TECHNISCHE UNIVERSITÄT CHEMNITZ

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Preprint 98-30

Fakultät für Mathematik

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Abstract

A possible decomposition approach for a convex bordering structured programming problem is suggested. Some properties of the decomposed problem and in particular of the resulting subproblems are described. For instance, these are equivalence, convexity, and solvability. The last part deals with the nonemptiness of the optimal sets of the subproblems. Finally, some assumptions are described ensuring that the optimal values are attained.

Key words convex programming, decomposition methods, bordering structure.

AMS subject classifications 65K05, 90C25, 49M27.

1 Introduction

The representation of a programming problem as a family of subproblems is called a decomposition of this problem. The mentioned subproblems are weakly connected through certain common variables respectively restrictions. For instance, applying a decomposition approach is suitable if it is possible to separate well-structured parts of the problem from variables respectively restrictions violating the structure. These variables are fixed temporarily. Consequently, efficient algorithms for solving the subproblems can be used. Clearly, the decomposed problem has to be equivalent to the original one. In particular an optimal solution of the original program has to be easy to compute from an optimal solution of the decomposed problem.

Consider the bordering structured programming problem

$$\begin{aligned} & f(x^0, x) := \sum_{i=0}^{k} f_i(x^i) \to \inf_{(x^0, x^1, \dots, x^k)} \qquad (1.1) \\ \text{s.t.} \quad & g_i(x^i) + g_i^0(x^0) \le \mathbf{0} \quad \text{for} \quad i \in \{1, \dots, k\}, \\ & \sum_{i=0}^{k} h_i(x^i) \le \mathbf{0}, \\ & x^i \in X^i \quad \text{for} \quad i \in \{0, 1, \dots, k\}. \end{aligned}$$

The variable $x := (x^1, \dots, x^k)$ is an element of $\mathbb{R}^{n_1+\dots+n_k}$ with $x^i \in \mathbb{R}^{n_i}$ for $i \in \{1, \dots, k\}$.

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Suppose that

$\mathbf{X}^i$ is a nonempty, closed, convex subset of $\mathbb{R}^{n_i}$ for $i \in \{0,1,\dots,k\}$;

f_i : $\mathbb{R}^{n_i} \to \mathbb{R} \cup \{+\infty\}$ is a convex and lower semicontinuous function on $\mathbb{R}^{n_i}$ and finite on $\mathbf{X}^i$ for $i \in \{0,1,\dots,k\}$;

g_{ij} : $\mathbb{R}^{n_i} \to \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous, proper function for $j \in \{1,\dots,m_i\}$ and $g_i := (g_{i1},\dots,g_{im_i})^\tau$ for $i \in \{1,\dots,k\}$;

g^0_{ij} : $\mathbb{R}^{n_0} \to \mathbb{R} \cup \{+\infty\}$ is a convex function for $j \in \{1,\dots,m_i\}$ and $g^0_i := (g^0_{i1},\dots,g^0_{im_i})^\tau$ for $i \in \{1,\dots,k\}$;

h_{ij} : $\mathbb{R}^{n_i} \to \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous, proper function for $j \in \{1,\dots,m_0\}$ and $h_i := (h_{i1},\dots,h_{im_0})^\tau$ for $i \in \{0,1,\dots,k\}$.

Therefore, the problem (1.1) is a convex programming problem. The common restrictions

$$\sum_{i=0}^{k} h_i(x^i) \leq \mathbf{0}$$

are splitted up by the introduction of new variables u. In this way the common variables x^0 and u are fixed in the subproblems.
Let $u := (u^1,\dots,u^k) \in \mathbb{R}^{k \cdot m_0}$. Consider the decomposed problem corresponding to (1.1)

$$\begin{aligned} \Phi(x^0,u) := \sum_{i=0}^{k} \Phi_i(x^0,u) &\to \inf_{(x^0,u)} \\ \text{s.t.} \quad \sum_{i=1}^{k} u^i + h_0(x^0) &\leq \mathbf{0}, \\ x^0 \in \mathbf{X}^0, \ u^i \in \mathbb{R}^{m_0} & \text{ for } i \in \{1,\dots,k\}, \end{aligned} \tag{1.2}$$

with

$$\Phi_0(x^0,u) := \begin{cases} f_0(x^0), & \text{if } x^0 \in \mathbf{X}^0, \\ +\infty, & \text{else,} \end{cases}$$

$$\Phi_i(x^0,u) := \begin{cases} \inf_{x^i}\{f_i(x^i) : x^i \in S^i(x^0,u)\}, & \text{if } S^i(x^0,u) \neq \emptyset, \\ +\infty, & \text{else} \end{cases}$$

and

$$S^i(x^0,u) := \{x^i \in \mathbf{X}^i : g_i(x^i) + g^0_i(x^0) \leq \mathbf{0}, \ h_i(x^i) - u^i \leq \mathbf{0}\} \ \text{ for } i \in \{1,\dots,k\}.$$

