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Efficiency in seminorm location problems

U. Krallert, G. Wanka

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EFFICIENCY IN SEMINORM LOCATION PROBLEMS

G.WANKA and U.KRALLERT*

Abstract

Connections between the solutions of a single objective location optimization problem and the efficiency sets of the belonging multiobjective location optimization problem in $\mathbf{R}^n$ have already been investigated extensively.

In this work connections between efficiency sets of multiobjective location optimization problems and solutions of single objective location optimization problems in Hausdorff locally convex topological vector spaces with seminorms as distance functions are given. If the single objective location optimization problems are replaced by multiobjective location optimization problems, e.g. because several seminorms or even families of seminorms are used simultaneously instead of only one seminorm for the single objective location optimization problem, then the ideal solution of this multiobjective location optimization problems should be considered. Then it is possible to produce a multiobjective location optimization problem which consists of collections of several criteria, e.g. with regard to the several seminorms. For only one seminorm the well-known multiobjective location optimization problem arises as a collection of single criteria. The application of Hilbertian seminorm families introduces the concept of projections. Then relations between efficiency sets, weak efficiency sets and definite sets of projections are shown which are partially generalizations of those by E.Carrizosa, E.Conde, F.R.Fernandez and J.Puerto (cf. [1]). Examples explain the results.

Key words: multiobjective location, efficiency, seminorm, semiscalar product.

*Faculty of Mathematics, Technical University Chemnitz, D-09107 Chemnitz, Germany

1 Introduction

For a certain time relations between multicriteria analysis and game theory, location theory and other branches have been studied. The number of the corresponding publications on this subject has still been increasing.

In [1] such connections between multicriteria analysis and location theory for location optimization problems with Euclidean distances in finite dimensional spaces have been considered. So different kinds of efficiency sets of the multiobjective location optimization problem arisen by extension of a single objective location optimization problem have been examined for relations to solutions of this single objective location optimization problem.

It should be remarked that connections between scalar or single objective optimization problems and multiple objective optimization problems with different intentions, especially with the concept of scalarization, are given in [2], [3], [4] and [5]. So for instance there are investigations of duality and optimality conditions.

Moreover, some papers are concerned with the numerical construction of the efficiency set (cf. [6], [7] and [8]), with approximation and location problems using gauges (cf. [9]) and with approximately efficient solutions for such multiobjective location and approximation problems (cf. [10] and [11]).

Now one of these relations from [1] shall be generalized (general spaces, general distance functions, especially seminorms and infinitely many criteria). In this case the so-called ideal solution of the multiobjective location optimization problem shall be considered by using the results of [12].

In section 2 the problem is explained and something about the existence of the ideal solution is remarked. Furthermore the different efficiency sets are defined there. Section 3 contains relations between efficient points, weakly efficient points of the multiobjective location optimization problem and the set of the ideal solutions if the seminorms are Hilbertians. Moreover, there is established the connection between the ideal solutions and some kinds of projections of sets to a convex set in the sense of semiscalar products. So these results represent especially a generalization of the assertions elaborated in [1] for more restrictive assumptions (cf. theorem 3.1 in this work with theorem 2 in [1]). Examples of these results follow in section 4. Conclusions and ideas for the future development are stated in section 5. At last the appendix and the reference list are presented.

2 Explaining the problem

Let A be an infinite set of location points a. Regarding A an optimal location s_0 from a non-empty set S of locations s should be chosen. Here, for each point $a \in A$ several, in general infinitely many, decision criteria depending on a parameter λ, $\lambda \in I$, should be observed.

As most general superset for A and S the Hausdorff locally convex topological vector space M is chosen here. Additionally it is required that S should be convex.

The optimal location point s_0 for an element $\underline{a}$ from A, also called the best approximation of $\underline{a}$ by S, shall be the point of S that has the smallest distance to $\underline{a}$ from all $s \in S$. Here, the distance functions are seminorms p_λ from the continuous family of seminorms $\{p_\lambda\}_{\lambda \in I}$. [1] As a special case the family of seminorms which induces the locally convex topology in M can be taken. So the point $\underline{a} \in A$ is approximated by using several or even infinitely many seminorms p_λ, $\lambda \in I$, simultaneously.

Thus for an element $\underline{a}$ from A with respect to the seminorms p_λ, $\lambda \in I$, the following multiobjective location optimization problem results:

It is looked for an element $s_0 \in S$, with:

$$p_\lambda(\underline{a} - s_0) \leq p_\lambda(\underline{a} - s) \quad \forall s \in S, \quad \forall \lambda \in I. \qquad \mathrm{MOP}_\lambda(\underline{a}, S)$$

Here, s_0 is called the ideal solution of $\mathrm{MOP}_\lambda(\underline{a}, S)$. That means s_0 is the optimal location or best approximation point simultaneously for all $\lambda \in I$. Then the set of all ideal solutions of the $\mathrm{MOP}_\lambda(\underline{a}, S)$ for an $\underline{a} \in A$ is denoted by $\mathrm{M}_\lambda(\underline{a}, S)$. The union of all sets of ideal solutions is $\mathrm{M}_\lambda(A, S)$:

$$\mathrm{M}_\lambda(A, S) = \bigcup_{\underline{a} \in A} \mathrm{M}_\lambda(\underline{a}, S).$$

If additionally all elements $a \in A$ are considered at the same time, the following problem arises:

$$v - \min_{s \in S}\{p_\lambda(a - s) \mid a \in A, \lambda \in I\}. \qquad \mathrm{MOP}_\lambda(A, S)$$

I.e., seminorm criteria for each of the on A depending criteria are to be considered.

[1] It should still be noted that a seminorm is a positive semidefinite, absolutely homogeneous and subadditive functional.

The above mentioned symbolic notation has still to be specified concerning the solution notion that is considered. This is, because in multiobjective optimization there are known several notions of solutions. The basic definition is that of the efficient solution. Later, this definition will be recalled and specified to the multiobjective optimization location problem $\mathrm{MOP}_\lambda(A, S)$.

If the existence of the ideal solution of $\mathrm{MOP}_\lambda(\underline{a}, S)$ is not guaranteed, that means it is due to examine efficiency sets for $\mathrm{MOP}_\lambda(\underline{a}, S)$, then the continuous seminorm family should be chosen as a family consisting of only one single continuous seminorm so as ideal solutions are present in any case. Consequently, the set I becomes a singleton and $\mathrm{MOP}_\lambda(\underline{a}, S)$ becomes a location optimization problem with exactly one objective function. It is well-known as the best approximation problem of $\underline{a}$ by S. Hence, $\mathrm{MOP}_\lambda(A, S)$ consists of only on A depending criteria. But this can be done only for a single seminorm which should be also proper if the seminorm family or then the single seminorm is not needed for inducing of the Hausdorff topology in M.[2] There are no complications if the single seminorm can also be improper, i.e. it can be a norm.

So the important special case of exactly one seminorm criterion without additionally inducing of the Hausdorff topology in M is contained in the investigations about several seminorm criteria. Of course, for this special case the existence of the ideal solutions need not be considered. But generally it does not exist an ideal solution for $\mathrm{MOP}_\lambda(\underline{a}, S)$. The existence is only possible under certain assumptions. In chapter 1 in [12] theorem 2.4. tells something about the existence of ideal solutions.

Now it is justified to investigate the relations to efficiency sets because the set of the ideal solutions of $\mathrm{MOP}_\lambda(\underline{a}, S)$ is not always the empty set.

