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Abstract. Looking for a new optimal location point for m given location points such that the sum of the distances between the new location point and the given location points becomes minimal is called 1-location median problem. If even n new location points are searched then it arises the n-location median problem.
In this work this n-location median problem with restrictions is investigated. Powers of several gauges are chosen as distance functions. The considerations happen in Hausdorff locally convex topological real vector spaces.
This generalized location problem can also be interpreted as a control-approximation problem with m state variables and n control variables.
After the formulation of the primal location or control-approximation problem some considerations to the gauges are performed.
Then a dual problem is given. As a relation between the primal and the dual problem a weak duality assertion follows. With the help of the duality theory of Fenchel and Rockafellar a strong duality assertion can also be derived.

Keywords: *location problem, control-approximation problem, gauge, duality*

AMS subject classification: Primary 49N15, 90B85, secondary 41A65, 90C25.

1. Introduction

In this paper convex programming problems of the type of the so-called control-approximation problems with respect to Hausdorff locally convex topological real vector spaces and with several control and state variables are considered. So m state variables $a_1, \ldots, a_m$ and n control variables $x_1, \ldots, x_n$ should be considered.

As typical for control-approximation problems there are measured the distances between the control and the state variables. Location problems can be considered as special cases of such problems (cf. [5]). Here additionally distances between the control variables themselves are included into the objective function which represents in general a function of these distances and has to be minimized. In location theory corresponding problems are refered to multifacility location. There are also problems which contain the optimization concerning the largest distance, e.g. the obnoxious facility problems (cf.[1]).

The starting point of this paper is to formulate a general control-approximation problem and multifacility location problem (in a specialized version) respectively by means of gauges instead of norms for measuring the distances as this done for classical

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location and control-approximation problems in normed spaces. And because the topic is duality there is derived a dual optimization problem as well as weak and strong duality assertions are started. Further it is remarked that this strong duality relation can be used to establish optimality conditions for the control-approximation problem or multifacility location problem (in a specialized version). Before the beginning of these investigations it is recalled the formulation of the classical multifacility problem and it is also given a vectorial formulation.

At that for the further handling two kinds of location problems follow. Let be given m location points $a_i, i = 1, \ldots, m,$ in a suitable space X. Then n new location points $\overset{\circ}{x}_j \in X, j = 1, \ldots, n,$ should be established, such that

(1) the sum of the distances between the given and the new location points and between the new location points among themselves is minimized, or

(2) the distances between the given and the new location points and between the new location points among themselves are minimized simultaneously.

So the given variables $a_i, i = 1, \ldots, m,$ should be approximated by the new variables $\overset{\circ}{x}_j \in X, j = 1, \ldots, n.$ The problem (1) is called the n-location median problem, in the literature it is also often named the multifacility problem. The problem (2) is the multifacility problem as a multiobjective problem for m given location points and n new location points.

If the considered variables are elements of a Banach space B or generally of a normed space and the distance functions are weighted norms, then problem (1) runs as follows:

$$\begin{aligned} \min_{(x_1,\ldots,x_n)\in B} \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\|x_j - a_i\| + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n} \tilde{\lambda}_{lj}\|x_l - x_j\| \\ = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}\|\overset{\circ}{x}_j - a_i\| + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n} \tilde{\lambda}_{lj}\|\overset{\circ}{x}_l - \overset{\circ}{x}_j\| \end{aligned} \tag{1.1}$$

with $\lambda_{ij} > 0\,; i = 1, \ldots, m\,, j = 1, \ldots, n;$ and $\tilde{\lambda}_{lj} > 0\,; 1 \le l < j \le n.$ In [3] and [5] problem (1.1) is investigated under special assumptions with respect to the considered spaces, e.g. for a finite dimensional space.

With weighted norms as distance functions problem (2) runs as follows in the Banach space or normed space B:

$$\begin{pmatrix} \lambda_{11}\|x_1 - a_1\| \\ \lambda_{12}\|x_2 - a_1\| \\ \lambda_{13}\|x_3 - a_1\| \\ \vdots \\ \lambda_{mn}\|x_n - a_m\| \\ \tilde{\lambda}_{12}\|x_1 - x_2\| \\ \tilde{\lambda}_{13}\|x_1 - x_3\| \\ \vdots \\ \tilde{\lambda}_{n-1,n}\|x_{n-1} - x_n\| \end{pmatrix} \longrightarrow \underset{(x_1,\ldots,x_n)\in B}{\text{v} - \min} \tag{1.2}$$

with $\lambda_{ij} > 0\,;\, i = 1,\ldots,m,\ j = 1,\ldots,n;$ and $\tilde{\lambda}_{lj} > 0\,;\, 1 \le l < j \le n$. Here v-min is a symbolic notation meaning to observe a vector minimum problem. In [12] problem (1.2) is investigated for $\lambda_{ij} = 1\,;\, i = 1,\ldots,m,\ j = 1,\ldots,n;$ and $\tilde{\lambda}_{lj} = 1\,;\, 1 \le l < j \le n$.

Now in this work merely the scalar location problem (n-location median problem) is investigated and for this the location problem (1.1) for the establishing of n new location points which can also be interpreted as control-approximation problems is generalized.

The variables $x_j, j = 1,\ldots,n;\ a_i, i = 1,\ldots,m$, are elements of different Hausdorff locally convex topological real vector spaces. Because of the inducing of the locally convex topology by seminorms in such spaces there are seminorms in any case, and they do not have to be norms. But the distance functions should be more general than norms or seminorms. Here different gauges which can be seminorms in a special case are to be chosen and as distance functions powers of these non-negatively weighted gauges are used. At the same time the exponents of the powers shall not be smaller than one. To attach to each different pair of variables a distance the variables of the pair are mapped by appropriate control operators in a common space. Some distances between the location points are perturbed by a linear and continuous functional. Additionally a set of restrictions is presented. From it the admissible location points are elected. The set contains cone and subset restrictions and its elements must fulfil operator inequalities.

This location problem or control-approximation problem is exactly formulated in section 2. Remarks to gauges are performed in section 3. A dual problem to the primal problem of section 2 is formulated in section 4. Section 5 contains weak and strong duality theorems for the dual problem. Section 6 finishes with summary and conclusions.