In this way we get k subproblems $\mathrm{P}^i(x^0,u)$

$$\inf_{x^i}\{f_i(x^i) : x^i \in S^i(x^0,u)\} \tag{$\mathrm{P}^i(x^0,u)$}$$

for each fixed (x^0,u).
If for $i \in \{1,\dots,k\}$ the set $S^i(x^0,u)$ is nonempty we have to solve $\mathrm{P}^i(x^0,u)$. By definition, $S^i(x^0,u)$ is the set of feasible solutions of $\mathrm{P}^i(x^0,u)$. We denote the corresponding set of optimal solutions by $\widetilde{S}^i(x^0,u)$.
Clearly, under the above assumption $\Phi_i(x^0,u) = -\infty$ is possible for some $i \in \{1,\dots,k\}$ and at some point (x^0,u). Therefore we enter into the agreement: If for $(x^0,u) \in \mathbb{R}^{n_0+k\cdot m_0}$ exists at least one $i \in \{0,1,\dots,k\}$ with $\Phi_i(x^0,u) = +\infty$ then we set $\Phi(x^0,u) := +\infty$.

The statements of the Theorem 1 show the equivalence of the problems (1.1) and (1.2). From the considerations in Section 2 it follows that under certain assumptions the subproblems

P^i and the decomposed problem (1.2) are convex programs. In section 3 some conditions are studied which guarantee that the optimal values of the subproblems are attained. Thus, the formulation of sufficient conditions for strong solvability of the bordering structured problem (1.1) is possible. The knowledge of such conditions is also useful for computation of subgradients of the function Φ.

In [1] and [2] some conditions of another type as in section 3 are described which ensure the nonemptiness of the optimal sets of programming problems. Properties of the objective functions are studied there. Some statements about nonempty optimal solution sets are also known from parametric optimization. However most of them need some sort of regularity of all the parameter depending programs which should be studied. With the assumptions for (1.1) used in section 2 it is only possible to ensure the existence of at least one regular subproblem $\mathrm{P}^i(.,.)$. Therefore, in section 3 another approach is chosen. It is proved that the existence of one subproblem $\mathrm{P}^i(x^0, u)$ with a nonempty bounded feasible set $S^i(x^0, u)$, respectively with a nonempty bounded optimal set $\widetilde{S}^i(x^0, u)$, guarantees nonempty optimal sets of all subproblems $\mathrm{P}^i(.,.)$. Finally, the case of an unbounded optimal set with a nonempty interior of its recession cone is studied.

A point $(x^0, x) \in \mathbb{R}^{n_0+n_1+\dots+n_k}$ is called a *feasible solution* of problem (1.1) if it satisfies the conditions $x^i \in \mathbf{X}^i$ for all $i \in \{0, 1, \dots, k\}$, $g_i(x^i) + g_i^0(x^0) \leq \mathbf{0}$ for all $i \in \{1, \dots, k\}$, $\sum_{i=0}^{k} h_i(x^i) \leq \mathbf{0}$ and $f(x^0, x) < +\infty$. The set of all feasible solutions of (1.1) is denoted by S_1. Let $f^\star$ denote the optimal value of (1.1). Analogously, $(x^0, u) \in \mathbb{R}^{n_0+k\cdot m_0}$ is a feasible solution of problem (1.2) if $x^0 \in \mathbf{X}^0$, $\sum_{i=1}^{k} u^i + h_0(x^0) \leq \mathbf{0}$ and $\Phi(x^0, u) < +\infty$. We denote the set of all feasible solutions of (1.2) by S_2 and the optimal value by $\Phi^\star$.

The problem (1.1) (respectively the problem (1.2)) is called *weakly solvable* if the set S_1 (respectively S_2) is nonempty and the optimal value $f^\star$ (respectively $\Phi^\star$) is finite. If additionally a point $(x^{0\star}, x^\star) \in S_1$ with $f(x^{0\star}, x^\star) = f^\star$ (respectively $(x^{0\star}, u^\star) \in S_2$ with $\Phi(x^{0\star}, u^\star) = \Phi^\star$) exists then (1.1) (respectively (1.2)) is called *strongly solvable*.

Under the above assumptions and the agreement we get the following statements about the equivalence of the problems (1.1), (1.2).

Theorem 1

(i) *Let $(x^0, x) \in S_1$ and $u^i := h_i(x^i)$ for $i \in \{1, \dots, k\}$. Then we have $(x^0, u) \in S_2$ and $f(x^0, x) \geq \Phi(x^0, u)$.*

(ii) *For $(x^0, u) \in S_2$ there exists a point $\hat{x} = (\hat{x}^1, \dots, \hat{x}^k)$ with $(x^0, \hat{x}) \in S_1$, $\hat{x}^i \in S^i(x^0, u)$ for $i \in \{1, \dots, k\}$ and $f(x^0, \hat{x}) \geq \Phi(x^0, u)$.*

(iii) *S_1 is nonempty if and only if S_2 is nonempty.*

(iv) *The equation $f^\star = \Phi^\star$ holds.*

(v) *(1.1) is weakly solvable if and only if (1.2) is weakly solvable.*

(vi) *If (1.1) is strongly solvable then (1.2) is strongly solvable.*

In general, the converse of statement (vi) is not true. This is illustrated by the example below.