As announced, the definition of the different notions of efficiency which shall be used follows:

Definition 2.1

(i) *The set of weakly efficient solutions of* $\mathrm{MOP}_\lambda(A, S)$ *is:*

$$\mathrm{WE}_{(\lambda \in I)}(A, S) = \{s_0 \in S \mid \nexists s \in S : p_\lambda(a - s) < p_\lambda(a - s_0) \;\; \forall a \in A\,, \;\; \forall \lambda \in I\}.$$

[2]In the appendix the hierarchy of locally convex topological vector spaces, Hausdorff locally convex topological vector spaces, spaces which are metricable and other spaces are shown.

(ii) *The set of the efficient solutions of* $\mathrm{MOP}_\lambda(A,S)$ *is:*

$$\mathrm{E}_{(\lambda\in I)}(A,S)=\left\{ s_0\in S \;\middle|\; \begin{array}{l} \nexists s\in S: \\ p_\lambda(a-s)\leq p_\lambda(a-s_0)\;\; \forall a\in A\,,\;\forall\lambda\in I \text{ and} \\ p_{\bar\lambda}(\bar a-s)<p_{\bar\lambda}(\bar a-s_0)\;\; \text{for some } \bar a\in A \text{ and} \\ \text{for some } \bar\lambda\in I \end{array} \right\}.$$

For definitions of efficiency types generally based on partial orders induced by a cone is referred to the book [13].

Furthermore, determined seminorms are used. For that purpose the following definitions are given:

Definition 2.2 *The function* $\langle\cdot,\cdot\rangle : M\times M\to\mathbb{R}$ *with the properties :*

(i) $\langle x,x\rangle\geq 0\quad\forall\, x\,\in\, M,$

(ii) $\langle \alpha x+\beta y,z\rangle=\alpha\,\langle x,z\rangle+\beta\,\langle y,z\rangle\quad\forall\, x,y,z\in M;\alpha,\beta\in\mathbb{R},$

(iii) $\langle x,y\rangle=\langle y,x\rangle\quad\forall\, x,y\in M,$

is called semiscalar product.

Definition 2.3 *A seminorm* p *for which a semiscalar product* $\langle\cdot,\cdot\rangle$ *exists with* $p(x)=\sqrt{\langle x,x\rangle}$ *is said to be a Hilbertian seminorm.*

With families of Hilbertian seminorms M becomes a Hausdorff pre-Hilbert-locally convex topological vector space.[3] Hence, the ideal solutions of $\mathrm{MOP}_\lambda(\underline{y},S)$ for each $\underline{y}\in M$ can be presented as λ-orthogonal projections on S, similar as in Hilbert spaces.

To get this and other similar properties as in Hilbert spaces, but now based on the semiscalar product, the additional definitions are listed:

Definition 2.4 *An element* x *from* M *is said to be* λ*-orthogonal to an element* y *from* M *with respect to a considered family of semiscalar products* $\{\langle\cdot,\cdot\rangle_\lambda\}_{\lambda\in I}$, *if and only if*

$$\langle x,y\rangle_\lambda=0\quad\forall\,\lambda\;\in\; I.$$

[3]For the characterization of the general spaces is referred to the survey in the appendix.

Definition 2.5 *If U is a subspace of M then the set*

$$U^{\perp_\lambda} := \{x \in M \mid \langle x, y \rangle_\lambda = 0 \ \ \forall \lambda \ \in \ I, \ \ \forall y \ \in \ U\}$$

is called the λ-orthogonal complement to U.

Remark 2.1 Each element x from M can be partitioned into two parts regarding a subspace U from M :

$$x = x_1 + x_2, \quad x_1 \in U, \quad x_2 \in U^{\perp_\lambda}.$$

This partition is unique for families of seminorms inducing a Hausdorff locally convex topology. The proof for the partition property is carried out in a manner like in Hilbert spaces.

Corresponding to the relation in Hilbert spaces that the orthogonal projection of an element y from M onto a convex set S is equal to the element of best approximation of y by S the following corollary is obtained:

Corollary 2.1 *If M is a Hausdorff pre-Hilbert-locally convex topological vector space with a family of Hilbertian seminorms $\{p_\lambda\}_{\lambda \in I}$ and $S \subset M$, with S is convex and non-empty, then it is valid for $\underline{y} \in M$ and a best approximation $s_0 \in S$ with $s_0 = s_0(\underline{y})$:*

$$\begin{aligned} & p_\lambda(\underline{y} - s_0) \leq p_\lambda(\underline{y} - s) \ \ \forall \ s \in S \,, \ \ \forall \ \lambda \in I & \mathrm{MOP}_\lambda(\underline{y}, S) \\ \Longleftrightarrow \ & \left\langle \underline{y} - s_0, s - s_0 \right\rangle_\lambda \leq 0 \ \ \forall \ s \in S \,, \ \ \forall \ \lambda \in I. & \mathrm{MOP}_{\lambda, \mathrm{P}_S}(\underline{y}, S) \end{aligned}$$

Remark 2.2 So for Hilbertian seminorm families $\{p_\lambda\}_{\lambda \in I}$ the ideal solutions of $\mathrm{MOP}_\lambda(\underline{a}, S)$ are λ-orthogonal projections. Also here the ideal solution is unique for families of seminorms inducing a Hausdorff locally convex topology.[4] The proof for this property is also carried out in the same manner as in Hilbert spaces.

3 Efficient and weakly efficient points

Because of corollary 2.1 with theorem 2.4. from chapter 1 of [12] a possibility for decisions about for instance the existence of projections of the convex hull of N (N as a subset of M), conv N, is given. So it is justified to consider the next theorem.

[4] If the family of seminorms consists of only one properly seminorm, then the ideal solution need not be unique. Hence also the λ-orthogonal partition of an element from M need not be unique.

Theorem 3.1 *If M is a Hausdorff pre-Hilbert-locally convex topological vector space with a family of continuous Hilbertian seminorms $\{p_\lambda\}_{\lambda \in I}$ and $A \subseteq M$, $S \subset M$, with S is convex, compact and non-empty, then:*
It holds:

a) *for each $\hat{A}$, $\hat{A} \subset A$, with* card $\hat{A} < \aleph_0$ *and for each $\hat{I}$, $\hat{I} \subset I$, with* card $\hat{I} < \aleph_0$:

$$\begin{aligned} \mathrm{proj}_{(\lambda \in \hat{I}),S}(\mathrm{conv}\ \hat{A}) \ &\subseteq \ \bigcap_{\lambda \in \hat{I}} \mathrm{WE}_{(\lambda \in \{\lambda\})}(\hat{A}, S) \ = \\ \bigcap_{\lambda \in \hat{I}} \mathrm{proj}_{(\lambda \in \{\lambda\}),S}(\mathrm{conv}\ \hat{A}) \ &\subseteq \ \lambda\text{-}\mathrm{E}_{(\lambda \in \hat{I})}(\hat{A}, S) \ \subseteq \\ \mathrm{WE}_{(\lambda \in \hat{I})}(\hat{A}, S) \ &\subseteq \ \mathrm{WE}_{(\lambda \in I)}(A, S),^{5} \end{aligned} \tag{1}$$

$$\begin{aligned} \mathrm{proj}_{(\lambda \in \hat{I}),S}(\mathrm{ri\ conv}\ \hat{A}) \ &\subseteq \ \mathrm{E}_{(\lambda \in \hat{I})}(\hat{A}, S) \ \subseteq \\ \lambda\text{-}\mathrm{E}_{(\lambda \in \hat{I})}(\hat{A}, S) \ &\subseteq \ \mathrm{WE}_{(\lambda \in \hat{I})}(\hat{A}, S) \ \subseteq \ \mathrm{WE}_{(\lambda \in I)}(A, S); \end{aligned} \tag{2}$$

b) *for each $\hat{A}$, $\hat{A} \subset A$, with* card $\hat{A} < \aleph_0$ *and for each λ from I:*

$$\mathrm{WE}_{(\lambda \in \{\lambda\})}(\hat{A}, S) \ = \ \mathrm{proj}_{(\lambda \in \{\lambda\}),S}(\mathrm{conv}\ \hat{A}). \tag{3}$$

