous on* $\mathbf{R}^m$*, hence the processes* $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, *are continuous in mean square on* $\mathbf{R}^m$*.*

The integral in (1) is assumed to exist in mean square sense, under weak conditions it coincides a.s. with the pathwise integral. From condition 2 of Assumption 1 it follows that the random vectors ${}^{\varepsilon}r(\cdot), \boldsymbol{\varepsilon} > \mathbf{0}$, are centered, i.e. $\mathbf{E}\{{}^{\varepsilon}r\} = 0$.

Definition 1 *Let* $R(\cdot)$ *be a continuous correlation function of a wide-sense homogeneous random field (not necessarily* $\mathbf{1}$*-correlated) and* $\boldsymbol{k} = (k_1, \ldots, k_m)$ *a multiindex with*

$$\int_{\mathbf{R}^m} |s_1|^{k_1} \ldots |s_m|^{k_m} |R(s_1, \ldots, s_m)|\, ds_m \ldots ds_1 < \infty.$$

Then

$$\mu_{\boldsymbol{k}} = \int_{\mathbf{R}^m} \boldsymbol{s}^{\boldsymbol{k}} R(\boldsymbol{s})\, d\boldsymbol{s}$$

is called the correlation moment of $\boldsymbol{k}$-th order *of the correlation function or the random process.*

From the relation $R(\boldsymbol{s}) = R^*(-\boldsymbol{s})$ it follows $\mu_{\boldsymbol{k}} = (-1)^{|\boldsymbol{k}|}\mu^*_{\boldsymbol{k}}$. In the special case of real valued random fields these relations read as $R(\boldsymbol{s}) = R(-\boldsymbol{s})$, $\mu_{\boldsymbol{k}} = (-1)^{|\boldsymbol{k}|}\mu_{\boldsymbol{k}}$ and especially $\mu_{\boldsymbol{k}} = 0$ for odd values $|\boldsymbol{k}|$.
In case of real valued orthotropic random fields, i.e. if $R(\cdot)$ satisfies the relation

$$R(s_1, \ldots, s_m) = R(|s_1|, \ldots, |s_m|), \quad (s_1, \ldots, s_m) \in \mathbf{R}^m,$$

the correlation moments vanish if at least one component of the multiindex $\boldsymbol{k}$ is odd.

Assumption 2 *The deterministic function $Q(\cdot)$ is $N-$ times continuously differentiable on $\mathcal{D}$, $N \in \mathbf{N}_0$, $Q^{(N)}(\cdot)$ is absolutely continuous on $\mathcal{D}$ and the derivatives of $Q(\cdot)$ up to the order $(N+1)$ belong to the function space $L^2(\mathcal{D}) \cap L^1(\mathcal{D})$.*

For such functions the Taylor expansion formula with exact integral representation of the remainder (cf. e.g. [1], Sec. 5.4) remains valid, for the integration by parts formula for absolutely continuous function see e.g. [2], Ch. IX, §7.
For sake of simplicity the expansion formulae will be explicitly derived for $m = 2$. The general case is also considered.

2 Asymptotic expansions of variances

First, the case of real valued random variables is considered. For functionals

$$^{\varepsilon}r(\omega) = \int_{\mathbf{R}^m} Q(\boldsymbol{s})\, {}^{\varepsilon}f(\boldsymbol{s}, \omega)\, d\boldsymbol{s}$$

with a real function $Q(\cdot)$ and real random fields $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, we derive

$$\begin{aligned}
\mathbf{E}\{{}^{\varepsilon}r^2\} &= \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} Q(\boldsymbol{s})Q(\boldsymbol{t})\mathbf{E}\{{}^{\varepsilon}f(\boldsymbol{s})\, {}^{\varepsilon}f(\boldsymbol{t})\}\, d\boldsymbol{s}\, d\boldsymbol{t} \\
&= \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} Q(\boldsymbol{s})Q(\boldsymbol{t})\, {}^{\varepsilon}R_{ff}(\boldsymbol{t}-\boldsymbol{s})\, d\boldsymbol{s}\, d\boldsymbol{t} \\
&= \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} Q(\boldsymbol{s})Q(\boldsymbol{t})R\left(\frac{\boldsymbol{t}-\boldsymbol{s}}{\boldsymbol{\varepsilon}}\right) d\boldsymbol{s}\, d\boldsymbol{t} \\
&= \boldsymbol{\varepsilon}^{\mathbf{1}} \int_{\mathbf{-1}}^{\mathbf{1}}\int_{\mathbf{R}^m} Q(\boldsymbol{s})Q(\boldsymbol{s}+\boldsymbol{\varepsilon}\boldsymbol{u})R(\boldsymbol{u})\, d\boldsymbol{s}\, d\boldsymbol{u}.
\end{aligned}$$

From Taylor expansion

$$Q(\boldsymbol{s}+\boldsymbol{\varepsilon}\boldsymbol{u}) = \sum_{j=0}^{N}\sum_{|\boldsymbol{k}|=j} \frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}}\boldsymbol{u}^{\boldsymbol{k}}}{\boldsymbol{k}!} D^{\boldsymbol{k}}Q(\boldsymbol{s}) + \tilde{\rho}_{N+1}(\boldsymbol{s}, \boldsymbol{u}, \boldsymbol{\varepsilon})$$

with the remainder term $\tilde{\rho}_{N+1}(\boldsymbol{s},\boldsymbol{u},\boldsymbol{\varepsilon})$ we find

$$\begin{aligned}\mathbf{E}\{{}^{\varepsilon}r^2\} &= \sum_{j=0}^{N}\sum_{|\boldsymbol{k}|=j}\frac{\boldsymbol{\varepsilon}^{\boldsymbol{k+1}}}{\boldsymbol{k}!}\int_{-\mathbf{1}}^{\mathbf{1}}\boldsymbol{u}^{k}R(\boldsymbol{u})\,d\boldsymbol{u}\int_{\mathbf{R}^m}Q(\boldsymbol{s})D^{k}Q(\boldsymbol{s})\,d\boldsymbol{s}+\rho_{N+1}(\boldsymbol{\varepsilon})\\ &= \sum_{j=0,\,j\text{ even}}^{N}\sum_{|\boldsymbol{k}|=j}\frac{\boldsymbol{\varepsilon}^{\boldsymbol{k+1}}}{\boldsymbol{k}!}\mu_{\boldsymbol{k}}\int_{\mathbf{R}^m}Q(\boldsymbol{s})D^{k}Q(\boldsymbol{s})\,d\boldsymbol{s}+\rho_{N+1}(\boldsymbol{\varepsilon})\end{aligned}$$

where $\mu_{\boldsymbol{k}}$ denotes the correlation moment of order $\boldsymbol{k}$.
In the special case $\varepsilon_i=\varepsilon$, $i=1,\ldots,m$, this reads

$$\mathbf{E}\{{}^{\varepsilon}r^2\} = \sum_{j=0,\,j\text{ even}}^{N}\varepsilon^{j+m}\sum_{|\boldsymbol{k}|=j}\frac{\mu_{\boldsymbol{k}}}{\boldsymbol{k}!}\int_{\mathbf{R}^m}Q(\boldsymbol{s})D^{k}Q(\boldsymbol{s})\,d\boldsymbol{s}+\rho_{N+1}(\varepsilon).$$

3 Expansion of correlation functions

Now expansions for correlation functions of real valued wide-sense homogeneous random fields arising from integral functionals

$${}^{\varepsilon}g(\boldsymbol{t}) = \int_{-\infty}^{\boldsymbol{t}}Q(\boldsymbol{t}-\boldsymbol{s})\,{}^{\varepsilon}f(\boldsymbol{s})\,d\boldsymbol{s}=\int_{\mathbf{0}}^{\infty}Q(\boldsymbol{u})\,{}^{\varepsilon}f(\boldsymbol{t}-\boldsymbol{u})\,d\boldsymbol{u}$$

are considered. For simplicity we start with the case of two-dimensional fields, i.e. $m=2$.

