On Ill-Posedness and Local Ill-Posedness of Operator Equations in Hilbert Spaces

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Abstract

In this paper, we study ill-posedness concepts of nonlinear and linear inverse problems in a Hilbert space setting. We define local ill-posedness of a nonlinear operator equation $F(x) = y_0$ in a solution point x_0 and the interplay between the nonlinear problem and its linearization using the Fréchet derivative $F'(x_0)$. To find an appropriate ill-posedness concept for

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the linearized equation we define intrinsic ill-posedness for linear operator equations $Ax = y$ and compare this approach with the ill-posedness definitions due to Hadamard and Nashed.

1 Introduction

In this paper we are going to analyse *inverse problems*, which can be written as an *operator equation*

$$F(x) = y_0\,, \tag{1.1}$$

where $F : D(F) \subseteq H_1 \longrightarrow H_2$ expresses a *continuous* and in our considerations preferably *nonlinear operator* between *separable infinite dimensional Hilbert spaces* H_1 and H_2 with norms $\|.\|_1$ and $\|.\|_2$, respectively, and with a domain $D(F)$. We are particularly interested in studying the phenomenon and the different varieties of *ill-posedness* for such equations. For *linear* problems

$$A\,x = y\,, \tag{1.2}$$

which can be considered as a particular case of (1.1) with $F(x) := Ax$, $A \in \mathcal{L}(H_1, H_2)$ (bounded linear operator from H_1 into H_2) and $D(F) = H_1$, the character of ill-posedness is *global* and depends only on properties of the operator A. For *nonlinear* operators F, the kind of ill-posedness may change with the solution point x_0 under consideration. Therefore we focus our attention to the *local* behaviour of equation (1.1) in a neighbourhood of x_0, where we assume $F(x_0) = y_0$ and impose some general assumptions:

Assumption 1.1 *Throughout this paper let hold the following properties:*

(i) *There exists a radius $r > 0$ such that*
$B_r(x_0) := \{x \in H_1 : \|x - x_0\|_1 \leq r\} \subseteq D(F)$.

(ii) *F is continuous, i.e., strong convergence $x_n \to \bar{x}$ in H_1 implies, for $x_n, \bar{x} \in D(F)$, strong convergence $F(x_n) \to F(\bar{x})$ in H_2 as $n \to \infty$.*

(iii) *F is weakly (sequentially) closed, i.e., weak convergence $x_n \rightharpoonup \tilde{x}$ in H_1 and $F(x_n) \rightharpoonup \tilde{y}$ in H_2, for $x_n \in D(F)$, imply $\tilde{x} \in D(F)$ and $F(\tilde{x}) = \tilde{y}$.*

(iv) *There exist, for all $x \in \mathrm{int}(B_r(x_0))$ (set of inner points of $B_r(x_0)$), Fréchet derivatives $F'(x) \in \mathcal{L}(H_1, H_2)$ of the operator F at the point x. Moreover, we assume that the mapping $x \mapsto F'(x)$ is continuous for all $x \in \mathrm{int}(B_r(x_0))$.*

Since we require for F the existence of Fréchet derivatives $F'(x)$, it makes sense to confront the nonlinear equation (1.1) with its linearization

$$F'(x_0)\, x = y \tag{1.3}$$

at the point x_0. Note that equation (1.3) can also be considered as a linear special case of (1.1) possessing the form (1.2) with $A = F'(x_0)$.

For linear as well as for nonlinear operator equations we have to distinguish the situations that the operators F and A are compact or noncompact, respectively.

Definition 1.2 *We call the operator F compact on $S \subseteq D(F)$, if F transforms every subset $\tilde{S} \subseteq S$, where $\tilde{S}$ is bounded in H_1, into a relatively compact subset $F(\tilde{S})$ in H_2.*

As we will see, the compactness of F essentially influences the kind of ill-posedness of a problem (1.1). It is easy to prove that the following two lemmas hold:

Lemma 1.3 *If an operator $F : D(F) \subseteq H_1 \to H_2$ is weakly (sequentially) closed and compact on $S \subseteq D(F)$, then it transforms weakly convergent sequences $x_n \rightharpoonup \tilde{x}$ in H_1 with $x_n, \tilde{x} \in S$ into strongly convergent sequences $F(x_n) \to F(\tilde{x})$ in H_2 as $n \to \infty$.*

Lemma 1.4 *If the weakly (sequentially) closed operator F is compact and injective on $B_r(x_0)$, then for a sequence $\{x_n\} \subset B_r(x_0)$ the weak convergence $F(x_n) \rightharpoonup F(x_0)$ in H_2 also implies the weak convergence $x_n \rightharpoonup x_0$ in H_1 as $n \to \infty$.*

HADAMARD's classical definition of well-posedness and ill-posedness (see Def. 2.1) is of global nature and applies in particular to linear problems (1.2). For a local analysis, which is appropriate for nonlinear operator equations (1.1), we suggest the following ill-posedness definition:

Definition 1.5 *We define a nonlinear operator equation (1.1) to be locally ill-posed in x_0 if, for arbitrarily small $\rho > 0$, there exists a sequence $\{x_n\} \subset B_\rho(x_0)$ satisfying the condition*

$$F(x_n) \to F(x_0) \quad \text{in} \quad H_2 \quad \text{as} \quad n \to \infty, \quad \text{but} \quad x_n \not\to x_0 \quad \text{in} \quad H_1. \tag{1.4}$$

Otherwise, we call the nonlinear equation (1.1) locally well-posed in x_0.

