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Abstract. A general convex multiobjective control- approximation problem is considered with respect to duality. The single objectives contain linear functionals and powers of norms as parts, measuring the distance between the control and the state variables. Moreover, linear inequality restrictions are included. A dual problem is established and weak and strong duality properties as well as necessary and sufficient optimality conditions are derived. So-called point-objective location problems and linear vector optimization problems turn out to be special cases of the investigated problem. Therefore the well-known duality results for linear vector optimization are obtained as special case.

Key Words. Multiobjective optimization, control-approximation problem, point-objective location, duality, optimality conditions.

1. Introduction

The paper deals with duality for multiobjective control-approximation problems, where the single objectives consist of the sum of a linear functional part $\langle l_i^*, u\rangle$ and a power of a norm part $\alpha_i^{n_i}\|x_i - S_i u\|^{n_i}$ measuring the distance between a state variables x_i and the linear mapping $S_i u$ of the control variable u. So the objective function for the i-th objective reads as

$$f_i(x_i, u) = \langle l_i^*, u\rangle + \alpha_i^{n_i}\|x_i - S_i u\|^{n_i}, \; i = 1, \dots, m.$$

The spaces underlying the variables x_i and u are assumed to be normed spaces.
The vectorial function $F(x, u) = (f_1(x_1, u), f_2(x_2, u), \dots, f_m(x_m, u))^T$, $x = (x_1, \dots, x_m)$, has to be minimized with respect to additional restrictions in the sense of multiobjective (multicriterial or vectorial) optimization.

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The basic solution notion is that one of so-called efficiency or Pareto-optimality which we are also considering here. Besides there are well-known some modifications of the definition of solutions for multiobjective optimization problems as e.g. weak efficiency, proper efficiency etc. In the present paper there are considered properly efficient solutions to the original control-approximation problem as well as Pareto-optimal solutions to the introduced dual problem. Scalar problems, i.e. such ones with only one objective function, where the objective function has got the from $\|x - Su\|$ are sometimes called (abstract) control-approximation problems. This is the reason to denote our problems as multiobjective or vectorial control-approximation problems.

Let us refer to some specializations of the problem formulation. If the vectorial objective function takes the from $F(x, u) = (\|x_1 - u\|, \|x_2 - u\|, \ldots, \|x_m - u\|)^T$ the arising vectorial optimization problem is said to be a so-called point-objective location problem (Ref. 1). Here the fixed location points (or demand points) x_i (representing the location of any clients) have to be approximated by a facility located at u, because each client wishes the facility to be as close as possible.

Since the above mentioned first paper by Wendell and Hurter (Ref. 1) a lot of papers have been published concerning different aspects of such multicriterial location problems (cf. Refs. 2-10). Among them there are investigations with respect to numerical algorithms for determination of the set of efficient points (e.g. Refs. 2,3,8), with respect to dominance properties (e.g. Refs. 1,4-6), geometrical properties and characterizations of efficient points (e.g. Refs. 5,9). There also have been studied problems with different norms (round and polyhedral) and so-called gauges, a generalization of norms to the non-symmetric case of distance measures (cf. Ref. 5,10). Since the late eighties there are considerations concerning multiobjective duality for such multiobjective location problems (cf. Refs. 11-14).

As well-known from scalar programming duality represents an useful tool also in vectorial optimization and it has got its own meaning for getting insight into the properties and structure of optimization problems.

Especially, duality can be used to obtain optimality conditions, bounds for the objective function values, geometrical characterizations and for the construction of numerical algorithms.

In recent years there has been created an extensive literature in this field. We only refer to the book by Jahn (Ref. 15).

The present paper generalizes the problems and results of the papers with Refs. 11-14.

Especially, we consider more general objective functions (powers of norms) restrictions and spaces. Moreover, we derive (different from Refs. 11-14) necessary and sufficient optimality conditions characterizing properly efficient and efficient solutions, respectively. In comparison with general and abstract duality concepts in vector optimization (cf. e.g. Ref. 15) we use the special structure of the control-approximation problem to establish the dual problem and to prove the weak and strong duality assertions and the optimality conditions.
Particularly, the results also contain as special case (omitting the norm parts within the objective function and simplifying the restrictions) the dual problem and the corresponding duality properties for the linear vector optimization problem (cf. Refs. 15-17).

2. Problem Formulation

We are looking for so-called properly efficient solutions of the objective set $\{F(x,u) : (x,u,v) \in A\}$ (we call this problem (P)) with the vectorial objective function

$$F(x,u) = \begin{pmatrix} f_1(x,u) \\ \vdots \\ f_m(x,u) \end{pmatrix} = \begin{pmatrix} \langle l_1^*, u \rangle \\ \vdots \\ \langle l_m^*, u \rangle \end{pmatrix} + \begin{pmatrix} \alpha_1^{n_1} \|x_1 - S_1 u\|_{X_1}^{n_1} \\ \vdots \\ \alpha_m^{n_m} \|x_m - S_m u\|_{X_m}^{n_m} \end{pmatrix} \qquad (1)$$

and the restriction set

$$A = \left\{ (x,u,v) : u \underset{\hat{K}}{\geq} 0,\ v \underset{K_0}{\geq} 0,\ Bu + Cv + f \underset{K_1}{\leq} 0,\ x_i \in W_i,\ i = 1,\dots,m \right\}. \qquad (2)$$