Example 1 The programming problem

$$
\begin{aligned}
f(x^0, x) := (x^0)^2 + \tfrac{1}{x^1} \rightarrow & \inf_{(x^0, x^1)} \\
\text{s.t.} \quad & x^0 - x^1 \leq 0, \\
& -x^0 - x^1 \leq 0, \\
& x^0 \in \mathbb{R},\ x^1 \in [1, \infty)
\end{aligned}
$$

is of type (1.1) with $k = 1$ and satisfies the above assumptions. The optimal value is $f^\star = 0$. Of course the problem is not strongly solvable. But the corresponding problem of type (1.2)

$$\begin{aligned} \Phi(x^0, u^1) := \Phi_0(x^0, u^1) + \Phi_1(x^0, u^1) &\to \inf_{(x^0, u^1)} \\ \text{s.t.} \quad u^1 - x^0 &\le 0, \\ x^0 \in \mathbb{R},\ u^1 &\in \mathbb{R}, \end{aligned}$$

with

$$\Phi_0(x^0, u^1) = (x^0)^2,$$

$$\Phi_1(x^0, u^1) = \begin{cases} \inf\limits_{x^1} \left\{\frac{1}{x^1} : x^1 \in S^1(x^0, u^1)\right\}, & \text{if } S^1(x^0, u^1) \ne \emptyset, \\ +\infty, & \text{else} \end{cases}$$

and

$$S^1(x^0, u^1) = \{x^1 \in [1, \infty) : -x^1 - u^1 \le 0,\ x^0 - x^1 \le 0\}$$

has an optimal solution $(x^{0\star}, u^{1\star}) = (0, 0)$ with $\Phi^\star = \Phi(0,0) = 0$. In other words, the problem is strongly solvable.

2 On some properties of the decomposed problem

Some properties of the function Φ_i ($i \in \{0, 1, \ldots, k\}$) are pointed out in this section. Under certain assumptions concerning the solvability of problem (1.1) and under regularity conditions these functions have values in $\mathbb{R} \cup \{+\infty\}$. Hence, convexity properties of the subproblems $\mathrm{P}^i(x^0, u)$ and the problem (1.2) are established.

Lemma 1 *The inequality $\Phi_0(x^0, u) > -\infty$ holds for any $(x^0, u) \in \mathbb{R}^{n_0 + k \cdot m_0}$. Φ_0 is a proper convex function.*

Consider a programming problem of the form

$$\begin{aligned} \phi(y) &\to \inf_y \\ \text{s.t.} \quad \psi_i(y) &\le 0 \ \text{ for } i \in \{1, \ldots, p\}, \\ y &\in Y \end{aligned}$$

with $\phi,\ \psi_i : \mathbb{R}^q \to \mathbb{R} \cup \{\infty\}$ convex and $Y \subseteq \mathbb{R}^q$ convex.
We say that this problem satisfies the *Slater condition* if there exists a point $\bar{y} \in \operatorname{ri} Y$ with $\psi_i(\bar{y}) \le 0$ for all $i \in \{1, \ldots, p\}$ and $\psi_i(\bar{y}) < 0$ for all nonaffine functions ψ_i. The point $\bar{y}$ is called *Slater point* if additionally the inequality $\psi_i(\bar{y}) < 0$ holds for each $i \in \{1, \ldots, p\}$.
Because of a clear description we only study the two cases that program (1.1) has a Slater point respectively (1.1) is a linear problem. Analogous statements are true if the mentioned Slater condition is fulfilled.

Lemma 2 *Let $j \in \{1, \ldots, k\}$ be fixed.*

(i) *If problem (1.1) is a linear programming problem (w.l.o.g. all restrictions are formulated as inequalities) then the subproblem $\mathrm{P}^j(x^0, u)$ with $S^j(x^0, u) \ne \emptyset$ is also a linear programming problem for all $(x^0, u) \in \mathbb{R}^{n_0 + k \cdot m_0}$.*

(ii) *If a Slater point $(\bar{x}^0, \bar{x})$ exists for the programming problem (1.1) then there exists a point $\bar{u} \in \mathbb{R}^{k \cdot m_0}$ such that a Slater point exists for $S^j(\bar{x}^0, \bar{u})$. Moreover, if (1.1) is weakly solvable then $(\bar{x}^0, \bar{u})$ is feasible for (1.2) and $\Phi(\bar{x}^0, \bar{u})$ is finite.*

It is known from duality theory that the set of feasible solutions of the dual problem of a convex and regular program with finite optimal value is nonempty. If (1.1) satisfies the conditions of Lemma 2(i) and additionally (1.1) is weakly solvable then the following properties are easy to prove. For an arbitrary feasible solution $(\hat{x}^0, \hat{u}) \in S_2$ the set of dual feasible solutions of subproblem $\mathrm{P}^j(\hat{x}^0, \hat{u})$ is nonempty, that means

$$\Big\{y \in \mathbb{R}_+^{m_j+m_0} : \inf_{x^j \in \mathbf{X}^j} \{f_j(x^j) + \sum_{l=1}^{m_j} y_l \cdot (g_{jl}(x^j) + g_{jl}^0(\hat{x}^0)) + \sum_{l=m_j+1}^{m_j+m_0} y_l \cdot (h_{jl}(x^j) - \hat{u}_l^j)\} > -\infty\Big\} \neq \emptyset.$$