Note that local well-posedness in x_0 in the sense of Definition 1.5 implies the existence of a positive radius ρ such that

$$F(x_n) \to F(x_0) \quad \text{in} \quad H_2, \quad x_n \in B_\rho(x_0) \quad \Rightarrow \quad x_n \to x_0 \quad \text{in} \quad H_1. \tag{1.5}$$

On the other hand, local ill-posedness in x_0 implies, for arbitrarily small $\rho > 0$, the existence of a radius $0 < \tilde{\rho} < \rho$ such that a sequence $\{x_n\}$ satisfying (1.4) belongs to the ring $B_\rho(x_0) \setminus B_{\tilde{\rho}}(x_0)$. A necessary condition for local well-posedness in x_0 is that x_0 is an isolated point of the set

$$X_0 := \{x \in D(F) : F(x) = y_0\} \tag{1.6}$$

of solutions to (1.1) for given right-hand side $y_0 \in H_2$, or in other words we have

$$X_0 \cap B_\rho(x_0) = \{x_0\} \quad \text{for } \rho > 0 \text{ sufficiently small.} \tag{1.7}$$

If

$$F_\rho^{-1}(y) := \{x \in B_\rho(x_0) : F(x) = y\}$$

determines the restricted multi-valued inverse mapping F^{-1} of F with domain $F(B_\rho(x_0))$ and range $B_\rho(x_0)$, then the well-posedness condition (1.5) expresses the continuity of F^{-1} at the point y_0 in the sense of

$$y_n \to y_0 \text{ in } H_2,\ y_n \in F(B_\rho(x_0)) \ \Rightarrow\ qdist(F_\rho^{-1}(y_n), x_0) \to 0 \text{ as } n \to \infty, \tag{1.8}$$

where the point-to-set quasi-distance is defined by $qdist(M, x_0) := \max\limits_{z \in M} \|z - x_0\|_1$.

Our Definition 1.5 of local ill-posedness has in mind inverse problems, where a really existing and uniquely determined physical quantity expressed by a Hilbert space element x_0 is to be identified from noisy data y_δ of y_0, where we assume $\|y_\delta - y_0\| \le \delta$ for a given noise level δ. Then we call the problem also locally ill-posed if (1.7) is injured. Namely, in such a case we have no chance just to select x_0 from X_0 even if the noise level tends to zero. The frequently used concept of x^*-minimum norm solutions $x_{mn} \in X_0$, where

$$\|x_{mn} - x^*\|_1 = \min\{\|\tilde{x} - x^*\|_1 : A\tilde{x} = y_0,\ \tilde{x} \in D(F)\} \tag{1.9}$$

for a given initial guess $x^* \in H_1$ (cf. e.g. [6]) is by our philosophy only a way to handle this aspect of ill-posedness.

The local character of Definition 1.5 becomes evident by considering the fact that the problem (1.1) is even locally well-posed in x_0, if we have a sequence $\{x_n\}$ in a ring $\{x \in H_1 : 0 < \rho < \|x - x_0\|_1 \le r\}$ satisfying the condition (1.4) provided that we have no sequence of this kind in the ball $B_\rho(x_0)$.

Local ill-posedness of a nonlinear problem (1.1) in a solution point x_0 is a disadvantageous property, since it complicates the stable approximation of this solution for given noisy data. The stronger the operator F is 'smoothing', the

more the ill-posedness finds its expression in instability effects. The following theorem shows that compact operators F tend to imply bad situations in general. The result of this theorem was already mentioned by ENGL, KUNISCH, NEUBAUER in the Appendix of [6].

Theorem 1.6 *Let the operator F be compact on $B_r(x_0)$. Then the nonlinear equation (1.1) is locally ill-posed in x_0.*

Proof: In the infinite dimensional Hilbert space H_1 there exists an infinite complete orthonormal system $\{e_n\}$ such that, for an arbitrarily chosen element $u \in H_1$, we have a decomposition $u = \sum_{n=1}^{\infty} (u, e_n)_1 \, e_n$ with $\|u\|_1^2 = \sum_{n=1}^{\infty} (u, e_n)_1^2 < \infty$ and $(u, e_n)_1 \to 0$ as $n \to \infty$. For $0 < \rho \leq r$, the sequence $x_n := x_0 + \rho \, e_n$ belongs to the ball $B_\rho(x_0)$. Consequently, we have $e_n \rightharpoonup 0$, $x_n \rightharpoonup x_0$ and $\|x_n - x_0\|_1 = \rho > 0$. By applying Lemma 1.3 we obtain $F(x_n) \to F(x_0)$. Therefore, this sequence $\{x_n\}$ satisfies condition (1.4). Since such sequences $\{x_n\}$ exist for arbitrarily small values $\rho > 0$, this yields the local ill-posedness of (1.1) in x_0

■

Example 1.7 As an example of the nonlinear equation (1.1) with F compact we consider, for a domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary, the *Uryson integral equation*

$$[F(x)](s) := \int_\Omega k(s, t, x(t)) \, dt \; = \; y_0(s) \quad (s \in \Omega). \tag{1.10}$$