Duality statements for special cases of the considered generalized location problem or control-approximation problem are treated in [6]. At that norms are elected as distance functions and the spaces are normed spaces. Duality for generalized location problems in reflexive Banach spaces with norms as distance functions and with restrictions is considered in [8]. Duality statements for location problems in reflexive Banach spaces with restrictions and with gauges as distance functions are investigated in [7], but there the location problems are multiobjective location problems. And in [8] and [7] all investigations happen without considerations of the distances between new location points among themselves.

At the end of the introduction a few remarks are made about symbols, notations and definitions. The set of the real numbers is abbreviated by $\mathbb{R}$; the extended set of real numbers is $\overline{\mathbb{R}}, \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. If P is a linear and continuous operator mapping the space G into the space H, then it is denoted by $P \in L(G \to H)$. The space H' is the algebraic dual space to H, the space of the linear functionals on the space H and the space H^* is the topological dual space to H, the space of the linear and continuous functionals on the space H; it holds $H^* \subseteq H'$. A cone K in a real vector space H is a subset from H, $K \subseteq H$, with the property $\beta k \in K \; \forall k \in K$ and $\forall \beta \in \mathbb{R}_+$. For any convex cone K a partial ordering is defined by $x \preceq y :\Longleftrightarrow x \underset{K}{\leqq} y :\Longleftrightarrow (y - x) \in K$. The dual cone K^* to a cone K is defined by $K^* := \{h^* \in H^* \mid \langle h^*, h\rangle \geq 0 \; \forall h \in K \subseteq H\}$. A subset N of a real vector space H is called absorbent if, for each $h \in H$, there exists an α such that $[0, \alpha] \cdot h \subseteq N$. If for a subset N of a real vector space H it is valid $\lambda N \subseteq N \; \forall \lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ then the set N is named circled. Let be r a number with $r \in (0, 1]$ then a subset N of a real vector space H is said to be an r-convex set if it satisfies $\lambda N + \mu N \subseteq N \; \forall \lambda, \mu \geq 0$ such that $\lambda^r + \mu^r = 1$.

2. The control-approximation problem

It is given a certain number of different Hausdorff locally convex topological real vector spaces $V_j, j = 1, \ldots, n$; $X_j, j = 1, \ldots n$; $Y_i, i = 1, \ldots, m$, and $Z_{ij}, i = 1, \ldots, m, j = 1 \ldots n$. In each space Y_i an element a_i which can also be interpreted as a location point or as a state variable is considered. In each space X_j an element x_j which can also be interpreted as a location point or as a control variable is searched. Each pair $(a_i, x_j), i = 1, \ldots, m, j = 1 \ldots n$, and each pair $(x_l, x_j), 1 \leq l < j \leq n$ is associated with a distance by means of a belonging distance function. Then it is looked for the infimum of the objective function which consists of different distances among certain restrictions. The control-approximation problem or multifacility location in the case of all occuring control operators $S_{ji}, i = 1, \ldots, m, j = 1, \ldots, n$, and $T_{lj}, 1 \leq l < j \leq n$, turn out to be the identity mapping is given by:

$$S(x, a) \longrightarrow \inf_{(x,a,v) \in M} \qquad (P)$$

with

$$S(x,a) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\lambda_{ij}{}^{\alpha_{ij}} [\gamma_{ij} (S_{ji} x_j - a_i)]^{\alpha_{ij}} + \langle l_{ij}{}^*, x_j \rangle\right) +$$
$$\sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\lambda}_{lj}{}^{\tilde{\alpha}_{lj}} [\tilde{\gamma}_{lj} (T_{lj} x_l - x_j)]^{\tilde{\alpha}_{lj}}.$$

The set M is defined through

$$M := \left\{ (x, a, v) \in (X, Y, V) \;\middle|\; \begin{array}{l} a_i \in W_i, i = 1, \ldots, m;\; x_j \underset{K_{X_j}}{\geqq} 0, v_j \underset{K_{V_j}}{\geqq} 0, j = \\ 1, \ldots, n;\; A_{ij} a_i + B_{ij} x_j + C_{ij} v_j + f_{ij} \underset{K_{Z_{ij}}}{\leqq} 0, \\ i = 1, \ldots, m, j = 1, \ldots, n \end{array} \right\}.$$

At that it holds:

- $x = (x_1, \ldots, x_n)^{\mathrm{T}}, a = (a_1, \ldots, a_m)^{\mathrm{T}}, v = (v_1, \ldots, v_n)^{\mathrm{T}}$;
- $(X, Y, V) = (X_1, \ldots, X_n; Y_1, \ldots, Y_m; V_1, \ldots, V_n)$;
- γ_{ij} is a gauge in the space $Y_i, i = 1, \ldots, m, j = 1, \ldots, n$, and $\tilde{\gamma}_{lj}$ is a gauge in the space $X_j, 1 \leq l < j \leq n$;
- $\lambda_{ij} \geq 0, i = 1, \ldots, m, j = 1, \ldots, n$; $\tilde{\lambda}_{lj} \geq 0, 1 \leq l < j \leq n$;
- $\alpha_{ij} \geq 1, i = 1, \ldots, m, j = 1, \ldots, n$; $\tilde{\alpha}_{lj} \geq 1, 1 \leq i < j \leq n$;
- $S_{ji} \in L(X_j \to Y_i), i = 1, \ldots, m, j = 1, \ldots, n$; $T_{lj} \in L(X_l \to X_j)$, $1 \leq l < j \leq n$;
- $l_{ij}{}^* \in X_j{}^*, i = 1, \ldots, m, j = 1, \ldots, n$, and $\langle \cdot, \cdot \rangle : X_j^* \times X_j \to \mathbb{R}, j = 1, \ldots, n$, is a bilinear form, the duality pairing between X_j^* and X_j;
- $W_i \subseteq Y_i$ is convex, $i = 1, \ldots, m$; $K_{X_j} \subseteq X_j, K_{V_j} \subseteq V_j, j = 1, \ldots, n; K_{Z_{ij}} \subseteq Z_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$ are closed and convex cones;
- $A_{ij} \in L(Y_i \to Z_{ij}), B_{ij} \in L(X_j \to Z_{ij}), C_{ij} \in L(V_j \to Z_{ij}), f_{ij} \in Z_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$.