3.1 The case $m=2$

The real valued wide-sense homogeneous random fields are defined by

$$\begin{aligned}{}^{\varepsilon}g(t_1,t_2) &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}Q(t_1-s_1,t_2-s_2)\,{}^{\varepsilon}f(s_1,s_2)\,ds_2\,ds_1\\ &= \int_{0}^{\infty}\int_{0}^{\infty}Q(u_1,u_2)\,{}^{\varepsilon}f(t_1-u_1,t_2-u_2)\,du_2\,du_1.\end{aligned}$$

For their correlation functions we derive

$$\begin{aligned}&{}^{\varepsilon}R_{gg}(\tau_1,\tau_2)=\mathbf{E}\{{}^{\varepsilon}g(t_1,t_2)\,{}^{\varepsilon}g(t_1+\tau_1,t_2+\tau_2)\}\\ &=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty Q(u_1,u_2)Q(v_1,v_2)\,{}^{\varepsilon}R_{ff}(\tau_1+u_1-v_1,\tau_2+u_2-v_2)dv_2dv_1du_2du_1\\ &= \Big\{\int_{-\infty}^{\tau_1}\int_{-\infty}^{\tau_2}\int_{\tau_1-w_1}^{\infty}\int_{\tau_2-w_2}^{\infty}+\int_{-\infty}^{\tau_1}\int_{\tau_2}^{\infty}\int_{\tau_1-w_1}^{\infty}\int_{0}^{\infty}+\int_{\tau_1}^{\infty}\int_{-\infty}^{\tau_2}\int_{0}^{\infty}\int_{\tau_2-w_2}^{\infty}\\ &\quad+\int_{\tau_1}^{\infty}\int_{\tau_2}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\Big\}\,Q(w_1+v_1-\tau_1,w_2+v_2-\tau_2)Q(v_1,v_2)\cdot\\ &\quad{}^{\varepsilon}R_{ff}(w_1,w_2)\,dv_2\,dv_1\,dw_2\,dw_1\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_1\varepsilon_2 \left\{ \int_{-\infty}^{\frac{\tau_1}{\varepsilon_1}} \int_{-\infty}^{\frac{\tau_2}{\varepsilon_2}} \int_{\tau_1-\varepsilon_1 u_1}^{\infty} \int_{\tau_2-\varepsilon_2 u_2}^{\infty} + \int_{-\infty}^{\frac{\tau_1}{\varepsilon_1}} \int_{\frac{\tau_2}{\varepsilon_2}}^{\infty} \int_{\tau_1-\varepsilon_1 u_1}^{\infty} \int_{0}^{\infty} + \right. \\
&\quad \left. + \int_{\frac{\tau_1}{\varepsilon_1}}^{\infty} \int_{-\infty}^{\frac{\tau_2}{\varepsilon_2}} \int_{0}^{\infty} \int_{\tau_2-\varepsilon_2 u_2}^{\infty} + \int_{\frac{\tau_1}{\varepsilon_1}}^{\infty} \int_{\frac{\tau_2}{\varepsilon_2}}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \right\} \cdot \\
&\quad Q(v_1-\tau_1+\varepsilon_1 u_1, v_2-\tau_2+\varepsilon_2 u_2) Q(v_1,v_2) R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 \\
&= \varepsilon_1\varepsilon_2 \left\{ \int_{-\infty}^{\frac{\tau_1}{\varepsilon_1}} \int_{-\infty}^{\frac{\tau_2}{\varepsilon_2}} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1,v_2) Q(v_1+\tau_1-\varepsilon_1 u_1, v_2+\tau_2-\varepsilon_2 u_2) \cdot \right. \\
&\quad R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 \\
&\quad + \int_{-\infty}^{\frac{\tau_1}{\varepsilon_1}} \int_{\frac{\tau_2}{\varepsilon_2}}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1, v_2-\tau_2+\varepsilon_2 u_2) Q(v_1+\tau_1-\varepsilon_1 u_1, v_2) \cdot \\
&\quad R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 \\
&\quad + \int_{\frac{\tau_1}{\varepsilon_1}}^{\infty} \int_{-\infty}^{\frac{\tau_2}{\varepsilon_2}} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1-\tau_1+\varepsilon_1 u_1, v_2) Q(v_1, v_2+\tau_2-\varepsilon_2 u_2) \cdot \\
&\quad R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 \\
&\quad + \int_{\frac{\tau_1}{\varepsilon_1}}^{\infty} \int_{\frac{\tau_2}{\varepsilon_2}}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1-\tau_1+\varepsilon_1 u_1, v_2-\tau_2+\varepsilon_2 u_2) Q(v_1,v_2) \cdot \\
&\quad \left. R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 \right\} \\
&= \varepsilon_1\varepsilon_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1+(\varepsilon_1 u_1-\tau_1)^+, v_2+(\varepsilon_2 u_2-\tau_2)^+) \cdot \\
&\quad Q(v_1+(\tau_1-\varepsilon_1 u_1)^+, v_2+(\tau_2-\varepsilon_2 u_2)^+) R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 \\
&= \varepsilon_1\varepsilon_2 \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1+(\varepsilon_1 u_1-\tau_1)^+, v_2+(\varepsilon_2 u_2-\tau_2)^+) \cdot \\
&\quad Q(v_1+(\tau_1-\varepsilon_1 u_1)^+, v_2+(\tau_2-\varepsilon_2 u_2)^+) R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 .
\end{aligned}
$$

The coefficients in the expansions are different for different signs of τ_1 and τ_2.