Here we define $x = (x_1, \dots, x_m)$, $\alpha_i \geq 0$, $n_i \geq 1$, $i = 1, \dots, m$. Moreover let U, Y, Z, X_i, $i = 1, \dots, m$, be normed spaces, $\hat{K} \subseteq U$, $K_0 \subseteq Y$, $K_1 \subseteq Z$ are assumed to be convex closed cones. $W_i \subset X_i$, $i = 1 \dots, m$, denote convex closed sets. Furthermore, it should be $(i = 1, \dots, m)$ $S_i \in L(U, X_i)$ (linear continuous mapping from U to X_i) $B \in L(U,Z)$, $C \in L(Y,Z)$, $f \in Z$, $l_i^* \in U^*$ (linear continuous functional, i.e. U^* denotes the topological dual space to U). $\|\cdot\|_{X_i}$ stands for a norm in X_i. E.g. $\langle l_i^*, u \rangle$ means the linear continuous functional l_i^* at point u. The convex cones generate partial orderings in the usual way, e.g. $u \underset{\hat{K}}{\geq} 0$ means $u \in \hat{K}$.

A tupel $(x,u,v) \in A$ is said to be (primal) admissible. The following definition states our present solution notion for (P).

Definition 2.1. An admissible point $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v})$ is said to be properly efficient to (P) if there exists a scalarizing vector $\overset{o}{\lambda} = (\overset{o}{\lambda}_1, \ldots, \overset{o}{\lambda}_m)^T, \overset{o}{\lambda}_i > 0, i = 1, \ldots, m$, such that

$$\overset{o}{\lambda}{}^T F(\overset{o}{x}, \overset{o}{u}) \leq \overset{o}{\lambda}{}^T F(x, u)$$

for all admissible points (x, u, v).

Of course, a properly efficient point (or solution) is also Pareto- optimal (or efficient) in the usual sense (cf. Ref. 18). Now let us set another multiobjective optimization problem (P^*) which later turns out to be as a dual problem to (P) according to its properties in relation to (P).

Let

$$G(p^*, \delta^*) = \begin{pmatrix} g_1(p^*, \delta^*) \\ \vdots \\ g_m(p^*, \delta^*) \end{pmatrix} \tag{3}$$

be the vectorial objective function to (P^*) with $p^* = (p_1^*, \ldots, p_m^*), p_i^* \in X_i^*, \delta^* = (\delta_1^*, \ldots, \delta_m^*)$, $\delta_i^* \in Z^*$ (X_i^*, Z^* are the topological dual spaces to X_i and Z^*, respectively), $i = 1, \ldots, m$. The coordinate functions of G read as

$$g_i(p^*, \delta^*) = \begin{cases} \inf\limits_{y_i \in W_i} \alpha_i \langle n_i p_i^*, y_i \rangle + (1 - n_i) \|p_i^*\|_{X_i^*}^{\frac{n_i}{n_i - 1}} - \langle \delta_i^*, f \rangle \text{ for } n_i > 1,\ \alpha_i > 0, \\ \\ \inf\limits_{y_i \in W_i} \alpha_i \langle p_i^*, y_i \rangle - \langle \delta_i^*, f \rangle \text{ for } n_i = 1 \text{ or } \alpha_i = 0. \end{cases} \tag{4}$$

The dual restriction set is given by

$$\begin{aligned} \hat{B} = \Big\{ (p^*, \delta^*) : \|p_i^*\|_{X_i^*} \leq 1 \text{ for } n_i = 1,\ \exists\, \lambda = (\lambda_1, \ldots, \lambda_m)^T, \\ \lambda_i > 0, \text{ with } \sum_{i=1}^m \lambda_i \delta_i^* \underset{K_1^*}{\leq} 0,\ \sum_{i=1}^m \lambda_i C^* \delta_i^* \underset{K_0^*}{\leq} 0, \\ \sum_{i=1}^m \lambda_i (n_i \alpha_i S_i^* p_i^* - l_i^* + B^* \delta_i^*) \underset{\hat{K}^*}{\leq} 0 \Big\}. \end{aligned} \tag{5}$$

Here $\|\cdot\|_{X_i^*}$ denotes the dual norm to $\|\cdot\|_{X_i}$ in X_i^*. The space indices will be ommited in the following. A point $(p^*, \delta^*) \in \hat{B}$ is said to be dual admissible. With K^* we denote the dual cone to $K \subset X$ (X is assumed to be a normed space) defined by

$$K^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0\, \forall x \in K\}.$$

K^* then generates a dual partial ordering in X^* in the usual way as mentioned before for K. Such orderings occur in the definition of the set $\hat{B}$ yielding inequality contraints.

T^* denotes the adjoint operator to the operator T. We consider Pareto-optimal (efficient) points (solutions) with respect to maximization.