3. Gauges

The distance functions $\gamma_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$, and $\tilde{\gamma}_{lj}, 1 \leq l < j \leq n$, in (P) are different gauges. Now a few remarks to the introduction of gauges with using assertions from [4] follow.

Let be H a real vector space and G a nonempty subset of H, $G \subseteq H$. The functional: γ_G: $H \longrightarrow \mathbb{R}_+$ with

$$\gamma_G(h) = \begin{cases} \infty & \text{for} \quad \{\lambda > 0 \,|\, h \in \lambda G\} = \emptyset, \\ \inf \{\lambda > 0 \,|\, h \in \lambda G\} & \text{else} \end{cases}$$

is called Minkowski functional of the set G.

Now it is defined $G_E := [0, 1] \cdot G$ (It holds $G = G_E$ if G is circled or G is r-convex, $r \in (0, 1]$ with $0 \in G$.) and G is assumed as absorbent. Then the functional

$$\gamma_G(h) = \inf \{\lambda > 0 \,|\, h \in \lambda G_E\} \tag{3.1}$$

is well-defined, that means $\mathrm{dom}(\gamma_G) = H$ and γ_G is continuous .

Henceforth the subset G is specified. The set G is absorbent and r-convex, $r \in (0, 1]$, i.e. $G = G_E$. Then according to definition (3.1) it holds for γ_G:

$$\begin{aligned} \gamma_G(h) &\geq 0 \;\; \forall h \in H, & (3.2) \\ \gamma_G(0) &= 0, & (3.3) \\ \mu\gamma_G(h) &= \gamma_G(\mu h) \;\; \forall \mu \in \mathbb{R}_+, \; \forall h \in H, & (3.4) \\ [\gamma_G(h_1 + h_2)]^r &\leq [\gamma_G(h_1)]^r + [\gamma_G(h_2)]^r \;\; \forall h_1, h_2 \in H, r \in (0, 1]. & (3.5) \end{aligned}$$

Here γ_G is said to be a r-gauge. If additionally the property

$$\gamma_G(\mu h) = |\mu| \; \gamma_G(h) \;\; \forall \mu \in \mathbb{R} \; \forall h \in H$$

is fulfilled, that happens if the set G is extra circled, i.e. the set is on the whole absorbent absolutely r-convex, then the r-gauge is named r-seminorm. Finally the r-seminorm becomes with

$$\gamma_G(h) = 0 \Longrightarrow h = 0$$

an r-norm. For $r = 1$ gauge, seminorm and norm are simply used instead of 1-gauge, 1-seminorm and 1-norm. An example for a proper gauge γ_G that is a gauge which is not a norm and not a seminorm is given with the set $G := \left\{ (x,y)^{\mathrm{T}} \in \mathbb{R}^2 \,|x \in [-1,2]\,, y \in \mathbb{R} \right\}$. The gauge γ_G in $H = \mathbb{R}^2$ is then defined with (3.1) by the help of G.

Here gauges, that means functionals with the properties (3.1) - (3.5) with $r = 1$ in (3.5), are only considered until to the end of this work. For the set G it holds then:

$$\{h \in H \,|\gamma_G(h) < 1\} \subseteq G \subseteq \{h \in H \,|\gamma_G(h) \le 1\}. \tag{3.6}$$

If the set G is even closed then (3.6) becomes

$$G = \{h \in H \,|\gamma_G(h) \le 1\}.$$

Now the dual gauge $\gamma_G{}^*$ to the gauge γ_G in the algebraic dual space H' is introduced by means of the polar G° of the set G. With the bilinear form $\langle \cdot,\cdot \rangle : H' \times H \longrightarrow \mathbb{R}$ the definition of G° is:

$$G^\circ := \left\{ h^* \in H' \;\middle|\; \sup_{h \in G} \langle h^*, h \rangle \le 1 \right\}.$$

And the dual gauge is:

$$\gamma_G{}^*(h^*) := \sup_{h \in G} \langle h^*, h \rangle. \tag{3.7}$$

It holds the generalized Cauchy-Schwarz inequality in the following manner for gauges:

$$\langle h^*, h \rangle \le \gamma_G(h) \cdot \gamma_G{}^*(h^*) \;\; \forall\, h \in H\,,\; \forall\, h^* \in H'. \tag{3.8}$$

So the gauges γ_G and $\gamma_G{}^*$ which are dual to each other can also be given by:

$$\gamma_G(h) = \sup_{h^* \in G^\circ} \langle h^*, h \rangle,$$
$$\gamma_G{}^*(h^*) = \inf \{\lambda > 0 \,|h^* \in \lambda G^\circ\}.$$

The different gauges γ_{ij} in the space Y_i, $i = 1,\ldots,m\,, j = 1,\ldots,n$, in the problem (P) are formed through the election of different absorbent and convex sets $G_{ij} \subset Y_i$, $i = 1,\ldots,m\,, j = 1,\ldots,n$, so that (3.1) - (3.5) with $r = 1$ are fulfilled. The belonging dual gauges $\gamma_{ij}{}^*$, $i = 1,\ldots,m\,, j = 1,\ldots,n$, are built with definition (3.7). It is handled in an analogous way for the introduction of the gauges $\tilde{\gamma}_{lj}$ in the space X_j, $1 \le l < j \le n$, and its corresponding dual gauges $\tilde{\gamma}^*_{lj}$.