• For fixed values $\tau_1 \geq \varepsilon_1 > 0, \tau_2 \geq \varepsilon_2 > 0$ we get

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau_1,\tau_2) &= \varepsilon_1\varepsilon_2 \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{\infty} \int_{0}^{\infty} Q(v_1,v_2) \cdot \\
&\qquad Q(v_1+\tau_1-\varepsilon_1 u_1, v_2+\tau_2-\varepsilon_2 u_2) R(u_1,u_2)\, dv_2\, dv_1\, du_2\, du_1 .
\end{aligned}
$$

Taylor expansion gives

$$
\begin{aligned}
&Q(v_1+\tau_1-\varepsilon_1 u_1, v_2+\tau_2-\varepsilon_2 u_2) \\
&\quad = \sum_{j=0}^{N} \sum_{k=0}^{j} \frac{(-\varepsilon_1 u_1)^k (-\varepsilon_2 u_2)^{j-k}}{k!(j-k)!} \frac{\partial^j Q(v_1+\tau_1, v_2+\tau_2)}{\partial v_1^k \partial v_2^{j-k}} + \tilde{\rho}_{N+1}(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\tau})
\end{aligned}
$$

and it yields

$$
{}^{\varepsilon}R_{gg}(\tau_1,\tau_2) = \sum_{j=0}^{N} (-1)^j \sum_{k=0}^{j} \frac{\varepsilon_1^{k+1} \varepsilon_2^{j-k+1}}{k!(j-k)!} \int_{-1}^{1} \int_{-1}^{1} u_1^k u_2^{j-k} R(u_1,u_2)\, du_2\, du_1 \cdot
$$

$$\int_0^\infty \int_0^\infty Q(v_1,v_2)\frac{\partial^j Q(v_1+\tau_1,v_2+\tau_2)}{\partial v_1^k \partial v_2^{j-k}}\,dv_2\,dv_1 + \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau})$$

$$= \sum_{j=0,\,j\,\text{even}}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\mu_{(k,j-k)}\cdot$$

$$\int_0^\infty \int_0^\infty Q(v_1,v_2)\frac{\partial^j Q(v_1+\tau_1,v_2+\tau_2)}{\partial v_1^k \partial v_2^{j-k}}\,dv_2\,dv_1 + \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}).$$

In the case $\varepsilon_1 = \varepsilon_2 = \varepsilon$ this leads to

$${}^{\varepsilon}R_{gg}(\tau_1,\tau_2) = \sum_{j=0,\,j\,\text{even}}^{N}\varepsilon^{j+2}\sum_{k=0}^{j}\frac{1}{k!(j-k)!}\mu_{(k,j-k)}\cdot$$

$$\int_0^\infty \int_0^\infty Q(v_1,v_2)\frac{\partial^j Q(v_1+\tau_1,v_2+\tau_2)}{\partial v_1^k \partial v_2^{j-k}}\,dv_2\,dv_1 + \rho_{N+1}(\varepsilon,\boldsymbol{\tau}).$$

• For fixed values $\tau_1 \geq \varepsilon_1 > 0, \tau_2 \leq -\varepsilon_2 < 0$ we derive

$${}^{\varepsilon}R_{gg}(\tau_1,\tau_2) = \varepsilon_1\varepsilon_2\int_{-1}^1\int_{-1}^1\int_0^\infty\int_0^\infty Q(v_1,v_2-\tau_2+\varepsilon_2 u_2)\cdot$$

$$Q(v_1+\tau_1-\varepsilon_1 u_1,v_2)R(u_1,u_2)\,dv_2\,dv_1\,du_2\,du_1.$$

Taylor expansion yields

$${}^{\varepsilon}R_{gg}(\tau_1,\tau_2) = \sum_{j=0}^{N}\sum_{k=0}^{j}(-1)^k\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\int_{-1}^1\int_{-1}^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1\cdot$$

$$\int_0^\infty\int_0^\infty \frac{\partial^k Q(v_1+\tau_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2-\tau_2)}{\partial v_2^{j-k}}\,dv_2\,dv_1 + \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau})$$

$$= \sum_{j=0,\,j\,\text{even}}^{N}\sum_{k=0}^{j}(-1)^k\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\mu_{(k,j-k)}\cdot$$

$$\int_0^\infty\int_0^\infty \frac{\partial^k Q(v_1+\tau_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2-\tau_2)}{\partial v_2^{j-k}}\,dv_2\,dv_1 + \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}).$$

For $\varepsilon_1 = \varepsilon_2 = \varepsilon$ this means

$${}^{\varepsilon}R_{gg}(\tau_1,\tau_2) = \sum_{j=0,\,j\,\text{even}}^{N}\varepsilon^{j+2}\sum_{k=0}^{j}\frac{(-1)^k}{k!(j-k)!}\mu_{(k,j-k)}\cdot$$

$$\int_0^\infty\int_0^\infty \frac{\partial^k Q(v_1+\tau_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2-\tau_2)}{\partial v_2^{j-k}}\,dv_2\,dv_1 + \rho_{N+1}(\varepsilon,\boldsymbol{\tau}).$$

• In the case $\tau_1 \geq \varepsilon_1 > 0, \tau_2 = 0$ we get

$${}^{\varepsilon}R_{gg}(\tau_1,0) = \varepsilon_1\varepsilon_2\left\{\int_0^\infty\int_0^\infty\int_{-1}^1\int_0^1 Q(v_1,v_2+\varepsilon_2 u_2)\cdot\right.$$

$$
\begin{aligned}
&Q(v_1+\tau_1-\varepsilon_1 u_1, v_2) R(u_1,u_2)\, du_2\, du_1\, dv_2\, dv_1 \\
&+\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^0 Q(v_1,v_2)\cdot \\
&Q(v_1+\tau_1-\varepsilon_1 u_1, v_2-\varepsilon_2 u_2) R(u_1,u_2)\, du_2\, du_1\, dv_2\, dv_1\big\}\,.
\end{aligned}
$$

Taylor expansion implies

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau_1,0) &= \sum_{j=0}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\Big\{(-1)^k\int_{-1}^1\int_0^1 u_1^k u_2^{j-k} R(u_1,u_2)\, du_2\, du_1\cdot \\
&\quad \int_0^\infty\int_0^\infty \frac{\partial^k Q(v_1+\tau_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\, dv_2\, dv_1 \\
&\quad +(-1)^j\int_{-1}^1\int_{-1}^0 u_1^k u_2^{j-k} R(u_1,u_2)\, du_2\, du_1\cdot \\
&\quad \int_0^\infty\int_0^\infty Q(v_1,v_2)\frac{\partial^j Q(v_1+\tau_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\, dv_2\, dv_1\Big\}+\rho_{N+1}(\varepsilon_1,\varepsilon_2,\tau_1) \\
&= \sum_{j=0}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\int_{-1}^1\int_0^1 u_1^k u_2^{j-k} R(u_1,u_2)\, du_2\, du_1\cdot \\
&\quad \Big\{\int_0^\infty\int_0^\infty Q(v_1,v_2)\frac{\partial^j Q(v_1+\tau_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\, dv_2\, dv_1 \\
&\quad +(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^k Q(v_1+\tau_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\, dv_2\, dv_1\Big\} \\
&\quad +\rho_{N+1}(\varepsilon_1,\varepsilon_2,\tau_1).
\end{aligned}
$$

If $\varepsilon_1=\varepsilon_2=\varepsilon$ it follows

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau_1,0) &= \sum_{j=0}^{N}\varepsilon^{j+2}\sum_{k=0}^{j}\frac{1}{k!(j-k)!}\int_{-1}^1\int_0^1 u_1^k u_2^{j-k} R(u_1,u_2)\, du_2\, du_1\cdot \\
&\quad \Big\{\int_0^\infty\int_0^\infty Q(v_1,v_2)\frac{\partial^j Q(v_1+\tau_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\, dv_2\, dv_1 \\
&\quad +(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^k Q(v_1+\tau_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\, dv_2\, dv_1\Big\} \\
&\quad +\rho_{N+1}(\varepsilon,\tau_1).
\end{aligned}
$$