Herein for $N \subseteq M$ the projection of N is:

$$\mathrm{proj}_{(\lambda \in I),S}\ N = \bigcup_{n \in N} \left\{ s_0 \in S \ \middle|\ \begin{array}{c} p_\lambda(n - s_0) \le p_\lambda(n - s) \\ \forall s \in S\ ,\ \forall \lambda \in I \end{array} \right\},^{6}$$

and the relative interior of N is:

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[5] So it follows for (1): $\bigcup_{\substack{\hat{A} \subset A, \mathrm{card}\hat{A} < \aleph_0 \\ \hat{I} \subset I, \mathrm{card}\hat{I} < \aleph_0}} \mathrm{proj}_{(\lambda \in \hat{I}),S}(\mathrm{conv}\hat{A}) \ \subseteq \ \mathrm{WE}_{(\lambda \in I)}(A, S)$.

[6] According to the declaration of the union of ideal solutions in section 2 it holds: $\mathrm{proj}_{(\lambda \in I),S}\ N \ = \ \mathrm{M}_\lambda(N, S)$.

$$\mathrm{ri}\, N = \{x \in N \mid \exists\, \epsilon > 0 \text{ with } (x + \epsilon B_1) \cap \mathrm{aff}(N) \subseteq N\}^{7}$$

and

$$\lambda - \mathrm{E}_{(\lambda \in I)}(A, S) = \left\{ s_0 \in S \;\middle|\; \begin{array}{l} \nexists s \in S : \\ p_\lambda(a - s) \leq p_\lambda(a - s_0) \quad \forall a \in A\,, \quad \forall \lambda \in I \text{ and} \\ p_{\bar\lambda}(a - s) < p_{\bar\lambda}(a - s_0) \quad \forall a \in A \text{ for some } \bar\lambda \in I \end{array} \right\}.$$

Remark 3.1 For card $A < \aleph_0$ and card $I < \aleph_0$ the inclusions (1) and (2) and for card $A < \aleph_0$ the equation (3) in theorem 3.1 are simplified in this way that all subsets $\hat{A} \subseteq A, \hat{I} \subseteq I$ have the demanded property card $\hat{A} < \aleph_0$ and card $\hat{I} < \aleph_0$.

The equation (3) in b) is a part of the theorem 2 in [1], but now for seminorms and general spaces.

Two propositions are needed for the proof of theorem 3.1. It follows the first proposition.

Proposition 3.1 *M is assumed to be a Hausdorff pre-Hilbert-locally convex topological vector space with a family of Hilbertian seminorms $\{p_\lambda\}_{\lambda \in I}$. For $x,\ y \in M$ and $N \subset M$ with* card $N < \aleph_0$ *and $w \in W$ with*

$$W = \left\{ (w_n)_{n \in N} \in \mathbb{R}^{\mathrm{card}\, N} \;\middle|\; w_n \geq 0 \ \forall\, n \in N\,,\ \sum_{n \in N} w_n = 1 \right\}$$

it holds for each $\lambda \in I$:

$$\begin{aligned} p_\lambda\left(\sum_{n \in N} w_n \cdot n - x \right) &\leq p_\lambda\left(\sum_{n \in N} w_n \cdot n - y \right) \\ \Longleftrightarrow \quad \sum_{n \in N} w_n \cdot p_\lambda^2(n - x) &\leq \sum_{n \in N} w_n \cdot p_\lambda^2(n - y).^{8} \end{aligned}$$

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[7]With B_1 the unit ball is denoted and aff N means the affine hull of the set N.

[8]Of course $p_\lambda^2(z)$ means $(p_\lambda(z))^2$ for an element z from M and for λ from I.

Proof A straightforward calculation using the definition of the semiscalar product yields the following equation

$$\sum_{n \in N} w_n \cdot p_\lambda^2(n - x) = \sum_{n \in N} w_n \cdot p_\lambda^2(n) - p_\lambda^2\left(\sum_{n \in N} w_n \cdot n\right) + p_\lambda^2\left(\sum_{n \in N} w_n \cdot n - x\right). \tag{4}$$

If equation (4) is applied to

$$\sum_{n \in N} w_n \cdot p_\lambda^2(n - x) \leq \sum_{n \in N} w_n \cdot p_\lambda^2(n - y),$$

so it yields

$$p_\lambda\left(\sum_{n \in N} w_n \cdot n - x\right) \leq p_\lambda\left(\sum_{n \in N} w_n \cdot n - y\right).$$

because of the non-negativity of the seminorms. The other direction is proved in the same fashion. □

The following proposition comes from [14] (cf. theorem 4.2.3.) and this proposition is also given in [15] (cf. Satz 2.103. with vector function $\mathbf{h} = 0$). It is also well-known as the generalized Gordan alternative theorem:

Proposition 3.2 *Convex functions $f_1, \ldots, f_m$ defined on a convex non-empty set $S \subset \mathbb{R}^n$ are given.*
Then either

$$f_k(s) < 0, \quad k = 1, \ldots, m,$$

has a solution $s \in S$ or

$$\sum_{i=1}^{m} \omega_i f_i(s) \geq 0 \quad \text{for all } s \in S \text{ for some } \omega_i \in \mathbb{R},\ \omega_i \geq 0,\ i = 1, \ldots, m, \text{ with at least one } \omega_j > 0,\ j \in \{1, \ldots, m\},$$

but never both.

The next part of the current section is built by the proof of the theorem 3.1.

Proof of Theorem 3.1

a) Let $s_0 \in \text{proj}_{(\lambda\in\hat{I}),S}(\text{conv } \hat{A})$ with $\hat{A} \subset A$, card $\hat{A} < \aleph_0$ and $\hat{I} \subset I$, card $\hat{I} < \aleph_0$. I.e, $\exists a_c \in \text{conv } \hat{A}$ with:

$$p_\lambda(a_c - s) \geq p_\lambda(a_c - s_0) \quad \forall \lambda \in \hat{I}, \ \forall s \in S. \tag{5}$$

In consequence of $a_c \in \text{conv } \hat{A}$ there exists $\omega_a \in \mathbb{R}_+^{\text{card}\hat{A}}$ with $\sum_{a\in\hat{A}} \omega_a = 1$ and $a_c = \sum_{a\in\hat{A}} \omega_a \cdot a$. So it holds:

$$p_\lambda\left(\sum_{a\in\hat{A}} \omega_a \cdot a - s\right) \geq p_\lambda\left(\sum_{a\in\hat{A}} \omega_a \cdot a - s_0\right) \quad \forall \lambda \in \hat{I}, \quad \forall s \in S. \tag{6}$$

By using proposition 3.1 it results for each $\lambda \in \hat{I}$:

$$\sum_{a\in\hat{A}} \omega_a \left(p_\lambda^2(a-s) - p_\lambda^2(a-s_0)\right) \geq 0 \quad \forall s \in S. \tag{7}$$

Thus, for this inequality a some generalized version of proposition 3.2 may be applied. Checking the proof of proposition 3.2 in [14] it is straightforward to establish a generalized version wherein the finitely dimensional space $\mathbb{R}^n$ may be substituted by a Hausdorff topological vector space M and $S \subset M$.