4. The dual problem

Similar to the investigations in [8] and following the Fenchel-Rockafellar approach of duality by means of the perturbation of the given optimization problem a dual control-approximation or location problem can be associated to (P). Thereby the derivation in [8] must be changed in the following way.
A perturbation function Φ is introduced by

$$\Phi(x,a,v,p,q) :=$$

$$\begin{cases} \sum_{i=1}^{m}\sum_{j=1}^{n}(\lambda_{ij}{}^{\alpha_{ij}}[\gamma_{ij}(S_{ji}x_j - a_i + p_{ij})]^{\alpha_{ij}} + \langle l_{ij}{}^*, x_j\rangle) + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\tilde{\lambda}_{lj}{}^{\tilde{\alpha}_{lj}}[\tilde{\gamma}_{lj}(T_{lj}x_l - x_j + \tilde{p}_{lj})]^{\tilde{\alpha}_{lj}} & \text{for } \begin{cases} A_{ij}a_i + B_{ij}x_j + C_{ij}v_j + f_{ij} \underset{K_{Z_{ij}}}{\leq} q_{ij},\ i=1,\dots,m, \\ j=1,\dots,n; \\ a_i \in W_i,\ i=1,\dots,m; \\ x_j \underset{K_{X_j}}{\geq} 0,\ v_j \underset{K_{V_j}}{\geq} 0, \\ j=1,\dots,n; \end{cases} \\ \infty & \text{else} \end{cases}$$

and $N(p,q) = \inf\limits_{(x,a,v)\in(X,Y,V)} \Phi(x,a,v,p,q)$.
Here it is $p = (p_{11}, p_{12}, p_{13}, \dots, p_{mn}, \tilde{p}_{12}, \tilde{p}_{13}, \dots, \tilde{p}_{n-1,n})$, $q = (q_{11}, q_{12}, \dots, q_{mn})$; $p_{ij} \in Y_i$, $q_{ij} \in Z_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$; $\tilde{p}_{lj} \in X_j$, $1 \leq l < j \leq n$. It is abbreviated by $(p,q) \in \left(\left[Y_n, \tilde{X}\right], Z\right)$. Then it holds

$$N(0,0) = \inf_{(x,a,v)\in M} S(x,a) = \inf(P).$$

It remains to determine

$$\sup(P^*) = \sup_{(p^*,q^*)\in([Y_n^*,\tilde{X}^*],Z^*)} [-\Phi^*(0,0,0,p^*,q^*)].$$

Here the conjugate function to $\Phi : (x,a,v,p,q) \in \left(X,Y,V,\left[Y_n,\tilde{X}\right],Z\right) \longmapsto \Phi(x,a,v,p,q) \in \overline{\mathbb{R}}$ is denoted by $\Phi^* : (x^*,a^*,v^*,p^*,q^*) \in \left(X^*,Y^*,V^*,\left[Y_n^*,\tilde{X}^*\right],Z^*\right) \longmapsto \Phi^*(x^*,a^*,v^*,p^*,q^*) \in \overline{\mathbb{R}}$ and defined by

$$\Phi^*(x^*,a^*,v^*,p^*,q^*) := \sup_{(x,a,v,p,q)\,\in\,(X,Y,V,[Y_n,\tilde{X}],Z)} \Bigg\{ \sum_{j=1}^{n}(\langle x_j{}^*, x_j\rangle + \langle v_j{}^*, v_j\rangle) + \sum_{i=1}^{m}\langle a_i{}^*, a_i\rangle + \sum_{i=1}^{m}\sum_{j=1}^{n}(\langle p_{ij}{}^*, p_{ij}\rangle + \langle q_{ij}{}^*, q_{ij}\rangle) + \sum_{l=1}^{n-1}\sum_{j=l+1}^{n}\langle \tilde{p}_{lj}{}^*, p_{lj}\rangle - \Phi(x,a,v,p,q) \Bigg\}.$$

At that $\left(X^*, Y^*, V^*, \left[Y_n^*, \tilde{X}^*\right], Z^*\right)$ is a collection of topological dual spaces to the collection of topological spaces $\left(X, Y, V, \left[Y_n, \tilde{X}\right], Z\right)$.

Then the result of the calculation of the above supremum (cf. [8] for analogous calculations with norms as distances) is the dual problem

$$R(p^*, q^*) \longrightarrow \sup_{(p^*, q^*) \in M^*} \qquad (P^*)$$

with

$$\begin{aligned} R(p^*, q^*) = & \sum_{\substack{i=1 \\ \alpha_{ij}>1}}^{m} \sum_{j=1}^{n} (1-\alpha_{ij}) \left[\gamma_{ij}^* (p_{ij}^*)\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} + \\ & \sum_{\substack{l=1 \\ \tilde{\alpha}_{lj}>1}}^{n-1} \sum_{j=l+1}^{n} (1-\tilde{\alpha}_{lj}) \left[\tilde{\gamma}_{lj}^* \left(\tilde{p}_{lj}^*\right)\right]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}} - \\ & \sum_{i=1}^{m} \sup_{a_i \in W_i} \left\langle \sum_{j=1}^{n} (\alpha_{ij} \lambda_{ij} p_{ij}^* + A_{ij}^* q_{ij}^*), a_i \right\rangle - \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} \langle q_{ij}^*, f_{ij} \rangle \end{aligned}$$

and

$$M^* := \left\{ \begin{array}{c|l} & \gamma_{ij}^* (p_{ij}^*) \leq 1 \text{ for } \alpha_{ij} = 1, i = 1, \dots, m, j = 1, \dots, n; \\ & \tilde{\gamma}_{lj}^* \left(\tilde{p}_{lj}^*\right) \leq 1 \text{ for } \tilde{\alpha}_{lj} = 1, 1 \leq l < j \leq n; \\ & q_{ij}^* \underset{K_{Z_{ij}}^*}{\leq} 0, i = 1, \dots, m, j = 1, \dots, n; \\ (p^*, q^*) \in & \sum_{i=1}^{m} C_{ij}^* q_{ij}^* \underset{K_{V_j}^*}{\leq} 0, j = 1, \dots, n; \\ \left(\left[Y_n^*, \tilde{X}^*\right], Z^*\right) & \sum_{i=1}^{m} (B_{ij}^* q_{ij}^* - \alpha_{ij} \lambda_{ij} S_{ji}^* p_{ij}^* - l_{ij}^*) + \\ & \sum_{l=1}^{j-1} \tilde{\alpha}_{lj} \tilde{\lambda}_{lj} \tilde{p}_{lj}^* - \sum_{l=j+1}^{n} \tilde{\alpha}_{jl} \tilde{\lambda}_{jl} T_{jl}^* \tilde{p}_{jl}^* \underset{K_{X_j}^*}{\leq} 0, \\ & j = 1, \dots, n \end{array} \right\}.$$

At that it holds:

- $(p^*, q^*) = \left(p_{11}^*, p_{12}^*, p_{13}^*, \dots, p_{mn}^*, \tilde{p}_{12}^*, \tilde{p}_{13}^*, \dots, \tilde{p}_{n-1,n}^*, q_{11}^*, q_{12}^*, \dots, q_{mn}^*\right)^T$;
- $p_{ij}^* \in Y_i^*$, $q_{ij}^* \in Z_{ij}^*$, $i = 1, \dots, m$, $j = 1, \dots, n$; $\tilde{p}_{lj}^* \in X_j^*$, $1 \leq l < j \leq n$; so it is valid: $\left(\left[Y_n^*, \tilde{X}^*\right], Z^*\right) = \Big(\underbrace{Y_1^*, \dots, Y_1^*}_{n \text{ times}}, \dots, \underbrace{Y_i^*, \dots, Y_i^*}_{n \text{ times}}, \dots, \underbrace{Y_m^*, \dots, Y_m^*}_{n \text{ times}}, X_2^*,$

$X_3^*, \ldots, X_n^*, X_3^*, X_4^*, \ldots, X_n^*, \ldots, X_{n-1}^*, X_n^*, X_n^*, Z_{11}^*, Z_{12}^*, \ldots, Z_{1n}^*, Z_{21}^*,$ $Z_{22}^*, \ldots, Z_{2n}^*, \ldots, Z_{ij}^*, \ldots, Z_{m1}^*, \ldots, Z_{mn}^*\big)$ with Y_i^*, X_j^*, Z_{ij}^* are topological dual spaces to $Y_i, X_j, Z_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$;

- γ_{ij}^* is the dual gauge to γ_{ij} in the space $Y_i^*, i = 1, \ldots, m, j = 1, \ldots, n$, and $\tilde{\gamma}_{lj}^*$ is the dual gauge to $\tilde{\gamma}_{lj}$ in the space $X_j^*, 1 \leq l < j \leq n$; at that the dual gauges γ_{ij}^* and $\tilde{\gamma}_{lj}^*$ are defined by (3.7) depending on the definition of γ_{ij} and $\tilde{\gamma}_{lj}$;
- $S_{ji}^* \in L(Y_i^* \to X_j^*)$ is the adjoint operator to S_{ji}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $T_{jl}^* \in L(X_l^* \to X_j^*)$ is the adjoint operator to T_{jl}, $1 \leq l < j \leq n$;
- $K_{X_j}^* \subset X_j^*$ is the dual cone to K_{X_j}, $K_{V_j}^* \subset V_j^*$ is the dual cone to $K_{V_j}, j = 1, \ldots,$ n and $K_{Z_{ij}}^* \subset Z_{ij}^*$ is the dual cone to $K_{Z_{ij}}, i = 1, \ldots, m, j = 1, \ldots, n$;
- $A_{ij}^* \in L(Z_{ij}^* \to Y_i^*), B_{ij}^* \in L(Z_{ij}^* \to X_j^*), C_{ij}^* \in L(Z_{ij}^* \to V_j^*)$ are the adjoint operators to A_{ij}, B_{ij}, C_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$;

The other occuring symbols were explained after the definition of M.

5. Duality assertions

Here a first duality assertion, the so-called weak duality, for the problems (P) and (P^*) is given with the following theorem:

Theorem 1. *For the objective functions S, R and the restriction sets M, M^* from the problems (P), (P^*) which are described in section 2 and 4 it holds:*

$$S(x,a) \geq R(p^*, q^*) \quad \forall\, (x,a,v) \in M \text{ and } \forall\, (p^*, q^*) \in M^*. \tag{5.1}$$

Proof of Theorem 1. Let be given $(x,a,v) \in M$ and $(p^*, q^*) \in M^*$. Because of the statement of problem (P) it is valid:

$$S(x,a) = \sum_{i=1}^{m} \sum_{j=1}^{n} \{\lambda_{ij}{}^{\alpha_{ij}} [\gamma_{ij}(S_{ji}x_j - a_i)]^{\alpha_{ij}} + \langle l_{ij}^*, x_j\rangle\} + \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\lambda}_{lj}{}^{\tilde{\alpha}_{lj}} [\tilde{\gamma}_{lj}(T_{lj}x_l - x_j)]^{\tilde{\alpha}_{lj}}. \tag{5.2}$$

For $a, b \in \mathbb{R}_+$ the Young inequality is fulfilled:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

If $\alpha_{ij} > 1$ then it is appointed $p := \alpha_{ij}$, $a := \lambda_{ij}\, \gamma_{ij}(S_{ji}x_j - a_i)$ and $b := \gamma_{ij}^*(p_{ij}^*)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and it results with inequality (3.8)

$$\alpha_{ij}\, \lambda_{ij}\, \langle p_{ij}^*, S_{ji}x_j - a_i\rangle + (1 - \alpha_{ij})\, [\gamma_{ij}^*(p_{ij}^*)]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} \leq \lambda_{ij}^{\alpha_{ij}}\, [\gamma_{ij}(S_{ji}x_j - a_i)]^{\alpha_{ij}} \quad i = 1, \ldots, m,\ j = 1, \ldots, n. \tag{5.3}$$

If $\alpha_{ij} = 1$ then it is originated directly from inequality (3.8). And on the ground of $(p^*, q^*) \in M^*$ it holds $\gamma_{ij}^* (p_{ij}^*) \leq 1$ such that the following inequality results

$$\lambda_{ij} \langle p_{ij}^*, S_{ji}x_j - a_i \rangle \leq \lambda_{ij} \gamma_{ij} (S_{ji}x_j - a_i), \quad i = 1, \ldots, m, \; j = 1, \ldots, n. \tag{5.4}$$

For the gauge $\tilde{\gamma}_{lj}$ and its dual gauge $\tilde{\gamma}_{lj}^*$, $1 \leq l < j \leq n$ there are analogous inequalities as (5.3) and (5.4). They can be produced also by using the Young inequality and inequality (3.8).