Provided that the kernel $k(s, t, x)$ is continuous with respect to x and we have nonnegative constants c_1 and c_2 and a function $f(s, t)$ with

$$\int_\Omega \int_\Omega |f(s, t)|^2 \, ds \, dt < \infty \tag{1.11}$$

such that

$$|k(s, t, x)| \leq f(s, t)(c_1 + c_2|x|) \quad (s, t \in \Omega, \, -\infty < x < \infty) \tag{1.12}$$

then the nonlinear Uryson integral operator $F : H_1 := L^2(\Omega) \longrightarrow H_2 := L^2(\Omega)$ is *compact on the whole space* $L^2(\Omega)$ (see e.g. KRASNOSELSKII's Theorem 7.1 in [13]). If the kernel $k(s,t,x)$ is smooth enough, the Fréchet derivative exists for all $x_0 \in L^2(\Omega)$ and attains the form

$$[F'(x_0)\,h](s) = \int_\Omega k'_x(s,t,x_0(t))\,h(t)\,dt \quad (s \in \Omega,\ h \in L^2(\Omega)) \tag{1.13}$$

of a compact linear Fredholm integral operator in $L^2(\Omega)$. A sufficient condition is e.g. that the second partial derviative with respect to x exists and is uniformly bounded, i.e., $|k''_{xx}(s,t,x)| \le const$ (see e.g. KANTOROVICH/AKILOV [14, p.548]).

As the Example 1.9 will show, local ill-posedness may also occur if F is not compact. However, before we give an example of local well-posedness (for the analytical background see also ZEIDLER [19, Sec. 16.4]).

Theorem 1.8 *Let $H := H_1 = H_2$ and $F := I + K$, where I is the identity operator and $K : B_r(x_0) \subset H \to H$ is a continuous, weakly (sequentially) closed and compact operator. Then equation (1.1) is locally well-posed in x_0 whenever x_0 is the uniquely determined solution of*

$$x + K(x) = y_0, \quad x \in B_r(x_0). \tag{1.14}$$

Proof: To prove well-posedness of (1.1) we will show that the implication (1.5) is satisfied for $\rho = r$. Under the assumptions stated above we have

$$F(x_n) \to F(x_0) \quad \text{in} \quad H, \quad x_n \in B_r(x_0) \quad \Rightarrow \quad x_n \rightharpoonup x_0 \quad \text{in} \quad H. \tag{1.15}$$

Namely, assume $F(x_n) \to F(x_0)$ in H as $n \to \infty$. Since F is weakly (sequentially) closed and $B_r(x_0)$ is weakly compact, there exists a subsequence $x_{n_k} \rightharpoonup \tilde{x} \in B_r(x_0)$, for which we have $F(\tilde{x}) = F(x_0)$. By the unique solvability of (1.14) this implies $\tilde{x} = x_0$ and $x_n \rightharpoonup x_0$ as $n \to \infty$. Then from Lemma 1.3 we obtain $K(x_n) \to K(x_0)$ and therefore the strong convergence $x_n \to x_0$ in H, i.e., (1.4) cannot be fulfilled ∎

Note that the uniqueness of solutions to (1.14) assumed in Theorem 1.8 can frequently be shown by applying Banach's fixed point theorem provided that the operator K is sufficiently contractive.

Example 1.9 In the paper [7] it is shown that the nonlinear *autoconvolution equation*

$$[F(x)](s) := \int_0^1 x(s-t)\,x(t)\,dt \;=\; y_0(s) \quad (0 \le s \le 1) \tag{1.16}$$

with the nonlinear autoconvolution operator $F : D(F) \subset H_1 := L^2(0,1) \longrightarrow H_2 := L^2(0,1)$ and domain $D(F) := \{x \in L^2(0,1) : x(t) \ge 0 \text{ a.e. in } [0,1]\}$ is *locally ill-posed everywhere, although F is not compact.* Note, however, two facts: First we have to mention that for all points $x_0 \in \text{int}(D(F))$ the Fréchet derivative $F'(x_0)$ of the noncompact nonlinear operator F of the form

$$[F'(x_0)\,h]\,(s) \;=\; \int_0^s x_0(s-t)\,h(t)\,dt \qquad (0 \le s \le 1,\; h \in L^2(0,1)) \tag{1.17}$$

is a *compact* linear Volterra integral operator in $L^2(0,1)$. On the other hand, the nonlinear autoconvolution operator becomes also compact if the Sobolev space $H_1 := W_2^1(0,1)$ of functions possessing quadratically integrable generalized derivatives is considered instead of $H_1 := L^2(0,1)$. This is a consequence of the compact embedding from $W_2^1(0,1)$ into $L^2(0,1)$. In any case, the autoconvolution operator F of this example satisfies all requirements of Assumption 1.1.