Analogously, it holds for the point $(\bar{x}^0, \bar{u})$ defined by Lemma 2(ii)

$$\Big\{y \in \mathbb{R}_+^{m_j+m_0} : \inf_{x^j \in \mathbf{X}^j} \{f_j(x^j) + \sum_{l=1}^{m_j} y_l \cdot (g_{jl}(x^j) + g_{jl}^0(\bar{x}^0)) + \sum_{l=m_j+1}^{m_j+m_0} y_l \cdot (h_{jl}(x^j) - \bar{u}_l^j)\} > -\infty\Big\} \neq \emptyset.$$

The existence of a point $(x^0, u) \in \mathbb{R}^{n_0+k\cdot m_0}$ with $\Phi_j(x^0, u) = -\infty$ contradicts the nonemptiness of the above sets of dual feasible solutions. Therefore, in combination with Lemma 1 and Lemma 2 follows:

Proposition 1 *For all $(x^0, u) \in \mathbb{R}^{n_0+k\cdot m_0}$ and $i \in \{0, 1, \ldots, k\}$ holds $\Phi_i(x^0, u) > -\infty$ if*
(i) (1.1) *is a weakly solvable linear programming problem or*
(ii) (1.1) *is weakly solvable and has a Slater point.*

Proposition 2 *Under the assumptions of Theorem 2 the functions Φ_i are proper convex on the convex set $\mathbf{X}^0 \times \mathbb{R}^{k\cdot m_0}$ for all $i \in \{0, 1, \ldots, k\}$. Hence, the subproblems $\mathrm{P}^i(x^0, u)$ are convex programming problems for all $(x^0, u) \in \mathbf{X}^0 \times \mathbb{R}^{k\cdot m_0}$. The problem* (1.2) *is a convex program, too.*

3 The sets of optimal solutions of the subproblems

In the following considerations let the index $j \in \{1, \ldots, k\}$ be fixed. Obviously, for a point $(x^0, u) \in \operatorname{dom} \Phi_j$ holds $S^j(x^0, u) \neq \emptyset$. In the previous sections we have studied properties of the problems $\mathrm{P}^j(x^0, u)$

$$\inf_{x^j}\{f_j(x^j) : x^j \in S^j(x^0, u)\}$$

with the feasible set $S^j(x^0, u) = \{x^j \in \mathbf{X}^j : g_j(x^j) + g_j^0(x^0) \leq \mathbf{0},\ h_j(x^j) - u^j \leq \mathbf{0}\}$ and the optimal value $\Phi_j(x^0, u)$. We consider the set $\widetilde{S}^j(x^0, u)$ of optimal solutions of $\mathrm{P}^j(x^0, u)$ in this section. The statements are proved by means of recession cones and recession functions. Firstly, we define these terms and mention some of their properties [2],[3]. Afterwards, we formulate some properties of the problems $\mathrm{P}^j(.,.)$. Finally, we prove a converse of statement (vi) of Theorem 1 under certain additional assumptions.

Let $C \subseteq \mathbb{R}^n$ be nonempty. The set

$$C_\infty := \{y \in \mathbb{R}^n : \exists\{x_m\} \subseteq C, \exists\{t_m\} \subseteq \mathbb{R} \text{ with } t_m \to +\infty \text{ and } \frac{x_m}{t_m} \to y \text{ for } m \to +\infty\}$$

is called *recession cone* of C. The cone C_∞ is closed. The set C is bounded if and only if $C_\infty = \{\mathbf{0}\}$. If C is a closed convex set then C_∞ is also convex and it holds

$$\begin{aligned} C_\infty &= \{y \in \mathbb{R}^n : \exists\, x \in C \text{ with } x + \lambda y \in C \text{ for all } \lambda \geq 0\} \\ &= \{y \in \mathbb{R}^n : \text{for all } x \in C \text{ holds } x + \lambda y \in C \text{ for all } \lambda \geq 0\}. \end{aligned}$$

Let $\{C_i \subseteq \mathbb{R}^n,\ i \in I\}$ be a collection of closed convex sets, where I is an arbitrary index set. If $\bigcap_{i\in I} C_i \neq \emptyset$ then $(\bigcap_{i\in I} C_i)_\infty = \bigcap_{i\in I}(C_i)_\infty$.
Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The function $\varphi_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ defined by $\operatorname{epi} \varphi_\infty = (\operatorname{epi} \varphi)_\infty$ is naturally called the *recession function* of φ.
If additionally φ is lower semicontinuous and convex then we have for $\lambda \in \mathbb{R}$

$$\{x \in \mathbb{R}^n : \varphi(x) \leq \lambda\}_\infty = \{x \in \mathbb{R}^n : \varphi_\infty(x) \leq 0\},$$

if the set $\{x \in \mathbb{R}^n : \varphi(x) \leq \lambda\}$ is nonempty.