If all these new gauge inequalities as e.g. (5.3) are applied to inequality (5.2) then it is valid:

$$\begin{aligned} S(x,a) \geq & \sum_{\substack{i=1 \\ \alpha_{ij}>1}}^{m} \sum_{j=1}^{n} (1 - \alpha_{ij}) \left[\gamma_{ij}^* (p_{ij}^*)\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} + \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\alpha_{ij}\lambda_{ij} \langle p_{ij}^*, S_{ji}x_j - a_i \rangle + \langle l_{ij}^*, x_j \rangle\right) + \\ & \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\alpha}_{lj}\tilde{\lambda}_{lj} \langle \tilde{p}_{lj}^*, T_{lj}x_l - x_j \rangle + \\ & \sum_{\substack{l=1 \\ \tilde{\alpha}_{lj}>1}}^{n-1} \sum_{j=l+1}^{n} (1 - \tilde{\alpha}_{lj}) \left[\tilde{\gamma}_{lj}^* \left(\tilde{p}_{lj}^*\right)\right]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}}. \end{aligned} \tag{5.5}$$

In addition to this the assumptions $(x, a, v) \in M$ and $(p^*, \delta^*) \in M^*$ also induce the inequalities

$$\langle q_{ij}^*, A_{ij}a_i + B_{ij}x_j + C_{ij}v_j + f_{ij} \rangle \geq 0; \quad i = 1, \ldots, m, j = 1, \ldots, n. \tag{5.6}$$

By means of some technical calculations the following identity can be pointed out:

$$\begin{aligned} & \sum_{j=1}^{n} \left(\left\langle \sum_{l=1}^{j-1} \tilde{\alpha}_{lj} \tilde{\lambda}_{lj} \tilde{p}_{lj}^* - \sum_{l=j+1}^{n} \tilde{\alpha}_{jl} \tilde{\lambda}_{jl} T_{jl}^* \tilde{p}_{jl}^*, x_j \right\rangle \right) = \\ & \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\alpha}_{lj} \tilde{\lambda}_{lj} \langle \tilde{p}_{lj}^*, x_j - T_{lj}x_l \rangle. \end{aligned}$$

The inequalities (5.5), (5.6) and this identity imply

$$\begin{aligned}S(x,a) \geq & \sum_{\substack{i=1\\ \alpha_{ij}>1}}^{m}\sum_{j=1}^{n}(1-\alpha_{ij})\left[\gamma_{ij}^{*}(p_{ij}^{*})\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} - \\ & \sum_{i=1}^{m}\left\langle \sum_{j=1}^{n} A_{ij}^{*}q_{ij}^{*} + \alpha_{ij}\lambda_{ij}p_{ij}^{*}, a_i \right\rangle - \\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\langle q_{ij}^{*}, f_{ij}\rangle - \sum_{i=1}^{m}\sum_{j=1}^{n}\langle C_{ij}^{*}q_{ij}^{*}, v_j\rangle + \\ & \sum_{j=1}^{n}\left\langle \sum_{i=1}^{m}(\alpha_{ij}\lambda_{ij}S_{ji}^{*}p_{ij}^{*} + l_{ij}^{*} - B_{ij}^{*}q_{ij}^{*}) + \sum_{l=1+j}^{n}\tilde{\alpha}_{jl}\tilde{\lambda}_{jl}T_{jl}^{*}\tilde{p}_{jl}^{*} - \right. \\ & \left. \sum_{l=1}^{j-1}\tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\tilde{p}_{lj}^{*}, x_j \right\rangle + \\ & \sum_{\substack{l=1\\ \tilde{\alpha}_{lj}>1}}^{n-1}\sum_{j=l+1}^{n}(1-\tilde{\alpha}_{lj})\left[\tilde{\gamma}_{lj}^{*}\left(\tilde{p}_{lj}^{*}\right)\right]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}}.\end{aligned} \tag{5.7}$$

Finally it holds

$$\left\langle \sum_{i=1}^{m} C_{ij}^{*}\, q_{ij}^{*}, v_j \right\rangle \leq 0\,, \ j=1,\dots,n\,;$$

$$\begin{aligned}\left\langle \sum_{i=1}^{m}(B_{ij}^{*}\, q_{ij}^{*} - \alpha_{ij}\,\lambda_{ij}\,S_{ji}^{*}\,p_{ij}^{*} - l_{ij}^{*}) + \sum_{l=1}^{j-1}\tilde{\alpha}_{lj}\,\tilde{\lambda}_{lj}\,\tilde{p}_{lj}^{*} - \right. \\ \left. \sum_{l=j+1}^{n}\tilde{\alpha}_{jl}\,\tilde{\lambda}_{jl}\,T_{jl}^{*}\,\tilde{p}_{jl}^{*}, x_j \right\rangle \leq 0\,, \ j=1,\dots,n,\end{aligned}$$

because of $(x,a,v)\in M$ and $(p^*,q^*)\in M^*$. So inequality (5.7) becomes

$$\begin{aligned}S(x,a) \geq & \sum_{\substack{i=1\\ \alpha_{ij}>1}}^{m}\sum_{j=1}^{n}(1-\alpha_{ij})\left[\gamma_{ij}^{*}(p_{ij}^{*})\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} + \\ & \sum_{\substack{l=1\\ \tilde{\alpha}_{lj}>1}}^{n-1}\sum_{j=l+1}^{n}(1-\tilde{\alpha}_{lj})\left[\tilde{\gamma}_{lj}^{*}\left(\tilde{p}_{lj}^{*}\right)\right]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}} - \\ & \sum_{i=1}^{m}\sup_{a_i\in W_i}\left\langle \sum_{j=1}^{n}(\alpha_{ij}\,\lambda_{ij}\,p_{ij}^{*} + A_{ij}^{*}\,q_{ij}^{*})\,,\, a_i \right\rangle - \\ & \sum_{i=1}^{m}\sum_{j=1}^{n}\langle q_{ij}^{*}\,,\, f_{ij}\rangle = R(p^*,q^*)\,.\end{aligned}$$

The proof of Theorem 1 is completed. ∎

The next theorem makes a strong duality assertion.