In the case of linear operator equations (1.2), where the analysis of ill-posedness phenomena has global character, one can define well-posed problems to be characterized by a *closed range* $R(A)$ of the operator A under consideration (see Definition 2.5 below). For a local analysis and in the case of a nonlinear and injective operator F the question whether we have local well-posedness or local ill-posedness in x_0 can be reduced to the question whether $y_0 = F(x_0)$ is an *inner point* of the sets $F(B_\rho(x_0))$:

Theorem 1.10 *Let F be injective on $B_r(x_0)$. Then problem (1.1) is locally well-posed in x_0 if and only if*

$$y_0 = F(x_0) \in \mathrm{int}(F(B_\rho(x_0))) \quad \textit{for all} \quad 0 < \rho \leq r. \tag{1.18}$$

Proof: For an arbitrarily fixed $0 < \rho < r$, we consider in the ill-posed case a sequence $\{x_n\}$ from the ring $B_r(x_0) \setminus B_\rho(x_0)$ satisfying condition (1.4). Then condition (1.18) would imply that $F(x_n) \in F(B_\rho(x_0))$ for sufficiently large n. This, however, contradicts the injectivity of F. On the other hand, in the well-posed case condition (1.5) reflects the continuity of the inverse operator F^{-1} at the point y_0. In view of the injectivity of F, F^{-1} restricted to the domain $F(B_r(x_0))$ is single-valued. This yields condition (1.18), since the operator F transforms a neighbourhood of x_0 into a neighbourhood of y_0 when F^{-1} is continuous at the point y_0 (cf. DIEUDONNÉ [4, 3.11.1]) ∎

From the above proof we also obtain that in the case where F is injective and condition (1.18) is valid, condition (1.4) cannot hold for any sequence $\{x_n\}$ in a ball $B_\rho(x_0)$ with $0 < \rho \leq r$. In the subscase of compact operators F we can prove the following theorem:

Theorem 1.11 *If F is compact and injective on $B_r(x_0)$, then for $0 < \rho \leq r$ there is no infinite dimensional subspace $\tilde{H}_2$ of H_2 such that*

$$y_0 = F(x_0) \in \mathit{intrel}_{M(y_0)}(F(B_\rho(x_0))).$$

The condition $y_0 \in \mathit{intrel}_{M(y_0)}(F(B_\rho(x_0)))$ denotes that y_0 belongs to the relative interior of $F(B_\rho(x_0))$ with respect to the linear manifold

$$M(y_0) := \{y \in H_2 : y = y_0 + \tilde{y},\ \tilde{y} \in \tilde{H}_2\}.$$

Proof: For an infinite dimensional Hilbert space $\tilde{H}_2$, the condition $y_0 \in \mathit{intrel}_{M(y_0)}(F(B_\rho(x_0)))$ would require that there exists a positive value β and an infinite system $\{e_n\} \subset \tilde{H}_2$ orthonormal in H_2 such that $y_0 + \beta e_n = F(x_n)$, $x_n \in$

$B_\rho(x_0)$, for all n. Then we would have $e_n \rightharpoonup 0$, $F(x_n) \rightharpoonup y_0$ and by Lemma 1.4 $x_n \rightharpoonup x_0$. On the other hand, $F(x_n) \not\to y_0$ contradicts the compactness of F (see Lemma 1.3) ∎

2 The linear case

In this section, we are going to embed the ill-posedness concept of *linear operator equations* (1.2) into our general considerations of equations (1.1).

We first refer to HADAMARD's definition.

Definition 2.1 *We call the linear equation (1.2) well-posed in Hadamard's sense if the continuous linear operator $A \in \mathcal{L}(H_1, H_2)$ is bijective and the inverse operator A^{-1} is continuous, i.e., $A^{-1} \in \mathcal{L}(H_2, H_1)$. Otherwise, we call the equation ill-posed in Hadamard's sense.*

As outlined for example in the books of LOUIS ([16, p.8]), COLTON, KRESS ([3, p.85]) and KIRSCH ([15, p.10]) we can distinguish three types of Hadamard's ill-posedness:

1) *Nonexistence* if A is not surjective;

2) *Nonuniqueness* if A is not injective;

3) *Instability* if the inverse operator A^{-1} is not continuous.

In the case of 1) equation (1.2) is solvable not for all $y \in H_2$, in the case of 2) the solution is not uniquely determined, and in the case of 3) the solutions do not depend continuously on the right-hand side y. Note that there is an interdependence between these three types of ill-posedness.

If equation (1.2) is well-posed in Hadamard's sense (see Definition 2.1), then the range $R(A) := \{y \in H_2 : y = Ax, x \in H_1\} = A(H_1)$ coincides with the whole space H_2 and is therefore a closed set in H_2. Moreover, in view of the following proposition, formulated by BAUMEISTER in [2, p.19], this closedness of the range $R(A)$ is also a criterion of stability if A is injective, but not surjective.

Proposition 2.2 *Let $A \in \mathcal{L}(H_1, H_2)$ be injective. Then the following conditions are equivalent:*

a) $R(A)$ is a closed subset in H_2.

b) $A^{-1} : R(A) \subseteq H_2 \to H_1$ is continuous.