Definition 2.2. A point $(\overset{o}{p}^*, \overset{o}{\delta}^*) \in \hat{B}$ is said to be efficient if there is no $(p^*, \delta^*) \in \hat{B}$ with $g_i(p^*, \delta^*) \geq g_i(\overset{o}{p}^*, \overset{o}{\delta}^*)$, $i = 1, \ldots, m$, and $g_j(p^*, \delta^*) > g_j(\overset{o}{p}^*, \overset{o}{\delta}^*)$ for at least one $j \in \{1, \ldots, m\}$.

The presented dual problem (P^*) is therefore a vectorial maximation problem and we will represent it shortly and in formal way as

$$(P^*) \quad G(p^*, \delta^*) \rightarrow \underset{(p^*, \delta^*) \in \hat{B}.}{v - \max}$$

Analogously the primal problem (P) should be represented in the short form

$$(P) \quad F(x, u) \rightarrow \underset{(x, u, v) \in A.}{v - \min}$$

3. Weak Duality

The following theorem expresses a connection between the both multiobjective optimization problems (P) and (P^*) which is usually refered to as weak duality. This property is also the reason that entitles us to call (P^*) as dual to (P).

Theorem 3.1. There are no points $(x, u, v) \in A$ (primal admissible) and $(p^*, \delta^*) \in \hat{B}$ (dual adimissible) such that

$$\begin{aligned} g_i(p^*, \delta^*) &\geq f_i(x, u), \quad i = 1, \ldots, m, \text{ and} \\ g_j(p^*, \delta^*) &> f_j(x, u) \end{aligned} \tag{6}$$

for at least one $j \in \{1, \ldots, m\}$.

The property claimed in theorem 3.1. thus generalizes in a natural way the weak duality relation for scalar optimization problems, namely the situation that the values of the dual goal function never exceed the values of the primal goal function, when the primal problem should be a minimum problem and the dual therefore turns out to be a maximum problem. From this explanation it is understandable to speake of weak duality with respect to the assertion of theorem 3.1.

Now we come to the proof of theorem 3.1.

Proof. Let us assume that the statement of theorem 3.1. does not hold.
Then there exist $(x,u,v) \in A$ and $(p^*,\delta^*) \in \hat{B}$ with

$$f_i(x,u) = g_i(p^*,\delta^*) - k_i,\ i = 1,\ldots,m,$$

where $k_i \geq 0$ and $k_j > 0$ for at least one $j \in \{1,\ldots,m\}$. According to the definition of the restriction set $\hat{B}$ numbers $\lambda_i > 0$, $1,\ldots,m$, are assigned to $(p^*,\delta^*) \in \hat{B}$.

With these numbers we get therefore

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i f_i(x,u) &= \sum_{i=1}^{m} \lambda_i g_i(p^*,\delta^*) - \sum_{i=1}^{m} \lambda_i k_i \\ &< \sum_{i=1}^{m} \lambda_i g_i(p^*,\delta^*) \end{aligned} \tag{7}$$

because of $\sum_{i=1}^{m} \lambda_i k_i > 0$.

Forthcoming we are showing the validity of the inverse inequality to (7) yielding a contradiction to our assumption.

From (5) it results

$$\sum_{i=1}^{m} \lambda_i \langle n_i \alpha_i S_i^* p_i^* - l_i^* + B^* \delta_i^*,\ u\rangle \leq 0 \quad \text{as well as} \quad \sum_{i=1}^{m} \lambda_i \langle C^* \delta_i^*,\ v\rangle \leq 0.$$

This leads to the estimate

$$\begin{aligned} &\sum_{i=1}^{m} \lambda_i f_i(x,u) = \sum_{i=1}^{m} \lambda_i (\alpha_i^{n_i} \|x_i - S_i u\|^{n_i} + \langle l_i^*, u\rangle) \geq \\ &\sum_{i=1}^{m} \lambda_i (\alpha_i^{n_i} \|x_i - S_i u\|^{n_i} + \langle l_i^*, u\rangle + \langle n_i \alpha_i S_i^* p_i^* - l_i^* + B^* \delta_i^*, u\rangle + \langle C^* \delta_i^*, v\rangle). \end{aligned} \tag{8}$$

We note down

$$\begin{aligned} \langle \alpha_i p_i^*,\ x_i - S_i u\rangle &\leq \|\alpha_i p_i^*\|\ \|x_i - S_i u\| \\ &= \|p_i^*\|(\alpha_i \|x_i - S_i u\|) \end{aligned}$$

and estimate the right hand side by means of the inequality

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q},\ a,b \geq 0,\ \frac{1}{p} + \frac{1}{q} = 1.$$

For $n_i > 1$ we set $p = n_i$, $q = \dfrac{p}{p-1} = \dfrac{n_i}{n_i - 1}$ and so we get after multiplication by n_i

$$\langle n_i \alpha_i p_i^*,\ x_i - S_i u\rangle \leq \alpha_i^{n_i} \|x_i - S_i u\|^{n_i} + (n_i - 1)\|p_i^*\|^{\frac{n_i}{n_i - 1}}.$$

Moreover, we use for $n_i = 1$

$$\alpha_i \|x_i - S_i u\| \geq \langle \alpha_i p_i^*,\, x_i - S_i u \rangle$$

since $\|p_i^*\| \leq 1$ for $n_i = 1$.