• For $\tau_1=\tau_2=0$ we find

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(0,0)=\varepsilon_1\varepsilon_2\cdot \\
&\Big\{\int_0^\infty\int_0^\infty\int_0^1\int_0^1 Q(v_1+\varepsilon_1 u_1, v_2+\varepsilon_2 u_2)Q(v_1,v_2)R(u_1,u_2)\, du_2\, du_1\, dv_2\, dv_1 \\
&+\int_0^\infty\int_0^\infty\int_0^1\int_{-1}^0 Q(v_1+\varepsilon_1 u_1, v_2)Q(v_1,v_2-\varepsilon_2 u_2)R(u_1,u_2)\, du_2\, du_1\, dv_2\, dv_1
\end{aligned}
$$

$$+\int_0^\infty\int_0^\infty\int_{-1}^0\int_0^1 Q(v_1,v_2+\varepsilon_2u_2)Q(v_1-\varepsilon_1u_1,v_2)R(u_1,u_2)\,du_2\,du_1\,dv_2\,dv_1$$
$$+\int_0^\infty\int_0^\infty\int_{-1}^0\int_{-1}^0 Q(v_1,v_2)Q(v_1-\varepsilon_1u_1,v_2-\varepsilon_2u_2)R(u_1,u_2)\,du_2\,du_1\,dv_2\,dv_1\Big\},$$

hence

$$^{\varepsilon}R_{gg}(0,0)=\sum_{j=0}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\cdot$$
$$\Big\{\int_0^1\int_0^1 u_1^ku_2^{j-k}R(u_1,u_2)\,du_2\,du_1\int_0^\infty\int_0^\infty\frac{\partial^jQ(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}Q(v_1,v_2)\,dv_2\,dv_1$$
$$+\ (-1)^{j-k}\int_0^1\int_{-1}^0 u_1^ku_2^{j-k}R(u_1,u_2)du_2du_1\int_0^\infty\int_0^\infty\frac{\partial^kQ(v_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}dv_2dv_1$$
$$+\ (-1)^k\int_{-1}^0\int_0^1 u_1^ku_2^{j-k}R(u_1,u_2)du_2du_1\int_0^\infty\int_0^\infty\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\frac{\partial^kQ(v_1,v_2)}{\partial v_1^k}dv_2dv_1$$
$$+\ (-1)^j\int_{-1}^0\int_{-1}^0 u_1^ku_2^{j-k}R(u_1,u_2)du_2du_1\int_0^\infty\int_0^\infty Q(v_1,v_2)\frac{\partial^jQ(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}dv_2dv_1\Big\}$$
$$+\quad\rho_{N+1}(\varepsilon_1,\varepsilon_2)$$

and finally

$$^{\varepsilon}R_{gg}(0,0)=2\sum_{j=0}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\cdot$$
$$\Big\{\int_0^1\int_0^1 u_1^ku_2^{j-k}R(u_1,u_2)\,du_2\,du_1\int_0^\infty\int_0^\infty\frac{\partial^jQ(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}Q(v_1,v_2)\,dv_2\,dv_1$$
$$+\ (-1)^k\int_{-1}^0\int_0^1 u_1^ku_2^{j-k}R(u_1,u_2)du_2du_1\int_0^\infty\int_0^\infty\frac{\partial^kQ(v_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}dv_2dv_1\Big\}$$
$$+\quad\rho_{N+1}(\varepsilon_1,\varepsilon_2).$$

For $\varepsilon_1=\varepsilon_2=\varepsilon$ this can be written as

$$^{\varepsilon}R_{gg}(0,0)=2\sum_{j=0}^{N}\varepsilon^{j+2}\sum_{k=0}^{j}\frac{1}{k!(j-k)!}\cdot$$
$$\Big\{\int_0^1\int_0^1 u_1^ku_2^{j-k}R(u_1,u_2)\,du_2\,du_1\int_0^\infty\int_0^\infty\frac{\partial^jQ(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}Q(v_1,v_2)\,dv_2\,dv_1$$
$$+\ (-1)^k\int_{-1}^0\int_0^1 u_1^ku_2^{j-k}R(u_1,u_2)du_2du_1\int_0^\infty\int_0^\infty\frac{\partial^kQ(v_1,v_2)}{\partial v_1^k}\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}dv_2dv_1\Big\}$$
$$+\quad\rho_{N+1}(\varepsilon).$$

• The values of $^{\varepsilon}R_{gg}(\tau_1,\tau_2)$ for other values τ_1,τ_2 can be found by permutation of indices and using the formula $^{\varepsilon}R_{gg}(\tau_1,\tau_2)=\ ^{\varepsilon}R_{gg}(-\tau_1,-\tau_2)$.

3.2 The case of arbitrary $m > 1$

In the general case of arbitrary $m > 1$ the real valued wide-sense homogeneous random fields are defined by

$$ {}^{\varepsilon}g(\boldsymbol{t}) \;=\; \int_{-\infty}^{\boldsymbol{t}} Q(\boldsymbol{t}-\boldsymbol{s})\,{}^{\varepsilon}f(\boldsymbol{s})\,d\boldsymbol{s} = \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{u})\,{}^{\varepsilon}f(\boldsymbol{t}-\boldsymbol{u})\,d\boldsymbol{u}. $$

Their correlation functions can be found by

$$ \begin{aligned} {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) &= \mathbf{E}\{\,{}^{\varepsilon}g(\boldsymbol{t})\,{}^{\varepsilon}g(\boldsymbol{t}+\boldsymbol{\tau})\} \\ &= \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{u})Q(\boldsymbol{v})\mathbf{E}\{\,{}^{\varepsilon}f(\boldsymbol{t}-\boldsymbol{u})\,{}^{\varepsilon}f(\boldsymbol{t}+\boldsymbol{\tau}-\boldsymbol{v})\}\,d\boldsymbol{v}\,d\boldsymbol{u} \\ &= \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{u})Q(\boldsymbol{v})\,{}^{\varepsilon}R_{ff}(\boldsymbol{\tau}+\boldsymbol{u}-\boldsymbol{v})\,d\boldsymbol{v}\,d\boldsymbol{u} \\ &= \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{u})Q(\boldsymbol{v})R\left(\frac{\boldsymbol{\tau}+\boldsymbol{u}-\boldsymbol{v}}{\boldsymbol{\varepsilon}}\right)\,d\boldsymbol{v}\,d\boldsymbol{u}, \end{aligned} $$

hence

$$ {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) \;=\; \boldsymbol{\varepsilon}^{\mathbf{1}} \int_{-\mathbf{1}}^{\mathbf{1}} \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{v}+(\boldsymbol{\varepsilon u}-\boldsymbol{\tau})^{+})Q(\boldsymbol{v}+(\boldsymbol{\tau}-\boldsymbol{\varepsilon u})^{+})R(\boldsymbol{u})\,d\boldsymbol{v}\,d\boldsymbol{u}. \quad (2) $$