Then it follows taking into consideration the non-negativity of the seminorms that for each $\lambda \in \hat{I}$ the inequality system

$$p_\lambda(a-s) < p_\lambda(a-s_0) \quad \forall a \in \hat{A} \tag{8}$$

has no solution in S. That means $s_0 \in \bigcap_{\lambda\in\hat{I}} \text{WE}_{(\lambda\in\{\lambda\})}(\hat{A}, S)$.

It is still to show that $s_0 \in \lambda\text{-E}_{(\lambda\in\hat{I})}(\hat{A}, S)$, if $s_0 \in \bigcap_{\lambda\in\hat{I}} \text{WE}_{(\lambda\in\{\lambda\})}(\hat{A}, S)$.

Therefore it is assumed that $s_0 \notin \lambda\text{-E}_{(\lambda\in\hat{I})}(\hat{A}, S)$; i.e. $\exists\, \tilde{s} \in S$ with

$$\begin{aligned} p_\lambda(a-\tilde{s}) &\leq p_\lambda(a-s_0) \quad \forall a \in \hat{A}, \ \forall \lambda \in \hat{I}, \\ p_{\bar{\lambda}}(a-\tilde{s}) &< p_{\bar{\lambda}}(a-s_0) \quad \forall a \in \hat{A} \text{ for some } \bar{\lambda} \in \hat{I}. \end{aligned} \tag{9}$$

Considering the inequality system (8) it must be held, if all $\bar{\lambda} \in \hat{I}$ in (9) build the subset $\bar{I} \subset \hat{I}$:

$$p_\lambda(a - \tilde{s}) \;=\; p_\lambda(a - s_0) \quad \forall\, a \,\in\, \hat{A} \quad \forall\, \lambda \,\in\, \hat{I} \setminus \bar{I}.$$

But $p_{\bar{\lambda}}(a - \tilde{s}) < p_{\bar{\lambda}}(a - s_0)$ $\forall a \in \hat{A}$ for some $\bar{\lambda} \in \hat{I}$ (or $\lambda \in \bar{I}$) implies a contradiction to (8). So it is valid $s_0 \in \lambda\text{--}\mathrm{E}_{(\lambda \in \hat{I})}(\hat{A}, S)$.

Finally it follows obviously that $\lambda\text{--}\mathrm{E}_{(\lambda \in \hat{I})}(\hat{A}, S) \subseteq \mathrm{WE}_{(\lambda \in \hat{I})}(\hat{A}, S) \subseteq \mathrm{WE}_{(\lambda \in I)}$ (A, S) because of definition 2.1.

With the generalized version of proposition 3.2 it yields that the insolubility of the system (8) for each $\lambda \in \hat{I}$, in other words $s_0 \in \bigcap_{\lambda \in \hat{I}} \mathrm{WE}_{(\lambda \in \{\lambda\})}(\hat{A}, S)$, is equivalent to:

For each $\lambda \in \hat{I}$ it holds for certain weights $\gamma_a^\lambda \in \mathbb{R}, \gamma_a^\lambda \geq 0$ $\forall a \in \hat{A}$ with $\gamma_{\underline{a}}^\lambda > 0$ for at least one $\underline{a} \in \hat{A}$:

$$\sum_{a \in \hat{A}} \gamma_a^\lambda \left(p_\lambda^2(a - s) \;-\; p_\lambda^2(a - s_0)\right) \;\geq\; 0 \quad \forall\; s \,\in\, S.$$

After normalization of the weights γ_a^λ for each $\lambda \in \hat{I}$ the resulted inequality is according to proposition 3.1 for each $\lambda \in \hat{I}$ equivalent to:

$$p_\lambda\left(\sum_{a \in \hat{A}} \omega_a^\lambda \cdot a \;-\; s\right) \;\geq\; p_\lambda\left(\sum_{a \in \hat{A}} \omega_a^\lambda \cdot a \;-\; s_0\right) \quad \forall\, s \,\in\, S.$$

It follows with $a_c^\lambda = \sum_{a \in \hat{A}} \omega_a^\lambda \cdot a$, $a_c^\lambda \in \mathrm{conv}\hat{A}$ since $\sum_{a \in \hat{A}} \omega_a^\lambda = 1$:

$$\text{For each } \lambda \in \hat{I} \text{ it is fulfilled}: \quad p_\lambda(a_c^\lambda \;-\; s) \;\geq\; p_\lambda(a_c^\lambda \;-\; s_0) \quad \forall\; s \;\in\; S. \tag{10}$$

So it arises $s_0 \in \bigcap_{\lambda \in \hat{I}} \mathrm{proj}_{(\lambda \in \{\lambda\}),S}(\mathrm{conv}\, \hat{A})$.

The opposite direction from (10) to the insolubility of (8) for each $\lambda \in \hat{I}$ does not require a changing of the weights ω_a^λ to γ_a^λ.

At that the equation

$$\bigcap_{\lambda \in \hat{I}} \mathrm{WE}_{(\lambda \in \{\lambda\})}(\hat{A}, S) \;=\; \bigcap_{\lambda \in \hat{I}} \mathrm{proj}_{(\lambda \in \{\lambda\}),S}(\mathrm{conv}\, \hat{A}) \tag{11}$$

is also proved.

If $s_0 \in \text{proj}_{(\lambda \in \hat{I}),S}(\text{ri conv } \hat{A})$ with $\hat{A} \subset A$, card $\hat{A} < \aleph_0$ and $\hat{I} \subset I$, card $\hat{I} < \aleph_0$, then $\exists\, a_c^p \in$ ri conv $\hat{A}$ with:

$$p_\lambda(a_c^p - s) \geq p_\lambda(a_c^p - s_0) \quad \forall \lambda \in \hat{I}, \; \forall s \in S.$$

It can be shown also for finite sets $\hat{A}$ in the Hausdorff pre-Hilbert-locally convex topological vector space M that an element $a_c^p \in$ ri conv $\hat{A}$ possess the representation $a_c^p = \sum\limits_{a \in \hat{A}} \omega_a^p \cdot a$ with $\sum\limits_{a \in \hat{A}} \omega_a^p = 1$ and $\omega_a^p > 0 \; \forall \, a \in \hat{A}$.