Proposition 2.2 is an immediate result of BANACH's *open mapping theorem* (see e.g. ZEIDLER [19, p.777])

By our philosophy mentioned in Sec. 1 noninjectivity implies always ill-posedness. However, in the literature we frequently find alternative concepts that define a linear problem *well-posed if and only if the Moore-Penrose generalized inverse $A^\dagger$ is bounded*. We summarize some such ideas following the concept of NASHED ([17]). A linear operator B between the Hilbert spaces H_2 and H_1 is called *inner inverse* of A if

$$ABA = A \tag{2.1}$$

On the other hand, a (nonzero) linear operator B is called *outer inverse* of A if

$$BAB = B. \tag{2.2}$$

If an inner or outer inverse is a continuous (bounded) linear operator $B \in \mathcal{L}(H_2, H_1)$, then we call it *bounded inner* or *bounded outer* inverse.

The setting $x := By$ yields a solution of (1.2) for every $y \in R(A)$ if and only if B is an inner inverse. If A is not injective, an inner inverse B selects for every $y \in R(A)$ a well-determined representative $x = By$ from the infinite set of solutions. In this sense, any inner inverse B describes a specific generalized solution concept.

Proposition 2.3 *[17, p.58]. The operator $A \in \mathcal{L}(H_1, H_2)$ has a bounded inner inverse B satisfying (2.1) if and only if $R(A)$ is a closed subset in H_2.*

If and only if $R(A)$ is *nonclosed*, every inner inverse B of A is a discontinuous (unbounded) operator. Using the concept of *minimum-norm solutions* we select

the solution x_{mn} with minimum norm $\|x_{mn}\|_1 = \min\{\|x\|_1 : Ax = y,\, x \in H_1\}$. We then have $x_{mn} = A^\dagger y$ with the *Moore-Penrose generalized inverse* $B = A^\dagger$ and its natural domain $D(A^\dagger) = R(A) \oplus R(A)^\perp$, where $A^\dagger$ is the zero-operator on $R(A)^\perp$. Note that $A^\dagger$ is an inner as well as an outer inverse of A.

Proposition 2.4 *[8, p.81]. $A^\dagger : D(A^\dagger) \subset H_2 \to H_1$ is a closed linear operator densely defined on H_2, which is continuous (bounded) if and only if $R(A)$ is a closed subset in H_2.*

Now it is evident that minimum-norm solutions stably depend on the data only if $R(A)$ is closed and therefore $A^\dagger$ is bounded. Restricted to this stability aspect the well-posedness definition can be diminished as follows:

Definition 2.5 *We call the linear equation (1.2) well-posed in Nashed's sense if the range $R(A)$ of $A \in \mathcal{L}(H_1, H_2)$ is a closed subset in H_2. Otherwise, we call the equation ill-posed in Nashed's sense.*

As the next proposition formulated by GROETSCH indicates, (1.2) is ill-posed in Nashed's sense for every nondegenerate compact linear operator A.

Proposition 2.6 *[8, p.82]. If A is compact, then $A^\dagger : D(A^\dagger) \subset H_2 \to H_1$ is a discontinuous (unbounded) operator if and only if $dim(R(A)) = \infty$.*

In [17] NASHED distinguishes the "less" ill-posed problems of type I (A noncompact with nonclosed range) in analogy to the full-rank case for matrices and the "more" ill-posed problems of type II (compact with infinite dimensional range) in analogy to the rank-deficient case for matrices.

Definition 2.7 *An ill-posed problem (1.2) in Nashed's sense is said to be of type I if there exists a family of bounded outer inverses $B = B_\alpha$, $\alpha > 0$ satisfying (2.2) and possessing infinite dimensional ranges $R(B_\alpha)$ such that*

$$\lim_{\alpha \to 0} \|B_\alpha A x - x\|_1 = 0 \quad \textit{for each} \quad x \in H_1. \tag{2.3}$$

Otherwise the problem is called ill-posed of type II.

Proposition 2.8 *[17, p.68]. Let the range $R(A)$ of $A \in \mathcal{L}(H_1, H_2)$ be nonclosed in H_2. Then the following statements are equivalent:*

a) The ill-posed problem (1.2) is of type I.

b) $R(A)$ contains a closed infinite dimensional subspace in H_2.

c) A is not compact.

Note that in the degenerate case of a compact linear operator A with $\dim(R(A)) < \infty$ we have $A^\dagger$ bounded and $R(A)$ closed in H_2. Hence, in that case (1.2) is well-posed in Nashed's sense. On the other hand, from point of view of the information contents contained in the data y this situation is extremely bad if we try to recover a unique parameter element x_0. Therefore, we refer to our concept of local ill-posedness formulated in Definition 1.5 as an alternative and consider in the following the linear equations (1.2) from that point of view. It is easy to show that the restricted range

$$A(B_\rho(x_0)) := \{y \in H_2 : y = Ax,\, x \in B_\rho(x_0)\}$$

is a *closed convex* subset in H_2 whenever $0 < \rho \leq r$. For the linear case $F(x) := Ax$ local well-posedness requires the *injectivity* of A. Provided A is injective, then condition (1.8) holds if and only if A is continuously invertible or, in other words, $R(A)$ is closed (see Proposition 2.2).

Corollary 2.9 *The linear equation (1.2) is either in all points $x_0 \in H_1$ locally well-posed in the sense of Definition 1.5 or locally ill-posed in all points. Local well-posedness occurs if and only if A is injective and $R(A)$ is closed in H_2.*

In the sequel we call linear equations *intrinsically well-posed* or *intrinsically ill-posed* if these equations considered as a special case of (1.1) are locally well-posed or locally ill-posed in the sense of Definition 1.5.