Then the first-order approximation can be written as

$$ {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) \;=\; \boldsymbol{\varepsilon}^{\mathbf{1}} \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{v}+\boldsymbol{\tau}^{-})Q(\boldsymbol{v}+\boldsymbol{\tau}^{+})\,d\boldsymbol{v} \int_{-\mathbf{1}}^{\mathbf{1}} R(\boldsymbol{u})\,d\boldsymbol{u} + o(\boldsymbol{\varepsilon}) $$

as $\boldsymbol{\varepsilon} \to \mathbf{0}$. In order to obtain higher order approximations we derive for fixed values $\boldsymbol{\tau} \geq \boldsymbol{\varepsilon} > \mathbf{0}$

$$ {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) \;=\; \boldsymbol{\varepsilon}^{\mathbf{1}} \int_{-\mathbf{1}}^{\mathbf{1}} \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{v})Q(\boldsymbol{v}+\boldsymbol{\tau}-\boldsymbol{\varepsilon u})R(\boldsymbol{u})\,d\boldsymbol{v}\,d\boldsymbol{u} $$

and from the Taylor expansion

$$ Q(\boldsymbol{v}+\boldsymbol{\tau}-\boldsymbol{\varepsilon u}) = \sum_{j=0}^{N}(-1)^{j} \sum_{|\boldsymbol{k}|=j} \frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}}\boldsymbol{u}^{\boldsymbol{k}}}{\boldsymbol{k}!} D^{\boldsymbol{k}}Q(\boldsymbol{v}+\boldsymbol{\tau}) + \tilde{\rho}_{N+1}(\boldsymbol{v},\boldsymbol{u},\boldsymbol{\tau},\boldsymbol{\varepsilon}) $$

we conclude

$$ \begin{aligned} {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) &= \sum_{j=0}^{N}(-1)^{j} \sum_{|\boldsymbol{k}|=j} \frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}+\mathbf{1}}}{\boldsymbol{k}!} \int_{-\mathbf{1}}^{\mathbf{1}} \boldsymbol{u}^{\boldsymbol{k}} R(\boldsymbol{u})\,d\boldsymbol{u} \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{v})D^{\boldsymbol{k}}Q(\boldsymbol{v}+\boldsymbol{\tau})\,d\boldsymbol{v} \\ &\quad + \rho_{N+1}(\boldsymbol{\tau},\boldsymbol{\varepsilon}) \\ &= \sum_{j=0,\,j\,\text{even}}^{N} \sum_{|\boldsymbol{k}|=j} \frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}+\mathbf{1}}}{\boldsymbol{k}!}\mu_{\boldsymbol{k}} \int_{\boldsymbol{0}}^{\infty} Q(\boldsymbol{v})D^{\boldsymbol{k}}Q(\boldsymbol{v}+\boldsymbol{\tau})\,d\boldsymbol{v} + \rho_{N+1}(\boldsymbol{\tau},\boldsymbol{\varepsilon}). \end{aligned} $$

For arbitrary fixed values $\boldsymbol{\tau}$ we can proceed in the following way.
Let among the m values $\tau_1, \ldots, \tau_m$ be m_+ positive, m_- negative and m_0 zero values. Further let $\{J_l, l = 1, \ldots, 2^{m_0}\}$ be a partition of $[-1,1]^m$, consisting of m-dimensional rectangles with sides $[-1,1]$ in the coordinates where $\tau_i \neq 0$ and sides $[-1,0]$ or $[0,1]$ in the coordinates where $\tau_i = 0$. The number of such different m-dimensional rectangles is 2^{m_0}.
For a given element J_l of this partition every multiindex $\boldsymbol{k} = (k_1, \ldots, k_m)$ can be written as the sum of four multiindices:

$$\boldsymbol{k} = \boldsymbol{k}_+ + \boldsymbol{k}_- + \boldsymbol{k}_{0l+} + \boldsymbol{k}_{0l-},$$

where $\boldsymbol{k}_+$ contains the entries of $\boldsymbol{k}$ in the coordinates with positive values τ_i and zero otherwise, $\boldsymbol{k}_-$ contains the entries of $\boldsymbol{k}$ in the coordinates with negative values τ_i and zero otherwise, $\boldsymbol{k}_{0l+}$ contains the entries of $\boldsymbol{k}$ in the coordinates with $\tau_i = 0$ and side $[0,1]$ for this J_l and zero otherwise and $\boldsymbol{k}_{0l-}$ contains the entries of $\boldsymbol{k}$ in the coordinates with $\tau_i = 0$ and side $[-1,0]$ for this J_l and zero otherwise.
Then the arguments u_i arise only
in the right occurence $Q(\boldsymbol{v} + \boldsymbol{\tau} - \boldsymbol{\varepsilon u})$ in (2) for the m_+ positive values τ_i,
in the left occurence $Q(\boldsymbol{v} + \boldsymbol{\varepsilon u} - \boldsymbol{\tau})$ in (2) for the m_- negative values τ_i ,
in the right occurence $Q(\boldsymbol{v} + \boldsymbol{\tau} - \boldsymbol{\varepsilon u})$ in (2) for these of the m_0 zero values τ_i with side $[-1,0]$ in J_l ,
in the left occurence $Q(\boldsymbol{v} + \boldsymbol{\varepsilon u} - \boldsymbol{\tau})$ in (2) for these of the m_0 zero values τ_i with side $[0,1]$in J_l.
This leads to the following general formula

$$\begin{aligned} {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) &= \sum_{j=0}^{N} \sum_{|\boldsymbol{k}|=j} \frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}+\mathbf{1}}}{\boldsymbol{k}!} (-1)^{|\boldsymbol{k}_+|} \sum_{l=1}^{2^{m_0}} (-1)^{|\boldsymbol{k}_{0l-}|} \int_{J_l} \boldsymbol{u}^{\boldsymbol{k}} R(\boldsymbol{u})\, d\boldsymbol{u} \cdot \\ &\quad \int_{\mathbf{0}}^{\infty} D^{\boldsymbol{k}_{0l+}} D^{\boldsymbol{k}_-} Q(\boldsymbol{v} + \boldsymbol{\tau}^-) D^{\boldsymbol{k}_{0l-}} D^{\boldsymbol{k}_+} Q(\boldsymbol{v} + \boldsymbol{\tau}^+)\, d\boldsymbol{v} + \rho_{N+1}(\boldsymbol{\varepsilon}, \boldsymbol{\tau}). \end{aligned}$$