By this (8) can be continued following

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i f_i(x,u) \;&\geq\; \sum_{\substack{i=1 \\ n_i>1}}^{m} \lambda_i (1-n_i)\|p_i^*\|^{\frac{n_i}{n_i-1}} \\
&+\; \sum_{i=1}^{m} \lambda_i \left(\langle n_i \alpha_i p_i^*,\, x_i - S_i u\rangle + \langle n_i \alpha_i p_i^*,\, S_i u\rangle + \langle B^*\delta_i^*,\, u\rangle + \langle C^*\delta_i^*, v\rangle \right) \\
&=\; \sum_{\substack{i=1 \\ n_i>1}}^{m} \lambda_i (1-n_i)\|p_i^*\|^{\frac{n_i}{n_i-1}} + \sum_{i=1}^{m} \lambda_i \langle \alpha_i n_i p_i^*,\, x_i\rangle \\
&+\; \sum_{i=1}^{m} \lambda_i \langle \delta_i^*,\, Bu + Cv\rangle.
\end{aligned} \tag{9}$$

Because of $Bu + Cv + f \underset{K_1}{\leq} 0$ and $\sum_{i=1}^{m} \lambda_i \delta_i^* \underset{K_1^*}{\leq} 0$ we have

$$\left\langle \sum_{i=1}^{m} \lambda_i \delta_i^*,\, Bu + Cv \right\rangle \geq \left\langle \sum_{i=1}^{m} \lambda_i \delta_i^*,\, -f \right\rangle.$$

We substitute this in (9) and obtain

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i f_i(x,u) \;&\geq\; \sum_{i=1}^{m} \lambda_i (1-n_i)\|p_i^*\|^{\frac{n_i}{n_i-1}} + \sum_{i=1}^{m} \lambda_i \left(\langle \alpha_i n_i p_i^*,\, x_i\rangle - \langle \delta_i^*,\, f\rangle \right) \\
&\geq\; \sum_{i=1}^{m} \lambda_i (1-n_i)\|p_i^*\|^{\frac{n_i}{n_i-1}} + \sum_{i=1}^{m} \lambda_i \Big(\inf_{y_i \in W_i} \langle \alpha_i n_i p_i^*,\, y_i\rangle - \langle \delta_i^*,\, f\rangle \Big) \\
&=\; \sum_{i=1}^{m} \lambda_i g_i(p^*,\, \delta^*).
\end{aligned}$$

This contradicts to (7) and finishes off the proof. □

4. Strong Duality

Coming from weak duality we refer to strong duality if there is an identity between certain primal and dual objective values analogously as in scalar optimization. Obviously these values then yield efficient points (i.e. solutions) to the both dual problems. The following theorem claims such a strong duality behaviour and turns out to be the main theorem of this paper.

Theorem 4.1. Let the existence of $(\bar{x}, \bar{u}, \bar{v}) \in A$ be assumed with the regularity condition $B\bar{u} + C\bar{v} + f \in \text{int}\,(-K_1)$. Furthermore it is supposed that $f \neq 0$.
Let $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v}) \in A$ be a properly efficient point to (P). Then there exists an efficient solution $(\overset{o}{p}{}^*, \overset{o}{\delta}{}^*) \in \hat{B}$ to (P^*) and it holds the strong duality relation

$$F(\overset{o}{x}, \overset{o}{u}) = G(\overset{o}{p}{}^*, \overset{o}{\delta}{}^*).$$

For $(\overset{o}{p}{}^*, \overset{o}{\delta}{}^*)$ there are the representation formulae (13), (20), with $(\overset{o}{p}{}^*, \overset{o}{\gamma}{}^*)$ as solution to $(P^*_{\overset{o}{\lambda}})$ (cf. (11)), the scalar dual problem to $(P_{\overset{o}{\lambda}})$ (cf. (10)).

Proof. Let $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v})$ be a properly efficient solution to (P). Then there exists an assigned scalarizing vector $\overset{o}{\lambda} = (\overset{o}{\lambda}_1, \ldots, \overset{o}{\lambda}_m)^T$, $\overset{o}{\lambda}_i > 0$, $i = 1, \ldots, m$, such that

$$\overset{o}{\lambda}{}^T\, F(x,u) \geq \overset{o}{\lambda}{}^T\, F(\overset{o}{x}, \overset{o}{u})$$

for all $(x, u, v) \in A$.

Therefore $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v})$ is a solution to the scalar control-approximation problem (cf. (1) for $F(x,u)$)

$$(P_{\overset{o}{\lambda}}) \quad \inf_{(x,u,v)\in A} \sum_{i=1}^{m} \overset{o}{\lambda}_i \left(\alpha_i^{n_i} \|x_i - S_i u\|^{n_i} + \langle l_i^*, u\rangle\right). \tag{10}$$