For this element a_c^p the inequalities (6) with the weights ω_a^p hold also. Hence it results after exerting proposition 3.1 for each $\lambda \in \hat{I}$ with $\omega_a^p > 0 \;\; \forall a \in \hat{A}$:

$$\sum_{a \in \hat{A}} \omega_a^p \left(p_\lambda^2(a - s) - p_\lambda^2(a - s_0) \right) \geq 0 \quad \forall s \in S. \tag{12}$$

Now the inequalities (12) are summed over all $\lambda \in \hat{I}$, then it results:

$$\sum_{a \in \hat{A},\, \lambda \in \hat{I}} \omega_a^p \left(p_\lambda^2(a - s) - p_\lambda^2(a - s_0) \right) \geq 0 \quad \forall s \in S. \tag{13}$$

Then it is valid $s_0 \in \text{E}_{(\lambda \in \hat{I})}(\hat{A}, S)$, i.e.:
There is no $s \in S$ with

$$p_\lambda(a - s) \leq p_\lambda(a - s_0) \quad \forall a \in \hat{A}, \; \forall \lambda \in \hat{I} \quad \text{and} \tag{14}$$

$$p_{\bar{\lambda}}(\bar{a} - s) < p_{\bar{\lambda}}(\bar{a} - s_0) \quad \text{for some } \bar{\lambda} \in \hat{I} \text{ and } \bar{a} \in A. \tag{15}$$

However, if an element s from S exists fulfilling the inequalities (14) and (15) then after building the squares in (14) and (15) and summing with the weights ω_a^p yields

$$\sum_{a \in \hat{A},\, \lambda \in \hat{I}} \omega_a^p \left(p_\lambda^2(a - s) - p_\lambda^2(a - s_0) \right) < 0 \quad \forall s \in S. \tag{16}$$

This is a contradiction to (12).

$\text{E}_{(\lambda \in \hat{I})}(\hat{A}, S) \subseteq \lambda - \text{E}_{(\lambda \in \hat{I})}(\hat{A}, S)$ is an immediate conclusion of the definitions of those sets.

b) The equation (3) is proved in the same fashion as the equation (11) merely without building of the intersection. The single steps are performed for each $\lambda \in I$ instead of for each $\lambda \in \hat{I}$, $\hat{I} \subset I$ card $\hat{I} < \aleph_0$.

□

The results from theorem 3.1 are presented in examples in the next chapter.

4 Examples

Example 4.1 *M, A,* and *S* are chosen in the following manner:

$$\begin{aligned} M &= \mathbb{R}^2, \\ S &= \left\{(x,y)^{\mathrm{T}} \in M \mid [\max(|x-3|, |y-3|) \leq 2] \wedge [x+y \geq 4]\right\}, \\ A &= \left\{(x,y)^{\mathrm{T}} \in M \mid x = -2,\ y \in [2,5]\right\}. \end{aligned}$$

Two several families of continuous seminorms are selected:

a) $\{p_\lambda\}_{\lambda \in I}: \qquad p_\lambda\left((x,y)^{\mathrm{T}}\right) = \lambda\,|x+y|\ ; \quad I := \mathbb{R}_+ \setminus \{0\}\,,\ \text{card}\, I = \aleph_1,$

b) $\{p_\mu\}_{\mu \in J}: \qquad p_\mu\left((x,y)^{\mathrm{T}}\right) = \mu\,|x|\ ; \quad J := \mathbb{R}_+ \setminus \{0\}\,,\ \text{card}\, J = \aleph_1.$

They are families of proper seminorms because they are not norms. Additionally they are also Hilbertian seminorms; the belonging semiscalar products are:

a) $\langle\cdot,\cdot\rangle_{\lambda \in I}: \quad \left\langle (x_1,y_1)^{\mathrm{T}}, (x_2,y_2)^{\mathrm{T}} \right\rangle_\lambda = \lambda^2 (x_1+y_1)(x_2+y_2)\ ; \quad I := \mathbb{R}_+ \setminus \{0\}\,,$

b) $\langle\cdot,\cdot\rangle_{\mu \in J}: \quad \left\langle (x_1,y_1)^{\mathrm{T}}, (x_2,y_2)^{\mathrm{T}} \right\rangle_\mu = \mu^2 x_1 x_2\,; \quad J := \mathbb{R}_+ \setminus \{0\}\,.$

It can be defined neighbourhoods for the seminorms. For instance a neighbourhood for the point $(-2,3)^{\mathrm{T}} \in M$ with the index value $\lambda = 1$ from the index set I and with the radius of the magnitude 2 is given by

$$\begin{aligned} U^2_{\lambda=1}\left((-2,3)^{\mathrm{T}}\right) &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \,\middle|\, p_{\lambda=1}\left((x,y)^{\mathrm{T}} - (-2,3)^{\mathrm{T}}\right) \leq 2\right\} \\ &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \mid -x-1 \leq y \leq -x+3\right\}. \end{aligned}$$

Another neighbourhood for the point $(-2,4)^{\mathrm{T}} \in M$ is given by

$$\begin{aligned} U^4_{\mu=4}\left((-2,4)^{\mathrm{T}}\right) &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \,\middle|\, p_{\mu=4}\left((x,y)^{\mathrm{T}} - (-2,4)^{\mathrm{T}}\right) \le 4\right\} \\ &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \mid -3 \le x \le -1\right\}. \end{aligned}$$

The set of the ideal solutions for the problem $\mathrm{MOP}_\lambda(A,S)$ is (cf. figure 1):

$$\begin{aligned} \mathrm{M}_\lambda(A,S) &= \mathrm{M}_\lambda(\mathrm{conv}A,S) = \mathrm{proj}_{(\lambda\in I),S}\, A = \mathrm{proj}_{(\lambda\in I),S}(\mathrm{conv}A) \\ &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \mid x+y=4,\ x\in[1,3]\right\}. \end{aligned}$$

It holds: $\mathrm{M}_\lambda(A,S) = \mathrm{E}_{(\lambda\in I)}(A,S) = \mathrm{WE}_{(\lambda\in I)}(A,S)$.

The set of the ideal solutions for the problem $\mathrm{MOP}_\mu(A,S)$ is:

$$\begin{aligned} \mathrm{M}_\mu(A,S) &= \mathrm{M}_\mu(\mathrm{conv}A,S) = \mathrm{proj}_{(\mu\in J),S}\, A = \mathrm{proj}_{(\mu\in J),S}(\mathrm{conv}A) \\ &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \mid x=1,\ y\in[3,5]\right\}. \end{aligned}$$

It holds again: $\mathrm{M}_\mu(A,S) = \mathrm{E}_{(\mu\in J)}(A,S) = \mathrm{WE}_{(\mu\in J)}(A,S)$.

A new seminorm family $\{p_\kappa\}_{\kappa\in K}$ is built with the union of the two seminorm families $\{p_\lambda\}_{\lambda\in I}$, $\{p_\mu\}_{\mu\in J}$:

$$\{p_\kappa\}_{\kappa\in K} = \{\{p_\lambda\}_{\lambda\in I},\ \{p_\mu\}_{\mu\in J}\}.$$

The set of the ideal solutions for the problem $\mathrm{MOP}_\kappa(A,S)$ is:

$$\mathrm{M}_\kappa(A,S) = \mathrm{M}_\kappa(\mathrm{conv}A,S) = \mathrm{proj}_{(\kappa\in K),S}\, A = \mathrm{proj}_{(\kappa\in K),S}(\mathrm{conv}A) = \left\{(1,3)^{\mathrm{T}}\right\}.$$

Here it holds:

$$\begin{aligned} \mathrm{M}_\kappa(A,S) &= \mathrm{E}_{(\kappa\in K)}(A,S) \text{ and} \\ \mathrm{WE}_{(\kappa\in K)}(A,S) &= \left\{(x,y)^{\mathrm{T}} \in \mathbb{R}^2 \mid (x+y=4, x\in[1,3]) \vee (x=1, y\in[3,5]\right\}. \end{aligned}$$