4 Expansion of covariance matrices

Now, the case of vector valued integral functionals is considered. For the family of random vectors

$${}^{\varepsilon}r(\omega) = \int_{\mathbf{R}^m} Q(\boldsymbol{s})\, {}^{\varepsilon}f(\boldsymbol{s}, \omega)\, d\boldsymbol{s}, \qquad \boldsymbol{\varepsilon} > \mathbf{0},$$

with a matrix valued function $Q(\cdot)$ and complex vector valued random fields $({}^{\varepsilon}f(\cdot)), \boldsymbol{\varepsilon} > \mathbf{0}$, we find

$$\mathbf{E}\{ {}^{\varepsilon}r\, {}^{\varepsilon}r^* \} = \int_{\mathbf{R}^m} \int_{\mathbf{R}^m} Q(\boldsymbol{s}) E\{ {}^{\varepsilon}f(\boldsymbol{s})\, {}^{\varepsilon}f^*(\boldsymbol{t}) \} Q^*(\boldsymbol{t})\, d\boldsymbol{s}\, d\boldsymbol{t}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} Q(\boldsymbol{s})\,{}^{\varepsilon}R_{ff}(\boldsymbol{t}-\boldsymbol{s})Q^*(\boldsymbol{t})\,d\boldsymbol{s}\,d\boldsymbol{t}\\
&= \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} Q(\boldsymbol{s})R\left(\frac{\boldsymbol{t}-\boldsymbol{s}}{\boldsymbol{\varepsilon}}\right)Q^*(\boldsymbol{t})\,d\boldsymbol{s}\,d\boldsymbol{t}\\
&= \boldsymbol{\varepsilon}^{\mathbf{1}}\int_{\mathbf{R}^m}\int_{\mathbf{-1}}^{\mathbf{1}} Q(\boldsymbol{s})R(\boldsymbol{u})Q^*(\boldsymbol{s}+\boldsymbol{\varepsilon u})\,d\boldsymbol{u}\,d\boldsymbol{s}.
\end{aligned}
$$

Taylor expansion gives

$$
Q^*(\boldsymbol{s}+\boldsymbol{\varepsilon u}) = \sum_{j=0}^{N}\sum_{|\boldsymbol{k}|=j}\frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}}\boldsymbol{u}^{\boldsymbol{k}}}{\boldsymbol{k}!}D^{\boldsymbol{k}}Q^*(\boldsymbol{s})+\tilde{\rho}_{N+1}(\boldsymbol{s},\boldsymbol{u},\boldsymbol{\varepsilon})
$$

and we conclude

$$
\begin{aligned}
\mathbf{E}\{{}^{\varepsilon}r\,{}^{\varepsilon}r^*\} &= \sum_{j=0}^{N}\sum_{|\boldsymbol{k}|=j}\frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}+\mathbf{1}}}{\boldsymbol{k}!}\int_{\mathbf{R}^m}Q(\boldsymbol{s})\int_{\mathbf{-1}}^{\mathbf{1}}\boldsymbol{u}^{\boldsymbol{k}}R(\boldsymbol{u})\,d\boldsymbol{u}\;D^{\boldsymbol{k}}Q^*(\boldsymbol{s})\,d\boldsymbol{s}+\rho_{N+1}(\boldsymbol{\varepsilon})\\
&= \sum_{j=0}^{N}\sum_{|\boldsymbol{k}|=j}\frac{\boldsymbol{\varepsilon}^{\boldsymbol{k}+\mathbf{1}}}{\boldsymbol{k}!}\int_{\mathbf{R}^m}Q(\boldsymbol{s})\mu_{\boldsymbol{k}}D^{\boldsymbol{k}}Q^*(\boldsymbol{s})\,d\boldsymbol{s}+\rho_{N+1}(\boldsymbol{\varepsilon}).
\end{aligned}
$$

In the special case $\varepsilon_i=\varepsilon$, $i=1,\ldots,m$, this reads as

$$
\mathbf{E}\{{}^{\varepsilon}r\,{}^{\varepsilon}r^*\} = \sum_{j=0}^{N}\varepsilon^{j+m}\sum_{|\boldsymbol{k}|=j}\frac{1}{\boldsymbol{k}!}\int_{\mathbf{R}^m}Q(\boldsymbol{s})\mu_{\boldsymbol{k}}D^{\boldsymbol{k}}Q^*(\boldsymbol{s})\,d\boldsymbol{s}+\rho_{N+1}(\varepsilon).
$$

5 Expansion of matrix correlation functions

In this section vector valued wide-sense homogeneous random fields arising from integral functionals

$$
{}^{\varepsilon}g(\boldsymbol{t}) = \int_{-\infty}^{\boldsymbol{t}}Q(\boldsymbol{t}-\boldsymbol{s})\,{}^{\varepsilon}f(\boldsymbol{s})\,d\boldsymbol{s}=\int_{\mathbf{0}}^{\boldsymbol{\infty}}Q(\boldsymbol{u})\,{}^{\varepsilon}f(\boldsymbol{t}-\boldsymbol{u})\,d\boldsymbol{u}
$$

are considered. For simplicity we start again with the case of two-dimensional fields, i.e. $m=2$.

5.1 The case $m=2$

The wide-sense homogeneous random fields are defined by

$$
\begin{aligned}
{}^{\varepsilon}g(t_1,t_2) &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}Q(t_1-s_1,t_2-s_2)\,{}^{\varepsilon}f(s_1,s_2)\,ds_1\,ds_2\\
&= \int_0^{\infty}\int_0^{\infty}Q(u_1,u_2)\,{}^{\varepsilon}f(t_1-u_1,t_2-u_2)\,du_1\,du_2.
\end{aligned}
$$

For their correlation functions we derive analogously to Section 2

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau_1,\tau_2) &= \mathbf{E}\{\,{}^{\varepsilon}g(t_1,t_2)\,{}^{\varepsilon}g^*(t_1+\tau_1,t_2+\tau_2)\} \\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty Q(u_1,u_2)\,{}^{\varepsilon}R_{ff}(\tau_1+u_1-v_1,\tau_2+u_2-v_2)\cdot \\
&\quad Q^*(v_1,v_2)\,dv_2\,dv_1\,du_2\,du_1 \\
&= \varepsilon_1\varepsilon_2\int_0^\infty\int_0^\infty\int_{-1}^1\int_{-1}^1 Q(v_1+(\varepsilon_1u_1-\tau_1)^+,v_2+(\varepsilon_2u_2-\tau_2)^+)\cdot \\
&\quad R(u_1,u_2)Q^*(v_1+(\tau_1-\varepsilon_1u_1)^+,v_2+(\tau_2-\varepsilon_2u_2)^+)\,du_2\,du_1\,dv_2\,dv_1.
\end{aligned}
$$

The different coefficients in the expansions for different signs of τ_1 and τ_2 are as follows.