Theorem 2. *Let be* $\infty > \inf\limits_{(x,a,v)\in M} S(x,a) = \inf(P) > -\infty$ *and there exists an admissible element* $(\bar{x}, \bar{a}, \bar{v}) \in M$ *with*

$$A_{ij}\, \bar{a}_i + B_{ij}\, \bar{x}_j + C_{ij}\, \bar{v}_j + f_{ij} \in \operatorname{int}\left(-K_{Z_{ij}}\right), i = 1, \ldots, m, j = 1, \ldots, n. \tag{5.8}$$

Then there is a solution $(\mathring{p}^*, \mathring{q}^*) \in M^*$ *of the dual problem* (P^*) *with the strong duality assertion*

$$\inf_{(x,a,v)\in M} S(x,a) = \max_{(p^*,q^*)\in M^*} R(p^*, q^*) = R(\mathring{p}^*, \mathring{q}^*). \tag{5.9}$$

Remark 1. Condition (5.8) is a regularity condition and it is also said to be a Slater condition.

Proof of Theorem 2. According to the assumption inf(P) is finite and it holds

$$\inf_{(x,a,v)\in M} S(x,a) = \inf_{(x,a,v)\in M} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\lambda_{ij}{}^{\alpha_{ij}} [\gamma_{ij} (S_{ji} x_j - a_i)]^{\alpha_{ij}} + \langle l_{ij}{}^*, x_j \rangle \right) + \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}} [\tilde{\gamma}_{lj} (T_{lj} x_l - x_j)]^{\tilde{\alpha}_{lj}} \right\} = \inf\,(P).$$

If the problem (P) is stable then there is a solution $(\mathring{p}^*, \mathring{q}^*) \in M^*$ of (P^*) according to the Fenchel-Rockafellar duality theory with (cf. [2]):

$$\inf_{(x,a,v)\in M} S(x,a) = \max_{(p^*,q^*)\in M^*} R(p^*, q^*) = R(\mathring{p}^*, \mathring{q}^*).$$

Indeed on the ground of the Slater condition (5.8) the stability of (P) can be proved. Here the fulfilment of two criteria for the stability of (P) must be shown:

1. $\infty > \inf\limits_{(x,a,v)\in M} S(x,a) > -\infty$,
2. the subdifferential of the function N in point $(p,q) = (0,0)$ is non-empty, $\partial N(0,0) \neq \emptyset$; the function N is here the infimum function of the perturbation function Φ from section 4, $N(p,q) = \inf\limits_{(x,a,v)\in(X,Y,V)} \Phi(x,a,v,p,q)$.

The first condition is an assumption of the theorem 2. The second condition is a conclusion that function N is convex and continuous in $(p,q) = (0,0)$. It is easy to show the convexity of N because it is built by convex functions, restrictions and perturbations. The continuity of N in $(p,q) = (0,0)$ is implied by the Slater condition

(5.8). So the problem (P) is stable and a solution $(\overset{\circ}{p}{}^*, \overset{\circ}{q}{}^*) \in M^*$ exists with:

$$
\begin{aligned}
\inf_{(x,a,v)\in M} S(x,a) = &\inf_{(x,a,v)\in M} \Bigg\{ \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\lambda_{ij}{}^{\alpha_{ij}} [\gamma_{ij}(S_{ji}x_j - a_i)]^{\alpha_{ij}} + \langle l_{ij}{}^*, x_j\rangle\right) + \\
&\sum_{l=1}^{n-1}\sum_{j=l+1}^{n} \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}} [\tilde{\gamma}_{lj}(T_{lj}x_l - x_j)]^{\tilde{\alpha}_{lj}} \Bigg\} \\
= &\sum_{\substack{i=1\\ \alpha_{ij}>1}}^{m}\sum_{j=1}^{n} (1-\alpha_{ij}) \left[\gamma_{ij}{}^*\left(\overset{\circ}{p}_{ij}{}^*\right)\right]^{\frac{\alpha_{ij}}{\alpha_{ij}-1}} + \\
&\sum_{\substack{l=1\\ \tilde{\alpha}_{lj}>1}}^{n-1}\sum_{j=l+1}^{n} (1-\tilde{\alpha}_{lj}) \left[\tilde{\gamma}_{lj}{}^*\left(\overset{\circ}{\tilde{p}}_{lj}{}^*\right)\right]^{\frac{\tilde{\alpha}_{lj}}{\tilde{\alpha}_{lj}-1}} - \\
&\sum_{i=1}^{m} \sup_{a_i\in W_i} \left\langle \sum_{j=1}^{n} \left(\alpha_{ij}\lambda_{ij}\overset{\circ}{p}_{ij}{}^* + A_{ij}{}^*\overset{\circ}{q}_{ij}{}^*\right), a_i \right\rangle - \\
&\sum_{i=1}^{m}\sum_{j=1}^{n} \langle \overset{\circ}{q}_{ij}{}^*, f_{ij}\rangle = R\left(\overset{\circ}{p}{}^*, \overset{\circ}{q}{}^*\right).
\end{aligned}
\tag{5.10}
$$

The proof of Theorem 2 is completed. ∎

Remark 2. If even the infimum value is assumed in problem (P) that means there is an element $(\overset{\circ}{x}, \overset{\circ}{a}, \overset{\circ}{v}) \in M$ with

$$\inf_{(x,a,v)\in M} S(x,a) = \min_{(x,a,v)\in M} S(x,a) = S(\overset{\circ}{x}, \overset{\circ}{a})$$

then with the Slater condition (5.8) the strong duality assertion is also fulfilled. So there is an element $(\overset{\circ}{p}{}^*, \overset{\circ}{q}{}^*) \in M^*$ with

$$S(\overset{\circ}{x}, \overset{\circ}{a}) = \min_{(x,a,v)\in M} S(x,a) = \max_{(p^*,q^*)\in M^*} R(p^*,q^*) = R\left(\overset{\circ}{p}{}^*, \overset{\circ}{q}{}^*\right). \tag{5.11}$$

Then with the Young inequality and inequality (3.8) the following optimality conditions can be derived from (5.11) in a similar way as in [11]:

(i) $\left\langle \sum_{i=1}^{m} C_{ij}{}^*\overset{\circ}{q}_{ij}{}^*, \overset{\circ}{v}_j \right\rangle = 0, j = 1,\ldots,n;$