Indeed, for the subsets

$$\begin{aligned} \hat{A} := \left\{(-2,2)^{\mathrm{T}}, (-2,5)^{\mathrm{T}}\right\} &\subset A \text{ and} \\ \hat{K} := \{(\lambda,\mu)\in(I,J) \mid \lambda=1, \mu=2\} &\subset K, \end{aligned}$$

i.e. $\left\{p_\kappa\left((x,y)^{\mathrm{T}}\right)\right\}_{\kappa\in\hat{K}} = \{|x+y|, 2|x|\}$, it follows according to (1) in theorem 3.1 the proper inclusion:

$$\left\{(1,3)^{\mathrm{T}}\right\} = \mathrm{proj}_{(\kappa\in\hat{K}),S}(\mathrm{conv}\hat{A}) \subset \mathrm{WE}_{(\kappa\in K)}(A,S).$$

For the subsets

$$\begin{aligned} \hat{A} &:= \left\{(-2,2)^{\mathrm{T}},(-2,5)^{\mathrm{T}}\right\} \subset A \text{ and} \\ \bar{K} &:= \{(\lambda,\mu)\in(I,J)\mid I:=\emptyset, \mu\in\{1,2\}\} \subset K, \end{aligned}$$

i.e. $\left\{p_\kappa\left((x,y)^{\mathrm{T}}\right)\right\}_{\kappa\in\bar{K}} = \{|x|, 2|x|\}$, it is valid:

$$\mathrm{proj}_{(\kappa\in\bar{K}),S}(\mathrm{conv}\hat{A}) = \kappa\!-\!\mathrm{E}_{(\kappa\in\bar{K})}(\hat{A},S) = \left\{(x,y)^{\mathrm{T}}\in\mathbb{R}^2 \mid x=1, y\in[3,5]\right\}.$$

Here also according to (1) in theorem 3.1 it follows a proper inclusion:

$$\kappa\!-\!\mathrm{E}_{(\kappa\in\bar{K})}(\hat{A},S) \subset \mathrm{WE}_{(\kappa\in K)}(A,S).$$

Finally for subsets

$$\begin{aligned} \hat{A} &:= \left\{(-2,2)^{\mathrm{T}},(-2,5)^{\mathrm{T}}\right\} \subset A \text{ and} \\ K_1 &:= \{(\lambda,\mu)\in(I,J)\mid\lambda\in\{1,2\}, J:=\emptyset\} \subset K, \\ K_2 &:= \{(\lambda,\mu)\in(I,J)\mid I:=\emptyset, \mu\in\{3,4\}\} \subset K, \end{aligned}$$

i.e. $\left\{p_\kappa\left((x,y)^{\mathrm{T}}\right)\right\}_{\kappa\in K_1} = \{|x+y|, 2|x+y|\}$, $\left\{p_\kappa\left((x,y)^{\mathrm{T}}\right)\right\}_{\kappa\in K_2} = \{3|x|, 4|x|\}$ it is valid:

$$\begin{aligned} \mathrm{proj}_{(\kappa\in K_1),S}(\mathrm{conv}\hat{A}) &= \mathrm{E}_{(\kappa\in K_1)}(\hat{A},S) = \left\{(x,y)^{\mathrm{T}}\in\mathbb{R}^2 \mid x+y=4,\ x\in[1,3]\right\}, \\ \mathrm{proj}_{(\kappa\in K_2),S}(\mathrm{conv}\hat{A}) &= \mathrm{E}_{(\kappa\in K_2)}(\hat{A},S) = \left\{(x,y)^{\mathrm{T}}\in\mathbb{R}^2 \mid x=1,\ y\in[3,5]\right\}. \end{aligned}$$

Then it holds according to (2) $\mathrm{E}_{(\kappa\in K_1)}(\hat{A},S) \subset \mathrm{WE}_{(\kappa\in K)}(A,S)$ and $\mathrm{E}_{(\kappa\in K_2)}(\hat{A},S) \subset \mathrm{WE}_{(\kappa\in K)}(A,S)$.

Already the union of $\mathrm{proj}_{(\kappa\in K_1),S}(\mathrm{conv}\hat{A})$ and $\mathrm{proj}_{(\kappa\in K_2),S}(\mathrm{conv}\hat{A})$ builds the set $\mathrm{WE}_{(\kappa\in K)}(A,S)$, that is why in footnote 5 it holds even the equality for the current example:

$$\mathrm{WE}_{(\kappa\in K)}(A,S) = \bigcup_{\substack{\hat{A}\subset A,\mathrm{card}\hat{A}<\aleph_0 \\ \hat{K}\subset K,\mathrm{card}\hat{K}<\aleph_0}} \mathrm{proj}_{(\kappa\in\hat{K}),\hat{S}}(\mathrm{conv}\hat{A}).$$

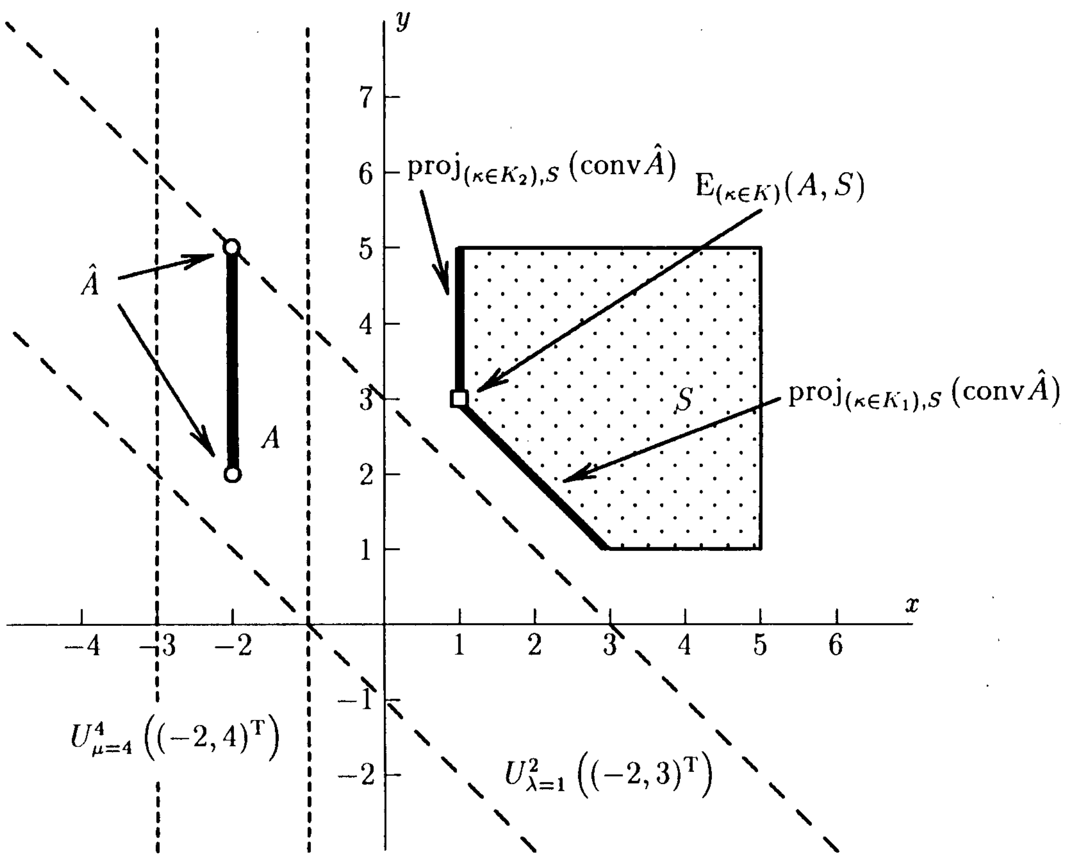


Figure 1: Solutions for $\mathrm{MOP}_\kappa(A,S)$

Remark 4.1 Generally in $\mathbb{R}^n$ proper seminorm families $\{p_\alpha\}_{\alpha\in\mathbb{R}^n}$ are defined through:

$$p_\alpha(v) = |\langle\alpha, v\rangle_{\mathbb{R}^n}|\,,\ \alpha\in\mathbb{R}^n$$

and corresponding families of semiscalar products $\{\langle\cdot,\cdot\rangle_\alpha\}_{\alpha\in\mathbb{R}^n}$ by means of:

$$\langle u,v\rangle_\alpha = u^{\mathrm{T}}(\alpha\alpha^{\mathrm{T}})v, \quad \alpha\in\mathbb{R}^n.$$