We assign to $(P_{\overset{o}{\lambda}})$ the following dual problem

$$(P^*_{\overset{o}{\lambda}}) \quad \sup_{(p^*,\gamma^*)\in D} \left\{ \sum_{\substack{i=1\\ n_i>1,\,\alpha_i>0}}^{m} \overset{o}{\lambda}_i\,(1-n_i)\|p_i^*\|^{\frac{n_i}{n_i-1}} + \sum_{i=1}^{m} \overset{o}{\lambda}_i \inf_{y_i\in W_i} \langle \alpha_i n_i p_i^*,\, y_i\rangle - \langle \gamma^*,\, f\rangle \right\}, \tag{11}$$

where $p^* = (p_1^*, \ldots, p_m^*)$, $p_i^* \in X_i^*$, $i = 1, \ldots, m$, $\gamma^* \in Z^*$, and the dual admissible set is given by

$$D = \left\{ (p^*, \gamma^*) \;:\; \|p_i^*\| \leq 1 \text{ for } n_i = 1,\ \gamma^* \underset{K_1^*}{\leq} 0,\ C^*\gamma^* \underset{K_0^*}{\leq} 0,\quad \sum_{i=1}^{m} \overset{o}{\lambda}_i\,(\alpha_i n_i S_i^* p_i^* - l_i^*) + B^*\gamma^* \underset{\hat{K}^*}{\leq} 0 \right\}. \tag{12}$$

We are entitled to refer to $(P^*_{\overset{o}{\lambda}})$ as dual problem because of the property of $(P_{\overset{o}{\lambda}})$ and $(P^*_{\overset{o}{\lambda}})$ that the infimum of $(P_{\overset{o}{\lambda}})$ is greater than or equal to the supremum of $(P^*_{\overset{o}{\lambda}})$. This weak duality can be verified by a direct estimation in a similar way as done in Ref. 19 for an analogous problem. There are no convexity assumptions necessary for this assertion. On the base of the regularity condition, as formulated in the theorem, it can be shown as in Ref. 19 for the modified problem that strong duality is fulfilled and that there exists a solution $(\overset{o}{p}^*, \overset{o}{\gamma}^*)$ to the dual problem $(P^*_{\overset{o}{\lambda}})$, i.e. it holds $\min (P_{\overset{o}{\lambda}}) = \max(P^*_{\overset{o}{\lambda}})$. This reads as

$$\sum_{i=1}^{m} \overset{o}{\lambda}_i \left(\alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} + \langle l_i^*, \overset{o}{u}\rangle\right) = \sum_{\substack{i=1 \\ n_i>1, \alpha_i>0}}^{m} \overset{o}{\lambda}_i (1-n_i) \| \overset{o}{p}_i^* \|^{\frac{n_i}{n_i-1}} + \sum_{i=1}^{m} \overset{o}{\lambda}_i \inf_{y_i \in W_i} \langle \alpha_i n_i \overset{o}{p}_i^*, y_i \rangle - \langle \overset{o}{\gamma}^*, f\rangle. \tag{13}$$

Using strong duality, especially equation (13), one can derive (cf. Ref. 20) the following optimality conditions for the solutions $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v})$ and $(\overset{o}{p}^*, \overset{o}{\gamma}^*)$ to $(P_{\overset{o}{\lambda}})$ and $(P^*_{\overset{o}{\lambda}})$, respectively.

i) $n_i > 1,\ \alpha_i > 0:$

$$\begin{aligned} &\langle \overset{o}{p}_i^*, \overset{o}{x}_i - S_i \overset{o}{u}\rangle = \alpha_i^{n_i-1} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i}, \\ &\| \overset{o}{p}_i^* \| = \alpha_i^{n_i-1} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i-1}, \end{aligned} \tag{14}$$

$n_i = 1,\ \alpha_i > 0:$

$$\langle \overset{o}{p}_i^*, \overset{o}{x}_i - S_i \overset{o}{u}\rangle = \| \overset{o}{x}_i - S_i \overset{o}{u} \|, \| \overset{o}{p}_i^* \| = 1 \text{ if } \overset{o}{x}_i \neq S_i \overset{o}{u}, \tag{15}$$

ii) $$\langle \overset{o}{p}_i^*, \overset{o}{x}_i \rangle = \inf_{y_i \in W_i} \langle \overset{o}{p}_i^*, y_i \rangle, \tag{16}$$

iii) $$\langle C^* \overset{o}{\gamma}^*, \overset{o}{v}\rangle = 0, \tag{17}$$

iv) $$\left\langle \sum_{i=1}^{m} \overset{o}{\lambda}_i (n_i \alpha_i S_i^* \overset{o}{p}_i^* - l_i^*) + B^* \overset{o}{\gamma}^*, \overset{o}{u} \right\rangle = 0, \tag{18}$$

v) $$\langle \overset{o}{\gamma}^*, B \overset{o}{u} + f \rangle = 0. \tag{19}$$

Now we define by means of $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v})$ and $(\overset{o}{p}^*, \overset{o}{\gamma}^*)$ an admissible point $(\overset{o}{p}^*, \overset{o}{\delta}^*)$, $\overset{o}{\delta}^* = (\overset{o}{\delta}_1^*, \ldots, \overset{o}{\delta}_m^*)$, to the vectorial dual problem (P^*) which turns out to be an efficient solution to (P^*) fulfilling the strong vectorial duality relation $F(\overset{o}{x}, \overset{o}{u}) = G(\overset{o}{p}^*, \overset{o}{\delta}^*)$.