With $a_1 = (a_{11}, \ldots, a_{1m_j})^T \in \mathbb{R}^{m_j}$ and $a_2 = (a_{21}, \ldots, a_{2m_0})^T \in \mathbb{R}^{m_0}$ the following sets are level sets of the functions g_{jl} and h_{jl}, respectively

$$C_{jl}(a_1) := \{x^j \in \mathbb{R}^{n_j} : g_{jl}(x^j) \leq a_{1l}\} \quad \text{for } l \in \{1, \cdots, m_j\} \text{ and}$$
$$C_{jl}(a_2) := \{x^j \in \mathbb{R}^{n_j} : h_{j(l-m_j)}(x^j) \leq a_{2(l-m_j)}\} \quad \text{for } l \in \{m_j+1, \cdots, m_j+m_0\}.$$

The sets $C_{jl}(a_1)$ and $C_{jl}(a_2)$ are convex and closed. With these notations Lemma 4 follows.

Lemma 4 *Let* $(x^0, u) \in \operatorname{dom} \Phi_j$.

(i) *Then* $S^j(x^0, u) = \bigcap_{l=1}^{m_j} C_{jl}(-g_j^0(x^0)) \bigcap \bigcap_{l=1}^{m_0} C_{j(l+m_j)}(u^j) \bigcap \mathbf{X}^j$.
This set is nonempty, closed and convex.

(ii) $(S^j(x^0, u))_\infty = \bigcap_{l=1}^{m_j} (C_{jl}(-g_j^0(x^0)))_\infty \bigcap \bigcap_{l=1}^{m_0} (C_{j(l+m_j)}(u^j))_\infty \bigcap (\mathbf{X}^j)_\infty$

(iii) $(S^j(\hat{x}^0, \hat{u}))_\infty = (S^j(\tilde{x}^0, \tilde{u}))_\infty$ *for* $(\hat{x}^0, \hat{u})$, $(\tilde{x}^0, \tilde{u}) \in \operatorname{dom} \Phi_j$.

Proof: Because of the definition of $S^j(x^0, u)$ the equations asserted in (i) hold. From $(x^0, u) \in \operatorname{dom} \Phi_j$ follows $S^j(x^0, u) \neq \emptyset$ and therefore, the intersection of the nonempty sets $C_{...}$ is closed and convex. By reason of the above noted property of the recession cone of a nonempty intersection of closed convex sets it follows (ii).
For $(\hat{x}^0, \hat{u})$, $(\tilde{x}^0, \tilde{u}) \in \operatorname{dom} \Phi_j$ we have $C_{jl}(-g_j^0(\hat{x}^0)) \neq \emptyset$, $C_{jl}(-g_j^0(\tilde{x}^0)) \neq \emptyset$ and therefore,

$$\begin{aligned}(C_{jl}(-g_j^0(\hat{x}^0)))_\infty &= \{x^j \in \mathbb{R}^{n_j} : g_{jl}(x^j) \leq -g_j^0(\hat{x}^0)\}_\infty = \{x^j \in \mathbb{R}^{n_j} : (g_{jl})_\infty(x^j) \leq 0\}_\infty \\ &= \{x^j \in \mathbb{R}^{n_j} : g_{jl}(x^j) \leq -g_j^0(\tilde{x}^0)\}_\infty = (C_{jl}(-g_j^0(\tilde{x}^0)))_\infty\end{aligned}$$

for all $l \in \{1, \cdots, m_j\}$. Analogously, $(C_{jl}(\hat{u}^j))_\infty = (C_{jl}(\tilde{u}^j))_\infty$ for $l \in \{m_j+1, \cdots, m_j+m_0\}$ holds and finally, the equation (iii) follows by use of (ii). □

Theorem 2 follows obviously from Lemma 4.

Theorem 2 *Let* $(x^0, u) \in \operatorname{dom} \Phi_j$. *If there exists a point* $(\bar{x}^0, \bar{u}) \in \operatorname{dom} \Phi_j$ *such that the feasible set* $S^j(\bar{x}^0, \bar{u})$ *of subproblem* $\mathrm{P}^j(\bar{x}^0, \bar{u})$ *is bounded, then* $S^j(x^0, u)$ *is a bounded set too. Moreover, the corresponding set* $\widetilde{S}^j(x^0, u)$ *is nonempty.*

Clearly, either $S^j(x^0, u)$ is unbounded or $S^j(x^0, u)$ is bounded for all $(x^0, u) \in \operatorname{dom} \Phi_j$. In the second case the optimal sets of the subproblems are nonempty, e.g. the optimal value $\Phi_j(x^0, u)$ is attained at a feasible point $x^j \in S^j(x^0, u)$ for all $(x^0, u) \in \operatorname{dom} \Phi_j$. In the following we formulate some statements in the other case. We consider some properties of the level sets of the convex, proper and lower semicontinuous function f_j.
Obviously, the optimal set of subproblem $\mathrm{P}^j(x^0, u)$ is the intersection of the feasible set of $\mathrm{P}^j(x^0, u)$ and a closed convex level set of f_j, i.e.

$$\widetilde{S}^j(x^0, u) = S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \Phi_j(x^0, u)\} \text{ for } (x^0, u) \in \operatorname{dom} \Phi_j.$$