Definition 2.10 *We call the linear equation (1.2) intrinsically well-posed if the continuous linear operator $A : H_1 \to H_2$ is injective and the inverse operator A^{-1} :*

$R(A) \subseteq H_2 \to H_1$ *is continuous. Otherwise, we call the equation intrinsically ill-posed.*

It is evident that an intrinsically well-posed linear equation (1.2) can be characterized by the following condition: For any sequence $\{h_n\} \subset H_1$ we have the implication

$$A\,h_n \to 0 \quad \text{in} \quad H_2 \quad \text{as} \quad n \to \infty \quad \Rightarrow \quad h_n \to 0 \quad \text{in} \quad H_1 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

On the other hand, a condition for an intrinsically ill-posed linear equation (1.2) can be formulated as follows: For any given $\rho > 0$ there exists a sequence $\{h_n\} \subset H_1$ with

$$A\,h_n \to 0 \quad \text{in} \quad H_2 \quad \text{as} \quad n \to \infty, \quad \text{but} \quad \|h_n\|_1 = \rho \quad (n = 1, 2, ...). \tag{2.5}$$

Note that *a linear problem (1.2) with A compact is always intrinsically ill-posed*, also if $\dim(R(A)) < \infty$ and hence $A^\dagger$ is bounded. On the other hand, the operator $A \in \mathcal{L}(H_1, H_2)$ of an intrinsically well-posed linear equation is not necessarily surjective, since $R(A)$ can be a closed subspace of H_2 that does not coincide with H_2. Hence, intrinsic well-posedness in the sense of Definition 2.10 is well-distinguished from Hadamard's well-posedness (Definition 2.1) and from Nashed's well-posedness (Definition 2.5).

The question whether y is an inner point of $A(B_\rho(x_0))$ is also interesting in the linear case. If A is injective with a closed range $R(A)$, then we have by Theorem 1.10 the condition $y \in int(A(B_\rho(x_0)))$ for all $0 < \rho \leq r$. Note that the radius ρ in the linear case is not important. Namely, we have $y \in int(A(B_\rho(x_0)))$ for all $\rho > 0$ if and only if this condition is satisfied for any fixed value $\rho = r > 0$. If A is compact, then $y \notin \text{intrel}_{M(y)}(A(B_\rho(x_0)))$ for every infinite dimensional linear manifold $M(y)$ (see Theorem 1.11). It remains to discuss the case of Nashed's ill-posedness of type I (A noncompact and $R(A)$ nonclosed). In the general remarks of Sec. 1 there are no statements on such a case. However, we can give the following characterization:

Theorem 2.11 *If A is noncompact and $R(A)$ is nonclosed in H_2, then there exists a linear manifold $M(y) := \{\hat{y} \in H_2 : \hat{y} = y + \tilde{y}, \tilde{y} \in \tilde{H}_2\}$ generated by an infinite dimensional linear subspace $\tilde{H}_2$ of H_2 such that $y = Ax_0 \in intrel_{M(y)}(A(B_\rho(x_0)))$ for all $\rho > 0$. However, we have $y \notin int(A(B_\rho(x_0)))$ for all $\rho > 0$.*

Proof: The existence of an infinite dimensional manifold $M(y)$ as required is a consequence of Proposition 2.8 (see in particular statement b) of this proposition). On the other hand, for $R(A)$ nonclosed, we have $R(A) \neq H_2$ and there exists in H_2 a nonzero element $\tilde{y} \notin R(A)$. Then y is no inner point of $A(B_\rho(x_0))$, since there are no range elements of the form $y + \alpha\tilde{y}$ for any real value $\alpha \neq 0$ ∎

3 Interdependeces between the nonlinear and the linearized problem

In this section we are going to study the interplay between the original nonlinear inverse problem (1.1) and its linearization (1.3) with respect to the well-posedness or ill-posedness of both problems. We focus our attention to the local well-posedness or ill-posedness of the nonlinear equation (cf. Definition 1.5) and to the intrinsic well-posedness or ill-posedness (cf. Definition 2.10) of the linearized version.

The appendix of the paper [6] contains examples explicitly given, where the nonlinear problem is well-posed, but the associated linearized problem is ill-posed and vice versa. However, that paper defines ill-posedness as the instable dependence of minimum-norm solutions upon the data. The examples therein do not apply to our concepts of local ill-posedness and intrinsic ill-posedness. Nevertheless, it should be conjectured that situations with a locally ill-posed equation (1.1) and an intrinsically well-posed linearization (1.3) as well as situations with a locally well-posed equation (1.1) and an intrinsically ill-posed linearization (1.3)

are possible. However, we have no explicit examples for such situations. On the other hand, in the following we can formulate sufficient conditions that exclude such situations a priori.