$$\overset{o}{\delta}_i^* = \begin{cases} \dfrac{\langle \alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*, \overset{o}{u}\rangle}{\langle \overset{o}{\gamma}^*, f\rangle} \overset{o}{\gamma}^* & \text{for } \langle \overset{o}{\gamma}^*, f \rangle \neq 0, \\ \dfrac{1}{m \overset{o}{\lambda}_i} \overset{o}{\gamma}^* + \langle \alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*, \overset{o}{u}\rangle \tilde{\gamma}^* & \\ \text{with } \tilde{\gamma}^* \in Z^* : \langle \tilde{\gamma}^*, f \rangle = 1 & \text{for } \langle \overset{o}{\gamma}^*, f \rangle = 0. \end{cases} \tag{20}$$

Remark 4.1. Such a $\tilde{\gamma}^*$ exists due to the Hahn-Banach continuation theorem.
In the next steps we show that $F(\overset{o}{x}, \overset{o}{u}) = G(\overset{o}{p}^*, \overset{o}{\delta}^*)$. We have to distinguish between several cases.

1. Let be $\langle \overset{o}{\gamma}^*, f\rangle \neq 0$, $n_i > 1$, $\alpha_i > 0$. Because of (3), (4), (14), (16) and (20) it holds

$$\begin{aligned} g_i(\overset{o}{p}^*, \overset{o}{\delta}^*) &= \inf_{y_i \in W_i} \alpha_i \langle n_i \overset{o}{p}_i^*, y_i\rangle + (1-n_i) \| \overset{o}{p}_i^* \|^{\frac{n_i}{n_i - 1}} - \langle \overset{o}{\delta}_i^*, f\rangle \\ &= \alpha_i \langle n_i \overset{o}{p}_i^*, \overset{o}{x}_i\rangle + (1-n_i)\alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} \\ &\quad - \frac{1}{\langle \overset{o}{\gamma}^*, f\rangle} \langle \alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*, \overset{o}{u}\rangle \langle \overset{o}{\gamma}^*, f\rangle \\ &= \langle \alpha_i n_i \overset{o}{p}_i^*, \overset{o}{x}_i - S_i \overset{o}{u}\rangle + (1-n_i)\alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} + \langle l_i^*, \overset{o}{u}\rangle \\ &= n_i \alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} + (1-n_i)\alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} + \langle l_i^*, \overset{o}{u}\rangle \\ &= \alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} + \langle l_i^*, \overset{o}{u}\rangle \\ &= f_i(\overset{o}{x}, \overset{o}{u}). \end{aligned}$$

2. Let be $\langle \overset{o}{\gamma}^*, f\rangle \neq 0$, $n_i = 1$ or $\alpha_i = 0$. Then with (15) and (16) it follows analogously to 1.

$$\begin{aligned} g_i(\overset{o}{p}^*, \overset{o}{\delta}^*) &= \inf_{y_i \in W_i} \alpha_i \langle \overset{o}{p}_i^*, y_i\rangle - \langle \overset{o}{\delta}_i^*, f\rangle \\ &= \alpha_i \langle \overset{o}{p}_i^*, \overset{o}{x}_i\rangle - \alpha_i \langle \overset{o}{p}_i^*, S_i \overset{o}{u}\rangle + \langle l_i^*, \overset{o}{u}\rangle \\ &= \alpha_i \| \overset{o}{x}_i - S_i \overset{o}{u} \| + \langle l_i^*, \overset{o}{u}\rangle \\ &= f_i(\overset{o}{x}, \overset{o}{u}). \end{aligned}$$

3. Let be $\langle \overset{o}{\gamma}^*, f\rangle = 0$, $n_i > 1$, $\alpha_i > 0$. This implies as for 1.

$$\begin{aligned} g_i(\overset{o}{p}^*, \overset{o}{\delta}^*) &= \alpha_i \langle n_i \overset{o}{p}_i^*, \overset{o}{x}_i\rangle + (1-n_i)\alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} \\ &\quad - \left\langle \frac{1}{m \overset{o}{\lambda}_i} \overset{o}{\gamma}^* + \langle \alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*, \overset{o}{u}\rangle \tilde{\gamma}^*, f \right\rangle \\ &= \alpha_i n_i \langle \overset{o}{p}_i^*, \overset{o}{x}_i\rangle + (1-n_i)\alpha_i^{n_i} \| \overset{o}{x}_i - S_i \overset{o}{u} \|^{n_i} \\ &\quad - \frac{1}{m \overset{o}{\lambda}_i} \langle \overset{o}{\gamma}^*, f\rangle - \langle \alpha_i n_i \overset{o}{p}_i^*, S_i \overset{o}{u}\rangle \langle \tilde{\gamma}^*, f\rangle + \langle l_i^*, \overset{o}{u}\rangle \langle \tilde{\gamma}^*, f\rangle. \end{aligned}$$

   Because of $\langle \tilde{\gamma}^*, f\rangle = 1$ and $\langle \overset{o}{\gamma}^*, f\rangle = 0$ the assertion results after further calculations as in 1.

4. Let be $\langle \overset{o}{\gamma}^*, f\rangle = 0$, $n_i = 1$ or $\alpha_i = 0$. The assertion follows analogously as in 2. using transformations as in 3.