• For fixed values $\tau_1 \geq \varepsilon_1 > 0, \tau_2 \geq \varepsilon_2 > 0$ we get analogously to the real case

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(\tau_1,\tau_2) \\
&= \sum_{j=0}^N(-1)^j\sum_{k=0}^j\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\int_0^\infty\int_0^\infty Q(v_1,v_2)\cdot \\
&\quad \int_{-1}^1\int_{-1}^1 u_1^ku_2^{j-k}R(u_1,u_2)\,du_2\,du_1\frac{\partial^jQ^*(v_1+\tau_1,v_2+\tau_2)}{\partial v_1^k\partial v_2^{j-k}}\,dv_2\,dv_1+\ \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}) \\
&= \sum_{j=0}^N\sum_{k=0}^j\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\int_0^\infty\int_0^\infty Q(v_1,v_2)\mu^*_{(k,j-k)}\frac{\partial^jQ^*(v_1+\tau_1,v_2+\tau_2)}{\partial v_1^k\partial v_2^{j-k}}\,dv_2\,dv_1 \\
&\quad +\rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}).
\end{aligned}
$$

In the case $\varepsilon_1 = \varepsilon_2 = \varepsilon$ this reads

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(\tau_1,\tau_2) \\
&= \sum_{j=0}^N\varepsilon^{j+2}\sum_{k=0}^j\frac{1}{k!(j-k)!}\int_0^\infty\int_0^\infty Q(v_1,v_2)\mu^*_{(k,j-k)}\frac{\partial^jQ^*(v_1+\tau_1,v_2+\tau_2)}{\partial v_1^k\partial v_2^{j-k}}\,dv_2\,dv_1 \\
&\quad +\rho_{N+1}(\varepsilon,\boldsymbol{\tau}).
\end{aligned}
$$

• For fixed values $\tau_1 \geq \varepsilon_1 > 0, \tau_2 \leq -\varepsilon_2 < 0$ we derive

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(\tau_1,\tau_2) \\
&= \sum_{j=0}^N\sum_{k=0}^j(-1)^k\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\int_0^\infty\int_0^\infty\frac{\partial^{j-k}Q(v_1,v_2-\tau_2)}{\partial v_2^{j-k}}\cdot \\
&\quad \int_{-1}^1\int_{-1}^1 u_1^ku_2^{j-k}R(u_1,u_2)\,du_2\,du_1\frac{\partial^kQ^*(v_1+\tau_1,v_2)}{\partial v_1^k}\,dv_2\,dv_1+\ \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}) \\
&= \sum_{j=0}^N\sum_{k=0}^j(-1)^k\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\cdot \\
&\quad \int_0^\infty\int_0^\infty\frac{\partial^{j-k}Q(v_1,v_2-\tau_2)}{\partial v_2^{j-k}}\mu_{(k,j-k)}\frac{\partial^kQ^*(v_1+\tau_1,v_2)}{\partial v_1^k}\,dv_2\,dv_1+\ \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}).
\end{aligned}
$$

Hence, for $\varepsilon_1 = \varepsilon_2 = \varepsilon$ it is valid

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(\tau_1,\tau_2)\\
&= \sum_{j=0}^{N}\varepsilon^{j+2}\sum_{k=0}^{j}\frac{(-1)^k}{k!(j-k)!}\cdot\\
&\quad\int_0^\infty\int_0^\infty \frac{\partial^{j-k}Q(v_1,v_2-\tau_2)}{\partial v_2^{j-k}}\mu_{(k,j-k)}\frac{\partial^k Q^*(v_1+\tau_1,v_2)}{\partial v_1^k}\,dv_2\,dv_1 + \rho_{N+1}(\varepsilon,\boldsymbol{\tau}).
\end{aligned}
$$

• In the case $\tau_1 \geq \varepsilon_1 > 0, \tau_2 = 0$ we get

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(\tau_1,0)\\
&= \sum_{j=0}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\Bigg\{(-1)^k\int_0^\infty\int_0^\infty\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\cdot\\
&\quad\int_{-1}^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1\frac{\partial^k Q^*(v_1+\tau_1,v_2)}{\partial v_1^k}\,dv_2\,dv_1\\
&\quad+(-1)^j\int_0^\infty\int_0^\infty Q(v_1,v_2)\cdot\\
&\quad\int_{-1}^1\int_{-1}^0 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1\frac{\partial^j Q^*(v_1+\tau_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\,dv_2\,dv_1\Bigg\} + \rho_{N+1}(\boldsymbol{\varepsilon},\tau_1).
\end{aligned}
$$

If $\varepsilon_1 = \varepsilon_2 = \varepsilon$ this gives

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(\tau_1,0)\\
&= \sum_{j=0}^{N}\varepsilon^{j+2}\sum_{k=0}^{j}\frac{1}{k!(j-k)!}\Bigg\{(-1)^k\int_0^\infty\int_0^\infty\frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\cdot\\
&\quad\int_{-1}^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1\frac{\partial^k Q^*(v_1+\tau_1,v_2)}{\partial v_1^k}\,dv_2\,dv_1\\
&\quad+(-1)^j\int_0^\infty\int_0^\infty Q(v_1,v_2)\cdot\\
&\quad\int_{-1}^1\int_{-1}^0 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1\frac{\partial^j Q^*(v_1+\tau_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\,dv_2\,dv_1\Bigg\} + \rho_{N+1}(\varepsilon,\tau_1).
\end{aligned}
$$

• For $\tau_1 = \tau_2 = 0$ we find

$$
\begin{aligned}
&{}^{\varepsilon}R_{gg}(0,0) = \sum_{j=0}^{N}\sum_{k=0}^{j}\frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\cdot\\
&\quad\Bigg\{\int_0^\infty\int_0^\infty\frac{\partial^j Q(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\int_0^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1 Q^*(v_1,v_2)\,dv_2\,dv_1\\
&+(-1)^{j-k}\int_0^\infty\int_0^\infty\frac{\partial^k Q(v_1,v_2)}{\partial v_1^k}\int_0^1\int_{-1}^0 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^{j-k}Q^*(v_1,v_2)}{\partial v_2^{j-k}}dv_2dv_1
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\int_{-1}^0\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^k Q^*(v_1,v_2)}{\partial v_1^k}dv_2dv_1\\
&+(-1)^j\int_0^\infty\int_0^\infty Q(v_1,v_2)\int_{-1}^0\int_{-1}^0 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^j Q^*(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}dv_2dv_1\Bigg\}\\
&+\quad \rho_{N+1}(\varepsilon_1,\varepsilon_2)\\
&=\quad \sum_{j=0}^N\sum_{k=0}^j \frac{\varepsilon_1^{k+1}\varepsilon_2^{j-k+1}}{k!(j-k)!}\cdot\\
&\quad\Bigg\{\int_0^\infty\int_0^\infty \frac{\partial^j Q(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\int_0^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1Q^*(v_1,v_2)\,dv_2\,dv_1\\
&+\quad\left[\int_0^\infty\int_0^\infty \frac{\partial^j Q(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\int_0^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1Q^*(v_1,v_2)\,dv_2\,dv_1\right]^*\\
&+(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\int_{-1}^0\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^k Q^*(v_1,v_2)}{\partial v_1^k}dv_2dv_1\\
&+\left[(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\int_{-1}^0\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^k Q^*(v_1,v_2)}{\partial v_1^k}dv_2dv_1\right]^*\Bigg\}\\
&+\quad \rho_{N+1}(\varepsilon_1,\varepsilon_2).
\end{aligned}
$$