Example 4.2 *M, A,* and *S* here are chosen in the following manner:

$$M \ = \ \mathrm{L}^2_{\mathrm{loc}}(\Omega)\,,\ \text{with}\ \Omega = \left(0,\frac{\pi}{2}\right),$$

$$
\begin{aligned}
S &= \left\{ f \in M \,\middle|\, f(x) = \begin{cases} -c\left(x - \frac{\pi}{3}\right) & \text{for } x < \frac{\pi}{3} \\ d\left(x - \frac{\pi}{3}\right) & \text{for } x \geq \frac{\pi}{3} \end{cases} ;\ c, d \in [0,1] \right\}, \\
A &= \left\{ f \in M \,\middle|\, f(x) = t + \tan^2 x, t \in \left[\frac{\pi}{3}, \pi\right] \right\}.
\end{aligned}
$$

M is the space of the equivalence classes of locally quadratic integrable functions and is a proper Hilbert-Fréchet space, i.e. M is a complete Hausdorff locally convex topological vector space, which is metricable but not normable. It holds additionally:

$$
\begin{aligned}
A &\not\subset \mathrm{L}^2(\Omega), \\
\text{but } A &\subset \mathrm{L}^2_{\mathrm{loc}}(\Omega).
\end{aligned}
$$

The family of proper seminorms $\{p_K\}_{K \subset \Omega, K \text{ compact}}$ is given by:

$$
p_K(f) = \left(\int_K |f(x)|^2 \, dx \right)^{\frac{1}{2}}.
$$

The following location problem $\mathrm{MOP}_K(A,S)$ is considered:

$$
\left(\int_K |a - s_0|^2 \, dx \right)^{\frac{1}{2}} \leq \left(\int_K |a - s|^2 \, dx \right)^{\frac{1}{2}} \quad \forall s \in S,\ \forall a \in A,\ \forall K \subset \Omega,\ K \text{ compact}.
$$

At that the set of the ideal solutions for the problem $\mathrm{MOP}_K(A,S)$ is:

$$
\begin{aligned}
\mathrm{M}_K(A,S) &= \mathrm{M}_K(\mathrm{conv}A, S) = \mathrm{proj}_{(K \subset \Omega, K \mathrm{compact}),S}\, A \\
&= \mathrm{proj}_{(K \subset \Omega, K \mathrm{compact}),S}\, (\mathrm{conv}A) = \left\{ f \in M \,\middle|\, f(x) = \left| x - \frac{\pi}{3} \right|, x \in \Omega \right\}.
\end{aligned}
$$

It holds

$$
\begin{aligned}
\mathrm{M}_K(A,S) &= \mathrm{E}_{(K \subset \Omega, K \mathrm{compact})}(A,S) \subset \mathrm{WE}_{(K \subset \Omega, K \mathrm{compact})}(A,S) \\
&= \left\{ f \in M \,\middle|\, \begin{array}{c} \left(f(x) = \begin{cases} -x + \frac{\pi}{3} & \text{for } x < \frac{\pi}{3} \\ d\left(x - \frac{\pi}{3}\right) & \text{for } x \geq \frac{\pi}{3} \end{cases} ;\ d \in [0,1] \right) \vee \\ \left(f(x) = \begin{cases} -c\left(x - \frac{\pi}{3}\right) & \text{for } x < \frac{\pi}{3} \\ x - \frac{\pi}{3} & \text{for } x \geq \frac{\pi}{3} \end{cases} ;\ c \in [0,1] \right),\ x \in \Omega \end{array} \right\}.
\end{aligned}
$$

For the subsets

$$\hat{A} := \left\{ f \in M \,\middle|\, f(x) = e + \tan^2 x, e \in \left\{ \frac{\pi}{3}, \frac{\pi}{2}, \frac{2}{3}\pi, \frac{3}{4}\pi \right\} \right\} \quad \subset \quad A \text{ and}$$

$$I_1 := \left\{ \left[\frac{\pi}{8}, \frac{\pi}{4}\right], \left[\frac{1}{2}, 1\right] \right\},\ I_2 := \left\{ \left[\frac{2}{5}\pi, \frac{4}{9}\pi\right], \left[\frac{6}{5}, \frac{13}{10}\right] \right\} \quad \subset \quad \Omega, \text{ i.e.,}$$

$$\{ \overset{2}{p}_K(f) \}_{K \in I_1} = \left\{ \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} |f(x)|^2 \, dx,\ \int_{\frac{1}{2}}^{1} |f(x)|^2 \, dx \right\},$$

$$\{ \overset{2}{p}_K(f) \}_{K \in I_2} = \left\{ \int_{\frac{2}{5}\pi}^{\frac{4}{9}\pi} |f(x)|^2 \, dx,\ \int_{\frac{6}{5}}^{\frac{13}{10}} |f(x)|^2 \, dx \right\},$$

it is valid according to (2):

$$\begin{aligned}
\text{proj}_{(K \in I_1),S}(\text{conv}\hat{A}) &= K\!-\!\text{E}_{(K \in I_1)}(\hat{A}, S) \ = \ \text{E}_{(K \in I_1)}(\hat{A}, S) \\
&= \left\{ f \in M \,\middle|\, \begin{array}{l} f(x) = \left\{ \begin{array}{lll} -x + \frac{\pi}{3} & \text{for } x < \frac{\pi}{3} \\ d\left(x - \frac{\pi}{3}\right) & \text{for } x \geq \frac{\pi}{3} \end{array} \right. ; \\ d \in [0,1],\ x \in \Omega \end{array} \right\} \\
&\subset \text{WE}_{(K \subset \Omega, K \text{compact})}(A, S),
\end{aligned}$$

$$\begin{aligned}
\text{proj}_{(K \in I_2),S}(\text{conv}\hat{A}) &= K\!-\!\text{E}_{(K \in I_2)}(\hat{A}, S) \ = \ \text{E}_{(K \in I_2)}(\hat{A}, S) \\
&= \left\{ f \in M \,\middle|\, \begin{array}{l} f(x) = \left\{ \begin{array}{lll} -c\left(x - \frac{\pi}{3}\right) & \text{for } x < \frac{\pi}{3} \\ x - \frac{\pi}{3} & \text{for } x \geq \frac{\pi}{3} \end{array} \right. ; \\ c \in [0,1],\ x \in \Omega \end{array} \right\} \\
&\subset \text{WE}_{(K \subset \Omega, K \text{compact})}(A, S),
\end{aligned}$$

$$\begin{aligned}
\text{proj}_{(K \in (I_1 \cup I_2)),S}(\text{conv}\hat{A}) &= K\!-\!\text{E}_{(K \in (I_1 \cup I_2))}(\hat{A}, S) \ = \ \text{E}_{(K \in (I_1 \cup I_2))}(\hat{A}, S) \\
&= \left\{ f \in M \,\middle|\, f(x) = \left| x - \frac{\pi}{3} \right|, x \in \Omega \right\} \\
&= \left\{ f \in M \,\middle|\, f(x) = \left| x - \frac{\pi}{3} \right|, x \in \Omega \right\} \\
&\subset \text{WE}_{(K \subset \Omega, K \text{compact})}(A, S).
\end{aligned}$$

Because of $\text{proj}_{(K \in I_1),S}(\text{conv}\hat{A}) \cup \text{proj}_{(K \in I_2),S}(\text{conv}\hat{A}) = \text{WE}_{(K \subset \Omega, K \text{compact})}(A, S)$ in footnote 5 the equality is also fulfilled.