Lemma 5 *Let* $(x^0, u) \in \operatorname{dom}\Phi_j$ *and* $(\bar{x}^0, \bar{u}) \in \operatorname{dom}\Phi_j$ *with* $\widetilde{S}^j(\bar{x}^0, \bar{u}) \neq \emptyset$. *Choose* $\alpha \in \mathbb{R}$ *with* $S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\} \neq \emptyset$.
Then $(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty = (S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\})_\infty$ *holds.*

Proof: Because of the nonemptiness of $\widetilde{S}^j(\bar{x}^0, \bar{u})$, Lemma 4 and the above mentioned properties of feasible sets, level sets and recession cones it follows

$$\begin{aligned}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty &= (S^j(\bar{x}^0, \bar{u}))_\infty \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \Phi_j(\bar{x}^0, \bar{u})\}_\infty \\ &= (S^j(x^0, u))_\infty \cap \{x^j \in \mathbb{R}^{n_j} : (f_j)_\infty(x^j) \leq 0\} \\ &= (S^j(x^0, u))_\infty \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}_\infty \\ &= (S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\})_\infty.\end{aligned}$$

□

Remark: Let $(x^0, u), (\bar{x}^0, \bar{u}) \in \operatorname{dom}\Phi_j$ with $\widetilde{S}^j(x^0, u) \neq \emptyset$ and $\widetilde{S}^j(\bar{x}^0, \bar{u}) \neq \emptyset$, then the equation $(\widetilde{S}^j(x^0, u))_\infty = (\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty$ holds.

Theorem 3 *If there exists a point* $(\bar{x}^0, \bar{u}) \in \operatorname{dom}\Phi_j$ *such that the set* $\widetilde{S}^j(\bar{x}^0, \bar{u})$ *is nonempty and bounded then* $\widetilde{S}^j(x^0, u)$ *is nonempty and bounded for all* $(x^0, u) \in \operatorname{dom}\Phi_j$.

Proof: Let $(x^0, u) \in \operatorname{dom}\Phi_j$ be arbitrarily chosen.
There exists $\alpha \in \mathbb{R}$ with $\Phi_j(x^0, u) = \inf_{x^j}\{f_j(x^j) : x^j \in S^j(x^0, u),\ f_j(x^j) \leq \alpha\}$. Obviously, $S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ is nonempty and closed. Due to the assumptions of Theorem 3 and Lemma 5 this intersection is bounded. The lower semicontinuous function f_j attains its infimum on this compact set. □

Theorems 2 and 3 show that the existence of at least one point $(\bar{x}^0, \bar{u}) \in \operatorname{dom}\Phi_j$ with a bounded feasible or a bounded nonempty optimal set of the subproblem $\mathrm{P}^j(\bar{x}^0, \bar{u})$ guarantees that the optimal value $\Phi_j(x^0, u)$ is attained at some feasible point $x^j \in S^j(x^0, u)$ for any $(x^0, u) \in \operatorname{dom}\Phi_j$. Because of this property we also get the following theorem.

Theorem 4 *If for* (1.1) *the optimal set is nonempty and bounded then the optimal value* $\Phi_i(x^0, u)$ *of the subproblem* $\mathrm{P}^i(x^0, u)$ *is attained for all* $(x^0, u) \in \operatorname{dom}\Phi_i$ *and* $i \in \{1, \ldots, k\}$.

Proof: Let $(x^{0\star}, x^\star)$ be an optimal solution of (1.1) and $u^{i\star} := h_i(x^{i\star})$ for $i \in \{1, \ldots, k\}$.
Therefore, $(x^{0\star}, u^\star) \in S_2$, $x^{i\star} \in S^i(x^{0\star}, u^\star)$ for $i \in \{1, \ldots, k\}$,

$$\sum_{i=0}^{k} f_i(x^{i\star}) = f(x^{0\star}, x^\star) = f^\star = \Phi^\star \leq \Phi(x^{0\star}, u^\star) = \sum_{i=0}^{k} \Phi_i(x^{0\star}, u^\star)$$

and $\Phi_i(x^{0\star}, u^\star) \leq f_i(x^{i\star})$ for $i \in \{0, 1, \ldots, k\}$. Thus, $\Phi_i(x^{0\star}, u^\star) = f_i(x^{i\star})$ for $i \in \{0, 1, \ldots, k\}$ and $x^{i\star} \in \widetilde{S}^i(x^{0\star}, u^\star)$ for $i \in \{1, \ldots, k\}$.
Let $j \in \{1, \ldots, k\}$ be arbitrarily fixed.
Because of $\widetilde{S}^j(x^{0\star}, u^\star) = \{x^j \in \mathbf{X}^j : f_j(x^j) = f_j(x^{j\star}),\ g_j(x^j) + g_j^0(x^{0\star}) \leq \mathbf{0},\ h_j(x^j) \leq u^{j\star}\}$ a point $(x^{0\star}, \tilde{x})$ is an element of the optimal set of (1.1), if $\tilde{x}$ is defined by $\tilde{x}^j \in \widetilde{S}^j(x^{0\star}, u^\star)$ and $\tilde{x}^i := x^{i\star}$ for $i \neq j$. The set

$$\{x^{0\star}\} \times \{x^{1\star}\} \times \ldots \times (\widetilde{S}^j(x^{0\star}, u^\star)) \times \ldots \times \{x^{k\star}\}$$

is a subset of the bounded optimal set of (1.1) and hence, $\widetilde{S}^j(x^{0\star}, u^\star)$ is bounded, too. The assertion follows in combination with Theorem 3. □

Consider the case that there is no point $(x^0, u) \in \operatorname{dom} \Phi_j$ with a bounded set $\widetilde{S}^j(x^0, u)$. With an additional assumption concerning the interior of the recession cone of $\widetilde{S}^j(x^0, u)$ we conclude the Theorem 5.