For $\varepsilon_1=\varepsilon_2=\varepsilon$ it follows

$$
\begin{aligned}
{}^\varepsilon R_{gg}(0,0) &= \sum_{j=0}^N \varepsilon^{j+2}\sum_{k=0}^j \frac{1}{k!(j-k)!}\cdot\\
&\quad\Bigg\{\int_0^\infty\int_0^\infty \frac{\partial^j Q(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\int_0^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1Q^*(v_1,v_2)\,dv_2\,dv_1\\
&+\quad\left[\int_0^\infty\int_0^\infty \frac{\partial^j Q(v_1,v_2)}{\partial v_1^k\partial v_2^{j-k}}\int_0^1\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)\,du_2\,du_1Q^*(v_1,v_2)\,dv_2\,dv_1\right]^*\\
&+(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\int_{-1}^0\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^k Q^*(v_1,v_2)}{\partial v_1^k}dv_2dv_1\\
&+\left[(-1)^k\int_0^\infty\int_0^\infty \frac{\partial^{j-k}Q(v_1,v_2)}{\partial v_2^{j-k}}\int_{-1}^0\int_0^1 u_1^k u_2^{j-k}R(u_1,u_2)du_2du_1\frac{\partial^k Q^*(v_1,v_2)}{\partial v_1^k}dv_2dv_1\right]^*\Bigg\}\\
&+\quad \rho_{N+1}(\varepsilon).
\end{aligned}
$$

• The values of ${}^\varepsilon R_{gg}(\tau_1,\tau_2)$ for other arguments τ_1,τ_2 can be found by permutation of indices and using the formula ${}^\varepsilon R_{gg}(\tau_1,\tau_2) = {}^\varepsilon R^*_{gg}(-\tau_1,-\tau_2)$.

5.2 The case of arbitrary $m > 1$

In case of arbitrary $m > 1$ the wide-sense homogeneous random fields are defined by

$$ {}^{\varepsilon}g(\boldsymbol{t}) \;=\; \int_{-\infty}^{\boldsymbol{t}} Q(\boldsymbol{t}-\boldsymbol{s})\,{}^{\varepsilon}f(\boldsymbol{s})\,d\boldsymbol{s} = \int_{0}^{\infty} Q(\boldsymbol{u})\,{}^{\varepsilon}f(\boldsymbol{t}-\boldsymbol{u})\,d\boldsymbol{u} $$

and their correlation functions can be found analogously to Section 3 by

$$ {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) \;=\; \boldsymbol{\varepsilon^{1}} \int_{\boldsymbol{-1}}^{\boldsymbol{1}} \int_{0}^{\infty} Q(\boldsymbol{v}+(\boldsymbol{\varepsilon u}-\boldsymbol{\tau})^{+}) R(\boldsymbol{u}) Q^{*}(\boldsymbol{v}+(\boldsymbol{\tau}-\boldsymbol{\varepsilon u})^{+})\,d\boldsymbol{v}\,d\boldsymbol{u}. $$

In the same way as in Section 3 and with the same notation it follows for fixed arguments $\boldsymbol{\tau}$

$$ \begin{aligned} {}^{\varepsilon}R_{gg}(\boldsymbol{\tau}) \;=\; & \sum_{j=0}^{N} \sum_{|\boldsymbol{k}|=j} \frac{\boldsymbol{\varepsilon^{k+1}}}{\boldsymbol{k!}} (-1)^{|\boldsymbol{k}_{+}|} \sum_{l=1}^{2^{m_0}} (-1)^{|\boldsymbol{k}_{0l-}|}\,. \\ & \int_{0}^{\infty} D^{\boldsymbol{k}_{0l+}} D^{\boldsymbol{k}_{-}} Q(\boldsymbol{v}+\boldsymbol{\tau}^{-}) \int_{J_l} \boldsymbol{u}^{\boldsymbol{k}} R(\boldsymbol{u})\,d\boldsymbol{u}\; D^{\boldsymbol{k}_{0l-}} D^{\boldsymbol{k}_{+}} Q^{*}(\boldsymbol{v}+\boldsymbol{\tau}^{+})\,d\boldsymbol{v} \\ & + \rho_{N+1}(\boldsymbol{\varepsilon},\boldsymbol{\tau}). \end{aligned} $$

6 The case of orthotropic fields and product kernel

In this section it is assumed that $({}^{\varepsilon}f(\cdot)), \varepsilon > 0$ are real valued random fields satisfying Assumption 1 with $\varepsilon_1 = \ldots = \varepsilon_m = \varepsilon$ and

$$ {}^{\varepsilon}R(\boldsymbol{s}) = R\left(\frac{|s_1|}{\varepsilon}, \ldots, \frac{|s_m|}{\varepsilon}\right), $$

i.e. $({}^{\varepsilon}f(\cdot)), \varepsilon > 0$ are orthotropic fields. Further, the kernel function $Q(\cdot)$ is assumed to possess the representation

$$ Q(s_1, \ldots, s_m) = \prod_{i=1}^{m} Q_i(s_i) $$

with some functions $Q_1(\cdot), \ldots, Q_m(\cdot)$ satisfying Assumption 2 on $\mathcal{D} = \mathbf{R}_{+}$ for some $N \geq 0$.
Then the leading expansion term of the correlation function ${}^{\varepsilon}R_{gg}(\tau)$ reads as

$$ {}^{\varepsilon}R_{gg}(\tau) = \varepsilon^{m} \mu_0 \prod_{i=1}^{m} \int_{0}^{\infty} Q_i(v) Q_i(v+|\tau|)\,dv + o(\varepsilon^{m}). $$

Higher-order expansion terms can be derived explicitly for the case

$$Q_i^{(j)}(0) = \lim_{u\to\infty} Q_i^{(j)}(u) = 0,\ j = 0, 1.$$

For $m = 2$ and arbitrary fixed arguments τ_1, τ_2 and $\varepsilon \to 0$ the following asymptotic expansion is valid

$$\begin{aligned} {}^{\varepsilon}R_{gg}(\tau_1,\tau_2) &= \varepsilon^2\mu_{(0,0)}\int_0^\infty Q_1(v)Q_1(v+|\tau_1|)\,dv\int_0^\infty Q_2(v)Q_2(v+|\tau_2|)\,dv \\ &\quad -\frac{\varepsilon^4}{2}\Big\{\mu_{(2,0)}\int_0^\infty Q_1'(v)Q_1'(v+|\tau_1|)\,dv\int_0^\infty Q_2(v)Q_2(v+|\tau_2|)\,dv \\ &\quad +\mu_{(0,2)}\int_0^\infty Q_1(v)Q_1(v+|\tau_1|)\,dv\int_0^\infty Q_2'(v)Q_2'(v+|\tau_2|)\,dv\Big\} \\ &\quad +o(\varepsilon^4). \end{aligned}$$

The proof uses properties of correlation moments and relations of the type

$$\int_0^\infty Q_i(v)Q_i'(v)\,dv = 0.$$

7 Conclusion

The results of [5] and [6] concerning asymptotic expansions of variances or correlation functions for real or vector valued integral functionals of stationary random processes are extended to the case of integral functionals of random fields. These expansion can be used in the investigation of stochastic characteristics of solutions to ordinary or partial differential equations with random inhomogeneous ε-correlated terms.

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