5 Conclusions and further development

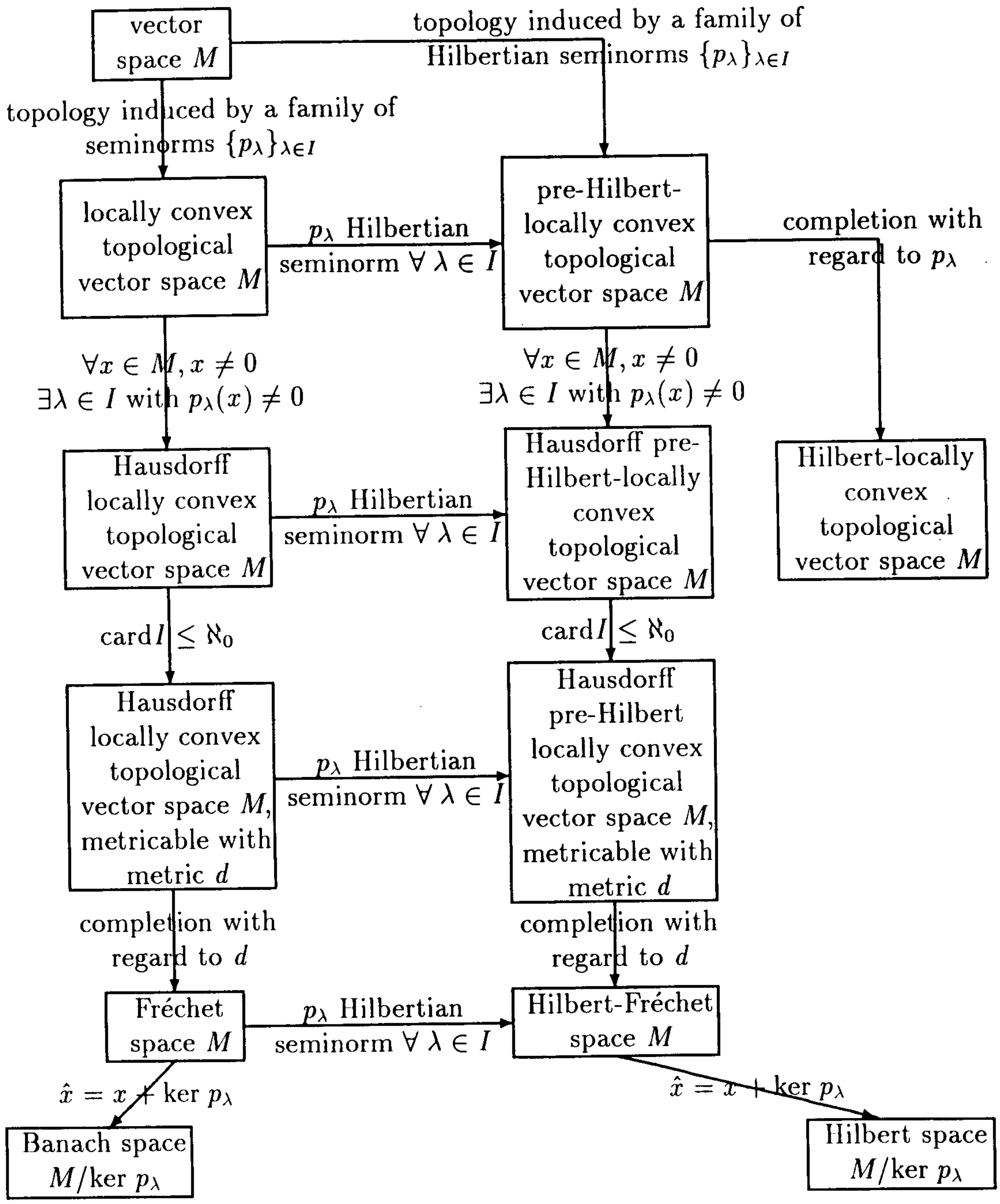
The presented thoughts have shown the existence of definite relations between solutions of single objective location optimization problems and multiobjective location optimization problems in Hausdorff locally convex topological vector spaces. In this case the investigated multiobjective location optimization problem occurs by considering the whole given set for whose elements the best approximation points from another set shall be chosen at the same time. If the single objective location optimization problems are extended to multiobjective location optimization problems, e.g. because families of seminorms are used simultaneously instead of only one seminorm for the single objective location optimization problem, then the ideal solution of this multiobjective location optimization problems are considered.

For single seminorm distances the ideal solutions become again solutions of a single objective location optimization problem. If the seminorms are Hilbertian, i.e, if M is a Hausdorff pre-Hilbert-locally convex topological vector space then the existence of the ideal solutions is equivalent to the existence of the λ-orthogonal projections.

The set of the weakly efficient points of the multiobjective location optimization problem can be bounded below with regard to inclusion relations by certain sets consisting of λ-orthogonal projections from a finite subset of the given set onto the set which contains the searched solutions or by certain weak efficiency or efficiency sets of finite subsets.

Definite assumptions guarantee the existence of the ideal solutions. The future research should follow this direction, because the decision tools about the existence of the ideal solutions are very abstract and not easy to handle. Furthermore the investigations shall be carried out with other distance concepts and in different general spaces.

Appendix



vector space M
topology induced by a family of Hilbertian seminorms $\{p_\lambda\}_{\lambda \in I}$
topology induced by a family of seminorms $\{p_\lambda\}_{\lambda \in I}$
locally convex topological vector space M
p_λ Hilbertian seminorm $\forall \lambda \in I$
pre-Hilbert-locally convex topological vector space M
completion with regard to p_λ
$\forall x \in M, x \neq 0$ $\exists \lambda \in I$ with $p_\lambda(x) \neq 0$
$\forall x \in M, x \neq 0$ $\exists \lambda \in I$ with $p_\lambda(x) \neq 0$
Hausdorff locally convex topological vector space M
p_λ Hilbertian seminorm $\forall \lambda \in I$
Hausdorff pre-Hilbert-locally convex topological vector space M
Hilbert-locally convex topological vector space M
$\mathrm{card} I \leq \aleph_0$
$\mathrm{card} I \leq \aleph_0$
Hausdorff locally convex topological vector space M, metricable with metric d
p_λ Hilbertian seminorm $\forall \lambda \in I$
Hausdorff pre-Hilbert locally convex topological vector space M, metricable with metric d
completion with regard to d
completion with regard to d
Fréchet space M
p_λ Hilbertian seminorm $\forall \lambda \in I$
Hilbert-Fréchet space M
$\hat{x} = x + \ker p_\lambda$
$\hat{x} = x + \ker p_\lambda$
Banach space $M/\ker p_\lambda$
Hilbert space $M/\ker p_\lambda$

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