Now we examine the admissibility of $(\overset{o}{p}^*, \overset{o}{\delta}^*)$ for (P^*), i.e. we have to show that $(\overset{o}{p}^*, \overset{o}{\delta}^*) \in \hat{B}$ (cf.(5)).

For $n_i = 1$ applies $\| \overset{o}{p}_i^* \| \leq 1$ since $(\overset{o}{p}^*, \overset{o}{\gamma}^*)$ solves $(P^*_{\overset{o}{\lambda}})$ (cf. (11) and (12)).

We need to compute $\sum\limits_{i=1}^{m} \overset{o}{\lambda}_i \overset{o}{\delta}_i^*$ and have to distinguish according to (20) between two cases.

Let firstly be $\langle \overset{o}{\gamma}^*, f\rangle \neq 0$. Then (17), (18) and (19) yield

$$\begin{aligned}
\sum_{i=1}^{m} \overset{o}{\lambda}_i \overset{o}{\delta}_i^* &= \sum_{i=1}^{m} \overset{o}{\lambda}_i \frac{1}{\langle \overset{o}{\gamma}^*, f\rangle} \langle \alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*, \overset{o}{u}\rangle \overset{o}{\gamma}^* \\
&= \frac{1}{\langle \overset{o}{\gamma}^*, f\rangle} \left\langle \sum_{i=1}^{m} \overset{o}{\lambda}_i (\alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*), \overset{o}{u} \right\rangle \overset{o}{\gamma}^* \\
&= \frac{1}{\langle \overset{o}{\gamma}^*, f\rangle} (-\langle B^* \overset{o}{\gamma}^*, \overset{o}{u}\rangle) \overset{o}{\gamma}^* = \frac{1}{\langle \overset{o}{\gamma}^*, f\rangle} \langle \overset{o}{\gamma}^*, C\overset{o}{v} + f\rangle \overset{o}{\gamma}^* = \overset{o}{\gamma}^* .
\end{aligned}$$

For the second case $\langle \overset{o}{\gamma}^*, f\rangle = 0$ there is the same result:

$$\begin{aligned}
\sum_{i=1}^{m} \overset{o}{\lambda}_i \overset{o}{\delta}_i^* &= \sum_{i=1}^{m} \overset{o}{\lambda}_i \frac{1}{m\overset{o}{\lambda}_i} \overset{o}{\gamma}^* + \sum_{i=1}^{m} \overset{o}{\lambda}_i \langle \alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*, \overset{o}{u}\rangle \tilde{\gamma}^* \\
&= \overset{o}{\gamma}^* + \left\langle \sum_{i=1}^{m} \overset{o}{\lambda}_i (\alpha_i n_i S_i^* \overset{o}{p}_i^* - l_i^*), \overset{o}{u} \right\rangle \tilde{\gamma}^* \\
&= \overset{o}{\gamma}^* - \langle B^* \overset{o}{\gamma}^*, \overset{o}{u}\rangle \tilde{\gamma}^* = \overset{o}{\gamma}^* + \langle \overset{o}{\gamma}^*, C\overset{o}{v} + f\rangle \tilde{\gamma}^* \\
&= \overset{o}{\gamma}^* + \langle C^* \overset{o}{\gamma}^*, \overset{o}{v}\rangle \tilde{\gamma}^* + \langle \overset{o}{\gamma}, f\rangle \tilde{\gamma}^* = \overset{o}{\gamma}^* .
\end{aligned}$$

Therefore we have in both cases that

$$\sum_{i=1}^{m} \overset{o}{\lambda}_i \overset{o}{\delta}_i^* = \overset{o}{\gamma}^* . \tag{21}$$

With $(\overset{o}{p}^*, \overset{o}{\gamma}^*) \in D$ (cf. (12)) and (21) one recognizes that $(\overset{o}{p}^*, \overset{o}{\gamma}^*) \in \hat{B}$.

From $F(\overset{o}{x}, \overset{o}{u}) = G(\overset{o}{p}^*, \overset{o}{\delta}^*)$ and the weak duality assertion of theorem 3.1. obviously follows the efficiency of $(\overset{o}{p}^*, \overset{o}{\delta}^*)$ to (P^*). □

As a detailed consideration of this proof shows, necessary optimality conditions can be given for the properly efficient solutions to (P). These turn out to be even sufficient for a properly efficient solution and the existence of an efficient solution to the vectorial dual problem (P^*) as well as for the strong duality property. This is summarized in the following theorem.

Theorem 4.2.