First in the case of a compact nonlinear operator F we can extend the Theorem 1.6 ensuring local ill-posedness of the nonlinear equation to an assertion on intrinsic ill-posedness for its linearization. The following lemma is an immediate consequence of Theorem 4.19 in [3]:

Lemma 3.1 *Let the nonlinear operator F be compact on $B_r(x_0)$. Then the Fréchet derivatives $F'(x)$ are compact linear operators for all $x \in \mathrm{int}(B_r(x_0))$.*

By our considerations of Sec. 2 this lemma yields as an immediate consequence:

Corollary 3.2 *If the nonlinear equation (1.1) being locally ill-posed in x_0 is characterized by an operator F, which is compact on $B_r(x_0)$, then the associated linearized equation (1.3) is intrinsically ill-posed.*

In a second step we temporarily assume that, with a constant $c_1 > 0$, it holds the inequality

$$\|F(x) - F(x_0)\|_2 \leq c_1 \|F'(x_0)(x - x_0)\|_2 \qquad \text{for all} \quad x \in B_r(x_0). \tag{3.6}$$

Theorem 3.3 *Let the operator F of equation (1.1) satisfy the inequality (3.6). If then (1.1) is locally well-posed in x_0, the associated linearized problem (1.3) is intrinsically well-posed. On the other hand, if (1.3) is intrinsically ill-posed, then (1.1) is locally ill-posed in x_0.*

Proof: Let F be locally well-posed in x_0. Then we have for a certain value $0 < \rho \leq r$ the condition

$$F(x_n) \to F(x_0) \quad \text{in} \quad H_2, \quad x_n \in B_\rho(x_0) \quad \Rightarrow \quad x_n \to x_0 \quad \text{in} \quad H_1. \tag{3.7}$$

Now assume the linearized problem (1.3) to be intrinsically ill-posed. Then, for any given $0 < \rho \leq r$, we have a sequence $\{x_n\} \subset H_1$ satisfying

$$F'(x_0)\,(x_n - x_0) \;\to 0 \text{ in } H_2 \quad \text{as} \quad n \to \infty, \quad \text{but} \quad \|x_n - x_0\|_1 = \rho \quad (n = 1, 2, ...) \tag{3.8}$$

(cf. (2.5)). Formula (3.6) implies for that sequence $F(x_n) \to F(x_0)$ in H_2 as $n \to \infty$. This, however, contradicts the condition (3.7). Consequently, the equation (1.3) is intrinsically well-posed. Vice versa, in the same manner we can conclude from (3.8) and (3.6) that intrinsic ill-posedness of (1.3) implies local ill-posedness of (1.1) in x_0 ∎

In a third step we temporarily assume that, with a constant $c_2 > 0$, it holds the inequality

$$\|F'(x_0)(x - x_0)\|_2 \;\leq\; c_2\,\|F(x) - F(x_0)\|_2 \qquad \text{for all} \quad x \in B_r(x_0). \tag{3.9}$$

Theorem 3.4 *Let the operator F of equation (1.1) satisfy the inequality (3.9). If then (1.1) is locally ill-posed in x_0, the associated linearized problem (1.3) is intrinsically ill-posed. On the other hand, if (1.3) is intrinsically well-posed, then (1.1) is locally well-posed in x_0.*

The proof of Theorem 3.4 is similar to the proof of Theorem 3.3. As a consequence of both theorems we obtain:

Corollary 3.5 *Let the operator F of equation (1.1) satisfy the estimations*

$$\underline{c}\|F'(x_0)(x-x_0)\|_2 \leq \|F(x)-F(x_0)\|_2 \leq \overline{c}\|F'(x_0)(x-x_0)\|_2 \quad \textit{for all} \quad x \in B_r(x_0), \tag{3.10}$$

where $0 < \underline{c} \leq \overline{c} < \infty$ are constants. Then (1.1) is locally ill-posed in x_0 if and only if the associated linearized problem (1.3) is intrinsically ill-posed.

Now we are going to discuss the conditions (3.6) and (3.9). Obviously, for $F'(x_0)$ injective, the condition (3.6) can be interpreted as a local Lipschitz continutity of the composite operator $F(F'(x_0))^{-1}$. That means, F is smoothing

enough in order to compensate the roughening effect of $F'(x_0)^{-1}$. For (3.6) and $F'(x_0)$ compact, weak convergence $x_n \rightharpoonup x_0$ implies strong convergence $F(x_n) \to F(x_0)$ as $n \to \infty$. In [7] it is shown for the autoconvolution problem in $L^2(0,1)$ (cf. Example 1.9) that this implication is violated. Therefore a condition (3.6) cannot hold for that example.

The condition (3.9), however, is a local Lipschitz continuity of $F'(x_0)F^{-1}$ whenever F^{-1} exists. This fact is closely related with the *local degree of ill-posedness* in x_0 of equation (1.1) characterized by the degree of ill-posedness of the linearized problem (1.3). For details we refer to the papers [10], [11] and [12].