(ii) $\langle \overset{\circ}{q}_{ij}{}^*, A_{ij}\overset{\circ}{a}_i + B_{ij}\overset{\circ}{x}_j + C_{ij}\overset{\circ}{v}_j + f_{ij}\rangle = 0, i = 1,\ldots,m, j = 1,\ldots,n;$

$$\alpha_{ij} > 1: \quad \begin{aligned} &\langle \overset{\circ}{p}_{ij}{}^*, S_{ji}\overset{\circ}{x}_j - \overset{\circ}{a}_i\rangle = \lambda_{ij}{}^{\alpha_{ij}-1}[\gamma_{ij}(S_{ji}\overset{\circ}{x}_j - \overset{\circ}{a}_i)]^{\alpha_{ij}}, \\ &\gamma_{ij}{}^*\left(\overset{\circ}{p}_{ij}{}^*\right) = \lambda_{ij}{}^{\alpha_{ij}-1}[\gamma_{ij}(S_{ji}\overset{\circ}{x}_j - \overset{\circ}{a}_i)]^{\alpha_{ij}-1}, \end{aligned}$$

$$\alpha_{ij} = 1: \quad \begin{aligned} &\langle \overset{\circ}{p}_{ij}{}^*, S_{ji}\overset{\circ}{x}_j - \overset{\circ}{a}_i\rangle = \gamma_{ij}(S_{ji}\overset{\circ}{x}_j - \overset{\circ}{a}_i), \\ &\gamma_{ij}{}^*\left(\overset{\circ}{p}_{ij}{}^*\right) = 1, \end{aligned}$$

$i = 1,\ldots,m,\ j = 1,\ldots,n;$

$$
\begin{array}{lll}
\text{(iii)} & \tilde{\alpha}_{lj} > 1: & \left\langle \overset{\circ}{\tilde{p}}_{lj}{}^{*}, T_{lj}\overset{\circ}{x}_{l} - \overset{\circ}{x}_{j} \right\rangle = \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}-1} \left[\tilde{\gamma}_{lj}\left(T_{lj}\overset{\circ}{x}_{l} - \overset{\circ}{x}_{j}\right)\right]^{\tilde{\alpha}_{lj}}, \\
 & & \tilde{\gamma}_{lj}{}^{*}\left(\overset{\circ}{\tilde{p}}_{lj}{}^{*}\right) = \tilde{\lambda}_{lj}^{\tilde{\alpha}_{lj}-1} \left[\tilde{\gamma}_{lj}\left(T_{lj}\overset{\circ}{x}_{l} - \overset{\circ}{x}_{j}\right)\right]^{\tilde{\alpha}_{lj}-1}, \\
 & \tilde{\alpha}_{lj} = 1: & \left\langle \overset{\circ}{\tilde{p}}_{lj}{}^{*}, T_{lj}\overset{\circ}{x}_{l} - \overset{\circ}{x}_{j} \right\rangle = \tilde{\gamma}_{lj}\left(T_{lj}\overset{\circ}{x}_{l} - \overset{\circ}{x}_{j}\right), \\
 & & \tilde{\gamma}_{lj}{}^{*}\left(\overset{\circ}{\tilde{p}}_{lj}{}^{*}\right) = 1, \\
 & 1 \le l < j \le n; &
\end{array}
$$

$$
\text{(iv)} \quad \left\langle \left[\sum_{i=1}^{m} \left(B_{ij}{}^{*}\overset{\circ}{q}_{ij}{}^{*} - \alpha_{ij}\lambda_{ij}S_{ij}{}^{*}\overset{\circ}{p}_{ij}{}^{*} - l_{ij}{}^{*}\right) + \sum_{l=1}^{j-1} \tilde{\alpha}_{lj}\tilde{\lambda}_{lj}\overset{\circ}{\tilde{p}}_{lj}{}^{*} - \sum_{l=j+1}^{n} \tilde{\alpha}_{jl}\tilde{\lambda}_{jl}T_{jl}{}^{*}\overset{\circ}{\tilde{p}}_{lj}{}^{*} \right], \overset{\circ}{x}_{j} \right\rangle = 0, \; j = 1, \ldots, n;
$$

$$
\text{(v)} \quad \sup_{a_i \in W_i} \left\langle \sum_{j=1}^{n} \left(\alpha_{ij}\lambda_{ij}\overset{\circ}{p}_{ij}{}^{*} + A_{ji}{}^{*}\overset{\circ}{q}_{ij}{}^{*}\right), a_i \right\rangle = \left\langle \sum_{j=1}^{n} \left(\alpha_{ij}\lambda_{ij}\overset{\circ}{p}_{ij}{}^{*} + A_{ji}{}^{*}\overset{\circ}{q}_{ij}{}^{*}\right), \overset{\circ}{a}_{i} \right\rangle, \; i = 1, \ldots, m.
$$

(5.12)

These necessary optimality conditions can be interpreted as a mixture and generalization of the classical Kolmogorov conditions in best approximation theory and of the maximum principles in optimal control theory (special version) and finally of the complementary slackness conditions in linear programming.

6. Conclusions and summary

Outgoing from results by former and recent researchers about location theory location problems or control-approximation problems were generalized. This happened in Hausdorff locally convex topological real vector spaces. As distance functions powers of gauges were used. A primal control-approximation problem (P) was formulated. For the problem (P) a dual problem (P^*) was produced. With the help of the Fenchel-Rockafellar theory of duality and former obtained results weak and strong duality assertions were derived. So the following results arose:

1. For all elements $(x, a, v) \in M$ and $(p^*, q^*) \in M^*$ it holds the weak duality assertion (5.1).
2. If $\inf(P)$ is finite and the Slater condition (5.8) is fulfilled then there is a solution $(\overset{\circ}{p}{}^{*}, \overset{\circ}{q}{}^{*}) \in M^*$ of the dual problem (P^*) so that the strong duality assertion (5.9) is satisfied and if the infimum of the objective function in problem (P) is attained then the strong duality assertion (5.11) and the optimality conditions (5.12) are also fulfilled.

In a forthcoming paper it should be tried to apply the derived results to the investigation of multiobjective multifacility location and control-approximation problems concerning vectorial duality. So the results contained in [9] and [10] are intended to become.

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