Lemma 6 *Let $(\bar{x}^0, \bar{u}) \in \operatorname{dom} \Phi_j$ be a point with the property $\operatorname{int}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty \neq \emptyset$. Furthermore, let $(x^0, u) \in \operatorname{dom} \Phi_j$ and choose $\alpha \in \mathbb{R}$ such that the intersection of the feasible set $S^j(x^0, u)$ and the level set $\{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ is nonempty.*

(i) *Then there exists $\tilde{x}^j \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ such that $\tilde{x}^j$ is an optimal solution of the subproblem $\mathrm{P}^j(\bar{x}^0, \bar{u})$, e.g. $\tilde{x}^j \in \widetilde{S}^j(\bar{x}^0, \bar{u})$.*

(ii) *The inequality $f_j(x^j) \geq \Phi_j(\bar{x}^0, \bar{u})$ holds for all elements x^j of the intersection $S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$.*

Proof: Let $x^{j1} \in \widetilde{S}^j(\bar{x}^0, \bar{u})$ and $x^{j2} \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ be arbitrarily chosen. Clearly, in the case $x^{j1} = x^{j2}$ the assertion (i) follows and the inequality of (ii) is true for x^{j2}.
Consider the case $x^{j1} \neq x^{j2}$. For an arbitrary $y^1 \in \operatorname{int}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty$ it holds

$$x^{j1} + \lambda y^1 \in \widetilde{S}^j(\bar{x}^0, \bar{u}) \text{ and } x^{j2} + \lambda y^1 \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\} \text{ for all } \lambda \geq 0$$

because of Lemma 5.
Furthermore, there exists $y^2 \in \operatorname{int}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty$ ($y^1 = y^2$ is possible) with a nonempty intersection of the semi-infinite intervals $x^{j1} + \lambda_2 y^2$, $\lambda_2 \geq 0$ and $x^{j2} + \lambda_1 y^1$, $\lambda_1 \geq 0$. An element of this intersection is denoted by $\tilde{x}^j$. Thus, there exist $\tilde{\lambda}_1$, $\tilde{\lambda}_2 \geq 0$ such that

$$\tilde{x}^j = x^{j1} + \tilde{\lambda}_2 y^2 = x^{j2} + \tilde{\lambda}_1 y^1.$$

W.l.o.g. let $\tilde{\lambda}_1$ be a positive number.
From $y^2 \in \operatorname{int}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty$ it follows $x^{j1} + \lambda y^2 \in \widetilde{S}^j(\bar{x}^0, \bar{u})$ for all $\lambda \geq 0$ and therefore, $\tilde{x}^j \in \widetilde{S}^j(\bar{x}^0, \bar{u})$. Moreover, $\tilde{x}^j$ is an element of $S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ because of $\tilde{x}^j = x^{j2} + \tilde{\lambda}_1 y^1$. The assertion (i) is proved.
To prove the claimed inequality consider the three elements x^{j2}, $\tilde{x}^j$ and $\tilde{x}^j + y^1$ of the intersection $S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$.
Due to $\tilde{\lambda}_1 > 0$ and $\tilde{x}^j = x^{j2} + \tilde{\lambda}_1 y^1$ it follows $\tilde{x}^j \neq x^{j2}$ and $\tilde{x}^j + y^1 = x^{j2} + (1 + \tilde{\lambda}_1) y^1$.
Hence, with $\beta := \frac{1}{1 + \tilde{\lambda}_1} \in (0, 1)$

$$\beta x^{j2} + (1 - \beta)(\tilde{x}^j + y^1) = \tilde{x}^j,$$

i.e. the point $\tilde{x}^j$ is a convex linear combination of the two other points.
The function f_j is convex on $S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\} \subseteq \mathbb{R}^{n_j}$ and thus,

$$f_j(\tilde{x}^j) \leq \beta f_j(x^{j2}) + (1 - \beta) f_j(\tilde{x}^j + y^1).$$

Note, that $\tilde{x}^j$, $\tilde{x}^j + y^1 \in \widetilde{S}^j(\bar{x}^0, \bar{u})$. The inequality $\Phi_j(\bar{x}^0, \bar{u}) \leq f_j(x^{j2})$ is true and the statement (ii) of Lemma 6 is proved because $x^{j2} \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ is arbitrarily chosen. □

Theorem 5 *If there exists a point $(\bar{x}^0, \bar{u}) \in \operatorname{dom} \Phi_j$ such that $\operatorname{int}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty$ is nonempty then $\widetilde{S}^j(x^0, u)$ is nonempty for all $(x^0, u) \in \operatorname{dom} \Phi_j$.*
Moreover, the equation $\Phi_j(x^0, u) = \Phi_j(\hat{x}^0, \hat{u})$ holds for (x^0, u), $(\hat{x}^0, \hat{u}) \in \operatorname{dom} \Phi_j$.