Under the assumptions stated above we have for the remainder in Taylor's formula

$$\|F(x) - F(x_0)\|_2 = \| \int_0^1 F'(x_0 + t(x - x_0))(x - x_0)\, dt\|_2 \quad \text{for all} \quad x \in B_r(x_0). \tag{3.11}$$

By considering that formula it becomes obvious that neither formula (3.6) nor formula (3.9) can generally be obtained without additional assumptions on F in a neighbourhood of x_0, since $\|F(x) - F(x_0)\|_2$ depends not only on $F'(x_0)$, but on the whole family of operators $F'(x_0 + t(x - x_0))$ for $0 \le t \le 1$. For an interesting idea of coupling the Fréchet derivatives (see formula (3.12) below) providing the condition (3.10) we refer to HANKE, NEUBAUER, SCHERZER [9] (see also Sec. 11.1 in the book [5] of ENGL, HANKE, NEUBAUER). The consequences of this idea will be formulated by the following theorem:

Theorem 3.6 *Let there exist, for given constants $0 < \rho < r$ and $K > 0$, a family $\{L_x : x \in B_\rho(x_0)\}$ of bounded linear operators $L_x \in \mathcal{L}(H_1, H_2)$ satisfying the conditions*

$$F'(x) = L_x F'(x_0) \quad \textit{and} \quad \|L_x - I\|_{\mathcal{L}(H_1,H_2)} \le K\|x - x_0\|_1 \quad \textit{for all} \quad x \in B_\rho(x_0). \tag{3.12}$$

Then we have a radius $\tilde{\rho} > 0$ such that

$$\underline{c}\|F'(x_0)(x-x_0)\|_2 \le \|F(x)-F(x_0)\|_2 \le \overline{c}\|F'(x_0)(x-x_0)\|_2 \quad \textit{for all} \quad x \in B_{\tilde{\rho}}(x_0), \tag{3.13}$$

where $0 < \underline{c} \le \overline{c} < \infty$. Consequently, the local ill-posedness in x_0 of the nonlinear equation (1.1) and the intrinsic ill-posedness of the corresponding linearized problem (1.3) are equivalent properties.

Proof: We can estimate

$$\begin{aligned}
\|F(x)-F(x_0)-F'(x_0)(x-x_0)\|_2 &= \| \int_0^1 (F'(x_0+t(x-x_0))-F'(x_0))(x-x_0)\,dt\|_2 \\
&= \| \int_0^1 (L_{x_0+t(x-x_0)} - I)F'(x_0)(x-x_0)\,dt\|_2 \\
&\le \int_0^1 \|L_{x_0+t(x-x_0)} - I\|_{\mathcal{L}(H_1,H_2)} \|F'(x_0)(x-x_0)\|_2\,dt \\
&\le K\,\|F'(x_0)(x-x_0)\|_2\,\|x-x_0\|_1 \quad \text{for all} \quad x \in B_\rho(x_0).
\end{aligned}$$

Now an inequality

$$\|F(x)-F(x_0)-F'(x_0)(x-x_0)\|_2 \;\le\; K\,\|F'(x_0)(x-x_0)\|_2\,\|x-x_0\|_1 \quad (x \in B_\rho(x_0)) \tag{3.14}$$

implies by the triangle inequality

$$\|F(x)-F(x_0)\|_2 \;\le\; (K\|x-x_0\|_1 + 1)\,\|F'(x_0)(x-x_0)\|_2 \quad (x \in B_\rho(x_0))$$

and for sufficiently small increment norms $\|x-x_0\|_1$

$$(1-K\|x-x_0\|_1)\,\|F'(x_0)(x-x_0)\|_2 \;\le\; \|F(x)-F(x_0)\|_2.$$

If $\tilde{\rho}$ is chosen such that $1-K\tilde{\rho} > 0$, then we have (3.13) with $\underline{c} = 1-K\tilde{\rho} > 0$ and $\overline{c} = K\tilde{\rho}+1$ ∎

The paper [9] presents a couple of examples of nonlinear inverse problems (1.1), where (3.12) is satisfied and hence Theorem 3.6 applies.

From (3.14) in the proof of Theorem 3.6 we can also derive an inequality of the form

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\|_2 \leq \check{K} \, \|F(x) - F(x_0)\|_2 \, \|x - x_0\|_1 \quad (x \in B_{\tilde{\rho}}(x_0)). \tag{3.15}$$

Vice versa, (3.15) implies for sufficiently small $\rho > 0$ the inequality (3.14). That means, an estimate (3.10) with the consequences of Corollary 3.5 holds if either (3.12) or alternatively one of the conditions (3.14) or (3.15) is satisfied. In the context of Proposition 3 in [11] a class of nonlinear inverse problems (1.1) is introduced, where (3.15) is valid. The books of ANGER [1] and GROETSCH [8] provide parameter identification problems in ordinary and partial differential equations as representatives of this class of problems (see [11, Example 1 and 2]). We close this paper by mentioning another class of examples, where the estimate (3.15) also holds:

Example 3.7 In the paper [18], TAUTENHAHN considers parameter identification problems, where a parameter function, in our terms $x = x_0$, is to be determined from observations of the state variable u satisfying the operator equation

$$T_1(u) + T_2(x, u) + T_3(x) \; = \; f. \tag{3.16}$$

The operators T_1 and T_3 in (3.16) are linear, whereas T_2 is a *bilinear* operator. Under some assumptions on T_1, T_2 and T_3 prescribed in [18] it can be shown that the parameter-to-solution mapping $x \mapsto u$ corresponding to the operator F in equation (1.1) of our considerations is Fréchet differentiable and there holds in a local sense an estimate of the form (3.15). Hence, for such bilinear identification problems, Corollary 3.5 is usable and ensures that the local ill-posedness of the nonlinear identification problem and the intrinsic ill-posedness of its linearization are equivalent properties. For the Fréchet derivative $F'(x_0)$, however, there exists a constructive approach using the Implicit Function Theorem applied to equation (3.16).

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