1. Under the assumption of theorem 4.1. let $(\overset{o}{x},\overset{o}{u},\overset{o}{v})$ and $(\overset{o}{p}^*,\overset{o}{\delta}^*)$ be associated properly efficient and efficient solutions to (P) and (P^*), respectively, with the corresponding scalarizing vector $\overset{o}{\lambda}$ (i.e. especially $F(\overset{o}{x},\overset{o}{u}) = G(\overset{o}{p}^*,\overset{o}{\delta}^*)$ according to theorem 4.1.). Then $(\overset{o}{x},\overset{o}{u},\overset{o}{v})$ and $(\overset{o}{p}^*,\overset{o}{\delta}^*)$ satisfy the following necessary optimality conditions:

   i) *for* $n_i > 1,\ \alpha_i > 0$:

   $$\langle \overset{o}{p}{}^*_i, \overset{o}{x}_i - S_i\,\overset{o}{u}\rangle = \alpha_i^{n_i-1} \| \overset{o}{x}_i - S_i\,\overset{o}{u}\|^{n_i}, \quad \| \overset{o}{p}{}^*_i \| = \alpha_i^{n_i-1} \| \overset{o}{x}_i - S_i\,\overset{o}{u}\|^{n_i-1},$$

   for $n_i = 1,\ \alpha_i > 0$:

   $$\langle \overset{o}{p}{}^*_i, \overset{o}{x}_i - S_i\,\overset{o}{u}\rangle = \| \overset{o}{x}_i - S_i\,\overset{o}{u}\| \ \textit{and}\ \| \overset{o}{p}{}^*_i \| = 1 \ \textit{if}\ \overset{o}{x}_i \neq S_i\,\overset{o}{u},$$

   ii) $\langle \overset{o}{p}{}^*_i, \overset{o}{x}_i\rangle = \min\limits_{y_i \in W_i} \langle \overset{o}{p}{}^*_i, y_i\rangle,$

   iii) $\left\langle \sum\limits_{i=1}^{m} \overset{o}{\lambda}_i\, C^*\, \overset{o}{\delta}{}^*_i, \overset{o}{v}\right\rangle = 0,$

   iv) $\left\langle \sum\limits_{i=1}^{m} \overset{o}{\lambda}_i\, (n_i\alpha_i S_i^*\, \overset{o}{p}{}^*_i - l_i^* + B^*\, \overset{o}{\delta}{}^*), \overset{o}{u}\right\rangle = 0,$

   v) $\left\langle \sum\limits_{i=1}^{m} \overset{o}{\lambda}_i \overset{o}{\delta}{}^*_i,\ B\,\overset{o}{u} + C\,\overset{o}{v} + f\right\rangle = 0.$

2. Let there exist admissible $(\overset{o}{x},\overset{o}{u},\overset{o}{v})$ for (P) and $(\overset{o}{p}^*,\tilde{\delta}^*)$ for (P^*) with an associated $\overset{o}{\lambda} = (\overset{o}{\lambda}_1,\ldots,\overset{o}{\lambda}_m)^T$, $\overset{o}{\lambda}_i > 0$, $i = 1,\ldots,m$ (cf. definition 2.1.) such that the conditions i), ..., v) are fulfilled.
   Then $(\overset{o}{x},\overset{o}{u},\overset{o}{v})$ is properly efficient to (P) and there exists an efficient element $(\overset{o}{p}^*,\overset{o}{\delta}^*)$ for (P^*). The functional $\overset{o}{\delta}^*$ has the representation (20) with $\overset{o}{\gamma}^* = \sum\limits_{i=1}^{m} \overset{o}{\lambda}_i\, \tilde{\delta}^*_i$.
   Moreover, it applies strong duality between $(\overset{o}{x},\overset{o}{u},\overset{o}{v})$ and $(\overset{o}{p}^*,\tilde{\delta}^*)$, i.e. $F(\overset{o}{x},\overset{o}{u}) = G(\overset{o}{p}^*,\tilde{\delta}^*)$. A regularity condition as in 1. is not required.

Proof. By means of (21) 1. follows immediately from (14), ..., (19).
To verify 2. let be $(\overset{o}{x},\overset{o}{u},\overset{o}{v}) \in A$ and $(\overset{o}{p}^*,\tilde{\delta}^*) \in \hat{B}$ and i), ..., v) in 1. are assumed to be fulfilled (with $\tilde{\delta}^*$ instead of $\overset{o}{\delta}^*$). We define $\overset{o}{\gamma}^* = \sum\limits_{i=1}^{m} \overset{o}{\lambda}_i\, \tilde{\delta}^*_i$. Then $(\overset{o}{p}^*,\overset{o}{\gamma}^*) \in D$ (admissible for $(P^*_{\overset{o}{\lambda}})$) and (14), ..., (19) are satisfied. Due to a result in Ref. 20 concerning necessary and sufficient optimality conditions for control-approximation problems of the type $(P_{\overset{o}{\lambda}})$ and the dual $(P^*_{\overset{o}{\lambda}})$ this implies that $(\overset{o}{x},\overset{o}{u},\overset{o}{v})$ and $(\overset{o}{p}^*,\overset{o}{\gamma}^*)$ are solutions to $(P_{\overset{o}{\lambda}})$ and $(P^*_{\overset{o}{\lambda}})$, respectively.

Furthermore there is strong duality $\min(P_{\overset{\circ}{\lambda}}) = \max(P^*_{\overset{\circ}{\lambda}})$.

Hence, $(\overset{o}{x}, \overset{o}{u}, \overset{o}{v})$ is properly efficient to (P) per definition (cf. definition 2.1.). Now one may apply the considerations of the proof of theorem 4.1. and so also the asserted representation for $\overset{o}{\delta}{}^*$ by means of (20) is obtained. □

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