Proof: Let (x^0, u) be an element of $\operatorname{dom}\Phi_j$. According to Lemma 6 there exists a point $\tilde{x}^j \in \widetilde{S}^j(\bar{x}^0, \bar{u})$ with $\tilde{x}^j \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$ for a number $\alpha \in \mathbb{R}$. It holds

$$f_j(x^j) \geq \Phi_j(\bar{x}^0, \bar{u}) = f_j(\tilde{x}^j) \geq \Phi_j(x^0, u)$$

for $x^j \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}$. Furthermore,

$$\Phi_j(x^0, u) = \inf_{x^j}\{f_j(x^j) : x^j \in S^j(x^0, u) \cap \{x^j \in \mathbb{R}^{n_j} : f_j(x^j) \leq \alpha\}\} \geq \Phi_j(\bar{x}^0, \bar{u}).$$

Thus, $\Phi_j(x^0, u) = \Phi_j(\bar{x}^0, \bar{u})$ holds and $\tilde{x}^j$ is an element of $\widetilde{S}^j(x^0, u)$. The assertions of the theorem follow because (x^0, u) is arbitrarily chosen. □

The case considered in the Theorem 5 ensures that all subproblems $\mathrm{P}^j(x^0, u)$ have the same optimal value $\Phi_j(x^0, u) = \textit{constant}$ and this value is always attained for some feasible point $x^j \in S^j(x^0, u)$, if $(x^0, u) \in \operatorname{dom}\Phi_j$. Because of the above remark with the notations of Theorem 5 it follows $(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty = (\widetilde{S}^j(x^0, u))_\infty$ and hence, $\operatorname{int}(\widetilde{S}^j(\bar{x}^0, \bar{u}))_\infty = \operatorname{int}(\widetilde{S}^j(x^0, u))_\infty$.

Remark: From Lemma 2, Proposition 1 and the properties of linear programming problems it follows that $\widetilde{S}^i(x^0, u)$ is nonempty for any point $(x^0, u) \in \operatorname{dom}\Phi_i$ for $i \in \{1, \ldots, k\}$ if the program (1.1) is a weakly solvable linear program.

Lemma 7 *Let* (1.2) *be strongly solvable. Then* (1.1) *is also strongly solvable if there exists* $x^i \in S^i(x^0, u)$ *with* $f_i(x^i) = \Phi_i(x^0, u)$, *i.e.* $\widetilde{S}^i(x^0, u) \neq \emptyset$, *for all* $(x^0, u) \in \operatorname{dom}\Phi_i$ *and for all* $i \in \{1, \ldots, k\}$.

Proof: Let $(x^{0\star}, u^\star)$ be an optimal solution of (1.2). There exists $x^{i\star} \in \widetilde{S}^i(x^{0\star}, u^\star)$ for all $i \in \{1, \ldots, k\}$. Thus, $(x^{0\star}, x^\star) \in S_1$. In combination with Theorem 1 it follows

$$f^\star = \Phi^\star = \sum_{i=0}^{k} \Phi_i(x^{0\star}, u^\star) = \sum_{i=0}^{k} f_i(x^{i\star})$$

and (1.1) is strongly solvable. □

Theorem 6 *Let* (1.2) *be strongly solvable. If the assumptions of Theorem 2, 3 or 5 are fulfilled for* $j \in \{1, \ldots, k\}$ *then* (1.1) *is strongly solvable, too.*

This assertion obviously follows from Lemma 7.

The theorems of this section are useful for a more clear description of the optimal sets of the subproblems which result from the decomposition approach suggested in section 1. In this way it was possible to formulate a converse of Theorem 1(vi). With this statement about the strong solvability of the bordering structured programming problem (1.1), the equivalence of (1.1) and the decomposed problem (1.2) could be described in more detail.
Although not all cases could be studied completely. The question remains open whether the existence of some subproblem with an unbounded optimal set with an empty interior of its recession cone allows conclusions about the nonemptiness of optimal sets of other subproblems. Do some conditions for the remaining case exist, which ensure statements analogous Theorem 2, 3 respectively 5 and finally, are generalizations of Theorem 4 respectively 6 possible?
The subject of further considerations could also be answering the question which special programming problems fulfil the conditions of the mentioned theorems. It could be possible to use properties of special function classes [1], [2].

At the end it should be mentioned again that a weakly solvable linear problem (1.1) possesses only subproblems $P^i(x^0, u)$ of the corresponding decomposed problem (1.2) with nonempty optimal sets, if $(x^0, u) \in \operatorname{dom} \Phi_i$ and $i \in \{1, \ldots, k\}$ clearly. With other words, in this case the converse of Theorem 1(vi) is true without any additional assumption concerning the existence of at least one subproblem $P^i(x^0, u)$ with a nonempty solution set for $i \in \{1, \ldots, k\}$.

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